Hadron Spectroscopy

Mathematical Techniques

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General Comments on my JLab talks

- An experimental physicist talking on theoretical topics and therefore elementary for theorists
- A lot of the formula in my talks—mostly based on intituitive perspectives and so
- Listen to what I say—not what I write—they are mostly meant for the experts, or for the those who would want to go over later
- I have given a series of numerous courses/seminars at TU/munich; and so, after some severe downsizing of the material, those pressented here are necessarily sketchy and rudimentary
- Just relax and enjoy and
- Consult the references lsited below—for further information

Introduction

- References
- The Poincaré Group
- Decay amplitudes for two- and three-body final states
- Comments on Helicity and Canonical approaches
- Reflectivity Operations
- Covariant Formulation of Helicity-Coupling Amplitudes
- Techniques of Partial-Wave Analysis—
  Extended Maximum-Likelihood Methods
- Ambiguities in the Partial-Wave Amplitudes

Not covered: massless particles and fermions

References

- S. Weinberg:
  'The Quantum Theory of Fields,' Cambridge, UK (1995), Volume I, Chapter 2: Relativistic Quantum Mechanics, p.49—
  - The Poincaré Algebra
  - One-Particle States (Mass>0 and $=0$)
  - $\boldsymbol{P}$-, $\boldsymbol{T}$-, $\boldsymbol{C}$-Operators
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Perliminaries

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  Rank-J Tensor for $|Jm\rangle$
  General $\gamma = E/m$ (the Lorentz factor) dependence
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The Poincaré Group

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'A Modern Introduction to Quantum Field Theory,' Michele Maggiore
Chapter 2 (Lorentz and Poincaré Symmetries in QFT)

The Poincaré Group

The general inhomogeneous Lorentz transformation (a, Λ)

(1) $$p'^{\,\mu} = a^\mu + \Lambda^\mu{}_\nu\, p^\nu \quad \text{and} \quad g_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\tau = g_{\rho\tau} \; :$$ The Lorentz Condition

[The latter comes from the condition $p'^{\,2}\big|_{a=0} = p^2$.] Our Lorentz metric $g^{\mu\nu} = g_{\mu\nu}$ has signature $(+,-,-,-)$ so that

(2)
$$g^\mu{}_\nu = \delta^\mu{}_\nu \quad \text{and} \quad g_\mu{}^\nu = \delta_\mu{}^\nu$$
$$\Lambda_\nu{}^\sigma\, \Lambda^\nu{}_\tau = g^\sigma{}_\tau \quad \text{or} \quad (\Lambda^{-1})^\sigma{}_\nu = \Lambda_\nu{}^\sigma$$

and

(3)
$$p^\mu = (E, p^1, p^2, p^3) = (E, p_x, p_y, p_z)$$
$$p_\mu = g_{\mu\nu}\, p^\nu = (E, p_1, p_2, p_3) = (E, -p_x, -p_y, -p_z)$$

Let w be the mass associated with p and adopt a notation in which p indicates *both* the four-momentum and the *magnitude* of the 3-momentum, i.e.

(4) $$E^2 = w^2 + p^2, \quad p^2 = p_x^2 + p_y^2 + p_z^2$$

Quantum Lorentz Transformations

The group multiplication law is

(5)
$$(a_1, \Lambda_1)\,(a_2, \Lambda_2) = (a_3, \Lambda_3),$$
$$a_3 = a_1 + \Lambda_1\, a_2, \quad \Lambda_3 = \Lambda_1\, \Lambda_2$$

The corresponding unitary representations (or unitary operators) follow the same rule

(6)
$$U(a, \Lambda) = U(a, 1)\, U(0, \Lambda) \equiv U(a)\, U(\Lambda),$$
$$U(a_1, \Lambda_1)\, U(a_2, \Lambda_2) = U(a_3, \Lambda_3)$$

Consider now an homogeneous Lorentz transformation $\Lambda(\vec{\beta})$ [$\vec{\beta} = \vec{p}/E$] *without* rotation which takes q to q'

(7)
$$q'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, q^{\nu}$$

where

(8)
$$\Lambda^0{}_0 = \gamma, \quad \Lambda^i{}_0 = \Lambda^0{}_i = \eta^i, \quad \Lambda^i{}_j = \delta^{ij} + \frac{\eta^i \eta^j}{\gamma + 1}$$

with $\gamma = E/w$ and $\eta^i = \gamma\beta^i = p^i/w$. Note

(9)
$$\gamma^2 = 1 + \eta^2 = \frac{1}{1 - \beta^2}, \quad \eta^2 = \eta_x^2 + \eta_y^2 + \eta_z^2$$

Quantum Lorentz Transformations

Denote $q = (E_q, \vec{q})$ and $q\,' = (E_q{}', \vec{q}\,')$ and see that

(10)
$$\begin{cases} E_q{}' = \gamma\, E_q + (\vec{\eta}\cdot\vec{q}) \\ \vec{q}\,' = \vec{q} + \left[E_q + \dfrac{(\vec{\eta}\cdot\vec{q})}{\gamma+1}\right]\,\vec{\eta} \end{cases}$$

The inverse of the Lorentz transormation $\Lambda(\vec{\beta}\,)$ is given by $\Lambda(-\vec{\beta}\,)$, i.e.

(11)
$$(\Lambda^{-1})^0{}_0 = \gamma, \quad (\Lambda^{-1})^i{}_0 = (\Lambda^{-1})^0{}_i = -\eta^i, \quad (\Lambda^{-1})^i{}_j = \delta^{ij} + \frac{\eta^i\eta^j}{\gamma+1}$$

so that

(12)
$$\begin{cases} E_q = \gamma\, E_q' - (\vec{\eta}\cdot\vec{q}\,') \\ \vec{q} = \vec{q}\,' + \left[-E_q' + \dfrac{(\vec{\eta}\cdot\vec{q}\,')}{\gamma+1}\right]\,\vec{\eta} \end{cases}$$

One can check for consistency by substituting q' into the equations above and see that the equalities hold.

Quantum Lorentz Transformations

Go over to infinitesimal transformations (a, Λ)

(13) $$a^\mu = \epsilon^\mu \quad \text{and} \quad \Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu \quad \text{or} \quad \Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu}$$

The unitary operators $U(a, \Lambda)$ become

$$\begin{cases} U(\epsilon) = U(\epsilon, 1) = 1 - i\,\epsilon_\mu\, P^\mu, \quad U^{-1}(\epsilon) = 1 + i\,\epsilon_\mu\, P^\mu \\ U(1+\omega) = U(0, 1+\omega) = 1 - \frac{1}{2}\, i\,\omega_{\mu\nu}\, J^{\mu\nu}, \quad U^{-1}(1+\omega) = 1 + \frac{1}{2}\, i\,\omega_{\mu\nu}\, J^{\mu\nu} \end{cases}$$

P and M are Hermitian, and ω and M are antisymmetric. That ω is antisymmetric can be worked out from the Lorentz Condition:

(14) $$\begin{aligned} g_{\rho\tau} &= g_{\mu\nu}\,(g^\mu{}_\rho + \omega^\mu{}_\rho)\,(g^\nu{}_\tau + \omega^\nu{}_\tau) \\ &= g_{\rho\tau} + \omega_{\rho\tau} + \omega_{\tau\rho} \quad \Longrightarrow \quad \omega_{\rho\tau} = -\omega_{\tau\rho} \end{aligned}$$

Poincaré group: 6 parameters $(\omega_{\mu\nu})$ + 4 parameters (ϵ^μ) = 10 parameters

Quantum Lorentz Transformations

Note that

(15) $$U(\Lambda)U(a) = U(\Lambda\, a)\, U(\Lambda)$$

so that, with $b = \Lambda\, a$,

(16) $$U^{-1}(\Lambda)\, U(b)\, U(\Lambda) = U(\Lambda^{-1}\, b)$$

For b infinitesimally small, one finds

(17) $$\begin{aligned} U(b) &= 1 - i\, b_\mu\, P^\mu \\ U(\Lambda^{-1}\, b) &= 1 - i\,(\Lambda^{-1}\, b)_\alpha\, P^\alpha \\ &= 1 - i\,(\Lambda^{-1})_\alpha{}^\mu\, b_\mu\, P^\alpha \\ &= 1 - i\,\Lambda^\mu{}_\alpha\, b_\mu\, P^\alpha \end{aligned}$$

using the identity $(\Lambda^{-1})_\alpha{}^\mu = \Lambda^\mu{}_\alpha$. And so one obtains

(18) $$U^{-1}(\Lambda)\, P^\mu\, U(\Lambda) = \Lambda^\mu{}_\alpha\, P^\alpha$$

Quantum Lorentz Transformations

Next, observe an obvious identity

(19) $$U^{-1}(\Lambda)\,U(\Sigma)\,U(\Lambda) = U(\Lambda^{-1}\,\Sigma\,\Lambda)$$

and set $U(\Sigma) = U(1+\omega)$

(20) $$\begin{cases} U(\Sigma) = 1 - \frac{1}{2}\,i\,\omega_{\mu\nu}\,J^{\mu\nu} \\ U(\Lambda^{-1}\,\Sigma\,\Lambda) = 1 - \frac{1}{2}\,i\,(\Lambda^{-1}\,\omega\,\Lambda)_{\alpha\beta}\,J^{\alpha\beta} \\ \qquad = 1 - \frac{1}{2}\,i\,(\Lambda^{-1})_{\alpha}{}^{\mu}\,\omega_{\mu\nu}\,\Lambda^{\nu}{}_{\beta}\,J^{\alpha\beta} \\ \qquad = 1 - \frac{1}{2}\,i\,\Lambda^{\mu}{}_{\alpha}\,\omega_{\mu\nu}\,\Lambda^{\nu}{}_{\beta}\,J^{\alpha\beta} \end{cases}$$

and find

(21) $$U^{-1}(\Lambda)\,J^{\mu\nu}\,U(\Lambda) = \Lambda^{\mu}{}_{\alpha}\,\Lambda^{\nu}{}_{\beta}\,J^{\alpha\beta}$$

Quantum Lorentz Transformations

The commutation relations can be obtained as follows. Observe

(22) $$U(a)U(b) = U(a+b)$$

and go over to infinitesimal translations

(23) $$\begin{aligned} U(a) &= 1 - i\,(a\cdot P) - \frac{1}{2}\,(a\cdot P)^2 \\ U(b) &= 1 - i\,(b\cdot P) - \frac{1}{2}\,(b\cdot P)^2 \\ U(a+b) &= 1 - i\,(a\cdot P + b\cdot P) - \frac{1}{2}\,(a\cdot P + b\cdot P)^2 \end{aligned}$$

to see that

(24) $$(a\cdot P)\,(b\cdot P) = (b\cdot P)\,(a\cdot P) \quad\Longrightarrow\quad a_\mu b_\nu\, P_\mu P_\nu = a_\mu b_\nu\, P_\nu P_\mu$$

and find

(25) $$[P^\mu, P^\nu] = 0$$

Quantum Lorentz Transformations

Consider next an infinitesimal Lorentz transformation $\Lambda = 1 + \omega$. For the purpose, it is convenient to recast ω, as it is antisymmetric,

(26)
$$\omega^\mu{}_\alpha = \frac{1}{2}\,\omega_{\alpha\beta}\,(g^{\mu\alpha}\,\vec{\delta}^{\,\beta}{}_\alpha - g^{\mu\beta})$$
$$\omega^\nu{}_\beta = \frac{1}{2}\,\omega_{\beta\alpha}\,(g^{\nu\beta}\,\vec{\delta}^{\,\alpha}{}_\beta - g^{\nu\alpha})$$

where $\vec{\delta}^{\,\beta}{}_\alpha$ acts on the variables on right and replaces the superscript α with β.
One sees that, from $U^{-1}(\Lambda)\,P^\mu\,U(\Lambda) = \Lambda^\mu{}_\alpha\,P^\alpha$ with $\Lambda = 1 + \omega$,

(27)
$$[P^\mu, J^{\alpha\beta}] = i\,(g^{\mu\alpha}\,P^\beta - g^{\mu\beta}\,P^\alpha)$$

and find, from $U^{-1}(\Lambda)\,J^{\mu\nu}\,U(\Lambda) = \Lambda^\mu{}_\alpha\,\Lambda^\nu{}_\beta\,J^{\alpha\beta}$ with $\Lambda = 1 + \omega$,

(28)
$$[J^{\mu\nu}, J^{\alpha\beta}] = -i\,(g^{\mu\alpha}\,J^{\nu\beta} - g^{\nu\alpha}\,J^{\mu\beta} + g^{\nu\beta}\,J^{\mu\alpha} - g^{\mu\beta}\,J^{\nu\alpha})$$

using the fact that both ω and M are antisymmetric.
This completes construction of the Lie algebra for the Poincaré group.

Quantum Lorentz Transformations

Define ‘relativistic spin’ via

(29) $$W^{\mu} = \frac{1}{2}\,\varepsilon^{\mu\,\alpha\,\beta\,\gamma}\,P_{\alpha}\,J_{\beta\gamma} = \frac{1}{2}\,\varepsilon^{\mu\,\alpha\,\beta\,\gamma}\,J_{\alpha\beta}\,P_{\gamma}$$

where ε is the 4-dimensional totally antisymmetric tensor with

$$\boxed{\varepsilon_{0123} = +1, \qquad \varepsilon^{0123} = -1}$$

Since W^{μ} is a four-vector, one must have

(30) $$U^{-1}(\Lambda)\,W^{\mu}\,U(\Lambda) = \Lambda^{\mu}{}_{\nu}\,W^{\nu}$$

But this formula can be derived independently, noting that

(31) $$\begin{aligned} U^{-1}(\Lambda)\,W^{\mu}\,U(\Lambda) &= \frac{1}{2}\,\varepsilon^{\mu\,\alpha\,\beta\,\gamma}\,U^{-1}(\Lambda)\,P_{\alpha}\,U(\Lambda)\,U^{-1}(\Lambda)\,J_{\beta\gamma}\,U(\Lambda) \\ &= \frac{1}{2}\,\varepsilon^{\mu\,\alpha\,\beta\,\gamma}\,\Lambda_{\alpha}{}^{\rho}\,\Lambda_{\beta}{}^{\sigma}\,\Lambda_{\gamma}{}^{\tau}\,P_{\rho}\,J_{\sigma\tau} \end{aligned}$$

Quantum Lorentz Transformations

One now makes use of an elegant identity (see Appendix, note on QLT)

(32) $$\varepsilon^{\nu\,\rho\,\sigma\,\tau} = \varepsilon^{\mu\,\alpha\,\beta\,\gamma}\,\Lambda_\mu{}^\nu\,\Lambda_\alpha{}^\rho\,\Lambda_\beta{}^\sigma\,\Lambda_\gamma{}^\tau$$

to show that

(33) $$\Lambda^\mu{}_\nu\,\varepsilon^{\nu\,\rho\,\sigma\,\tau} = \varepsilon^{\mu\,\alpha\,\beta\,\gamma}\,\Lambda_\alpha{}^\rho\,\Lambda_\beta{}^\sigma\,\Lambda_\gamma{}^\tau$$

Sustituting this to the equation above, one obtains the desired result.

$P^\mu\,P_\mu$ and $W^\mu\,W_\mu$ are invariants (i.e. they commute with all the operators), given by w^2 and $-w^2\,j(j+1)$, i.e.

(34) $$\begin{cases} P^\mu\,P_\mu = w^2 \\ W^\mu\,W_\mu = -w^2\,j(j+1) \end{cases}$$

Quantum Lorentz Transformations

Define

(35)
$$\begin{cases} J^i = \frac{1}{2}\,\varepsilon^{ijk}\,J^{jk}\,: & \text{Angular Momentum Operator} \\ K^i = J^{0i}\,: & \text{Boost Operator} \end{cases}$$

so that, using $P^i = -P_i$, $J_{ij} = +J^{ij}$, $J_{0i} = -J^{0i}$ and $J_{i0} = +J^{0i}$,

(36)
$$\begin{cases} W^0 = \vec{P}\cdot\vec{J} = \vec{J}\cdot\vec{P} \\ \vec{W} = P^0\,\vec{J} - \vec{P}\times\vec{K} = \vec{J}\,P^0 + \vec{K}\times\vec{P} \end{cases}$$

Now construct a general (relativistic) spin operator by

(37)
$$wS^i = W^i - \frac{1}{P^0 + w}P^i\,W^0$$

so that

(38)
$$w\vec{S} = P^0\vec{J} - \vec{P}\times\vec{K} - \frac{1}{P^0+w}\vec{P}\,(\vec{P}\cdot\vec{J})$$

Quantum Lorentz Transformations

The commutation relations are

(39) $$[P^\mu, P^\nu] = 0, \quad [P^0, J^i] = [P^0, S^i] = 0, \quad [J^i, A^j] = i\,\varepsilon^{ijk}\,A^k$$

where

(40) $$A^i = \{P^i,\ J^i,\ K^i,\ S^i\}$$

And

(41) $$[P^0, K^i] = i\,P^i, \quad [P^i, K^j] = i\,\delta_{ij}\,P^0, \quad [P^i, S^j] = 0$$
$$[S^i, S^j] = i\,\varepsilon^{ijk}\,S^k, \quad [K^i, K^j] = -i\,\varepsilon^{ijk}\,J^k$$

Define a general orbital angular momentum operator L via

(42) $$\vec{J} = \vec{L} + \vec{S}$$

so that

(43) $$w\vec{L} = -(P^0 - w)\vec{J} + \vec{P} \times \vec{K} + \frac{1}{P^0 + w}\vec{P}\,(\vec{P} \cdot \vec{J})$$

and

(44) $$[J^i, L^j] = [L^i, L^j] = i\,\varepsilon^{ijk}\,L^k, \quad [P^0, L^i] = [L^i, S^j] = 0$$
$$[P^i, L^j] = i\,\varepsilon^{ijk}\,P^k, \quad \vec{P} \cdot \vec{L} = 0$$

Quantum Lorentz Transformations

Recapitulate

(45)
$$\begin{cases} W^0 = \vec{J}\cdot\vec{P} \\ \vec{W} = \vec{J}P^0 + \vec{K}\times\vec{P} \\ \vec{J} = \vec{L} + \vec{S} \\ w\vec{S} = \vec{J}P^0 + \vec{K}\times\vec{P} - \dfrac{1}{P^0+w}\vec{P}(\vec{J}\cdot\vec{P}) \\ w\vec{L} = -\vec{J}(P^0 - w) - \vec{K}\times\vec{P} + \dfrac{1}{P^0+w}\vec{P}(\vec{J}\cdot\vec{P}) \end{cases}$$

Note that $P^0\vec{J}$ has been changed to $\vec{J}P^0$, since they commute. Note also that $\vec{P}\times\vec{K}$ has been changed to $-(\vec{K}\times\vec{P})$; this is possible because $[K^i, P^j] = 0$ for $i \neq j$. When the operators above act on states with a given momentum p^μ, it is clear that $(P^0, \vec{P})$ can be changed to $(E, \vec{p})$. We shall denote the resulting operators by writing $W^\mu(p)$, $S^\mu(p)$ and $L^\mu(p)$. We see that

(46)
$$P_\mu W^\mu = p_\mu W^\mu(p) = 0$$

If a state is at rest, then $\vec{S}$ is equivalent to $\vec{J}$. The actions of S^μ and $S^\mu(p)$ are identical, when applied to the states for which the P^μ operator has an eigenvalue of p^μ.

Single-Particle States

Define

(47)
$$J^1 = J_x, \quad J^2 = J_y, \quad J^3 = J_z,$$
$$J_\pm = J_x \pm i\, J_y, \quad \vec{J}^2 = J_x^2 + J_y^2 + J_z^2$$

The standard representation of angular momentum states are given by

(48)
$$\begin{cases} \vec{J}^2 \, |jm\rangle = j(j+1)\, |jm\rangle \\ J_z \, |jm\rangle = m \, |jm\rangle \\ J_\pm \, |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)} \, |jm \pm 1\rangle \end{cases}$$

Rest states $|k, jm\rangle$ for a single particle with mass w and spin j are eigenstates of P, J^2 and J_z with corresponding eigenvalues $k = (w, 0, 0, 0)$, $j(j+1)$ and m. They have the usual transformation property under rotation R

(49)
$$U(R)\, |k, jm\rangle = \sum_{m'} |k, jm'\rangle D^j_{m'm}(R)$$

One- and Two-Particle States

The actions of the relativistic spin

(50) $$W_\mu W^\mu = (W^0)^2 - (\vec{W}\cdot\vec{W})$$

on the rest states are, with

(51) $$P^\mu\,|k,jm\rangle = k^\mu\,|k,jm\rangle$$ $$P^0|k,jm\rangle = w\,|k,jm\rangle, \quad \vec{P}\,|k,jm\rangle = 0$$

(52) $$W_\mu W^\mu\,|k,jm\rangle = -w^2\,j(j+1)\,|k,jm\rangle$$

So $W_\mu W^\mu$ has the eigenvalue indicated above for all, massive and relativistic single-particle states. Note that

(53) $$\begin{cases} \vec{J}^{\,2}\,|k,jm\rangle = j(j+1)\,|k,jm\rangle \\ \vec{S}^{\,2}\,|k,jm\rangle = j(j+1)\,|k,jm\rangle \\ \vec{L}^{\,2}\,|k,jm\rangle = 0 \end{cases}$$

One- and Two-Particle States

Consider now a boost $B_z(p)$ along the z-axis which takes k to $p = (E, 0, 0, p)$, i.e.

(54) $$p^\mu = [B_z(p)]^\mu{}_\nu \, k^\nu, \quad B_z(p) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}$$

where $\tanh\alpha = \beta = p/E$, $\cosh\alpha = \gamma = E/w$ and $\sinh\alpha = \gamma\beta = p/w$. Define

(55) $$\mathring{p} = (E, \vec{p}) = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$$

and

(56) $$\mathring{p}^{\,\mu} = [R(\hat{p})]^\mu{}_\nu \, p^\nu, \quad R(\hat{p}) = R_z(\phi)\, R_y(\theta)$$

and

(57) $$R_y(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R_z(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One- and Two-Particle States

The relevant unitary operators are

(58)
$$U[B_z(p)] = \exp[-i\,\alpha\,K_z]$$
$$U[R(\hat{p})] = U[R_z(\phi)R_y(\theta)] = \exp(-i\,\phi\,J_z)\,\exp(-i\,\theta\,J_y)$$

Define

(59)
$$U[\mathcal{L}(\overset{\circ}{p})] = U[R(\hat{p})]\,U[B_z(p)]$$
$$U[L(\overset{\circ}{p})] = U[R(\hat{p})]\,U[B_z(p)]\,U^{-1}[R(\hat{p})]$$

The canonical and helicity states can now be defined via

(60)
$$\begin{cases} |\vec{p}, jm\rangle = U[L(\overset{\circ}{p})]\,|k, jm\rangle = U[R(\hat{p})]\,U[B_z(p)]\,U^{-1}[R(\hat{p})]\,|k, jm\rangle \\ |\vec{p}, \lambda\rangle = U[\mathcal{L}(\overset{\circ}{p})]\,|k, \lambda\rangle = U[R(\hat{p})]\,U[B_z(p)]\,|k, \lambda\rangle = U[L(\overset{\circ}{p})]\,U[R(\hat{p})]\,|k, \lambda\rangle \end{cases}$$

where $|k, \lambda\rangle = |k, jm\rangle$ with $\lambda = m$.

One- and Two-Particle States

The helicity and canonical states are related through

$$|\vec{p},\lambda\rangle = \sum_m D^j_{m\lambda}(\phi,\theta,0)|\vec{p},jm\rangle, \quad |\vec{p},jm\rangle = \sum_\lambda D^{j\,*}_{m\lambda}(\phi,\theta,0)\,|\vec{p},\lambda\rangle \tag{61}$$

from orthonormality of the D-functions. The ket states are normalized according to

$$\begin{aligned}\langle\vec{p},jm|\vec{p}\,',j'm'\rangle &= 2E\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{j\,j'}\,\delta_{m\,m'} = \tilde{\delta}(\vec{p}-\vec{p}\,')\,\delta_{j\,j'}\,\delta_{m\,m'}\\ \langle\vec{p},j\lambda|\vec{p}\,',j'\lambda'\rangle &= 2E\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{j\,j'}\,\delta_{\lambda\,\lambda'} = \tilde{\delta}(\vec{p}-\vec{p}\,')\,\delta_{j\,j'}\,\delta_{\lambda\,\lambda'}\end{aligned} \tag{62}$$

Together with the invariant volume element

$$\tilde{\mathrm{d}}p = \frac{\mathrm{d}^3p}{(2\pi)^3(2E)} \tag{63}$$

the closure relations can be written

$$\begin{aligned}\sum_{jm}\int|\vec{p},jm\rangle\,\tilde{\mathrm{d}}p\,\langle\vec{p},jm| &= I\\ \sum_{j\lambda}\int|\vec{p},j\lambda\rangle\,\tilde{\mathrm{d}}p\,\langle\vec{p},j\lambda| &= I\end{aligned} \tag{64}$$

One- and Two-Particle States

Let $\mathcal{R}$ be a general rotation $R(\alpha\beta\gamma)$

(65) $$U(\mathcal{R}) = U[R(\alpha\beta\gamma)] = \exp[-i\,\alpha\,J_z]\,\exp[-i\,\beta\,J_y]\,\exp[-i\,\gamma\,J_z]$$

The canonical and helicity states transform under a pure rotation $\mathcal{R}\,(\vec{p} \to \vec{p}\,')$

(66) $$\begin{aligned} U(\mathcal{R})\,|\vec{p}, jm\rangle &= \sum_{m'} |\mathcal{R}\vec{p}, jm'\rangle\, D^j_{m'm}(\mathcal{R}) = \sum_{m'} |\vec{p}\,', jm'\rangle\, D^j_{m'm}(\mathcal{R}) \\ U(\mathcal{R})\,|\vec{p}, \lambda\rangle &= |\mathcal{R}\,\vec{p}, \lambda\rangle = |\vec{p}\,', \lambda\rangle \end{aligned}$$

Canonical states transform as if they were rest-frame states and helicity is conserved under a pure rotation.

One- and Two-Particle States

One needs similar transformation laws for $\vec{S}$ and $\vec{L}$. For the purpose, one defines

(67)
$$\begin{aligned} wS^\mu(p) &= U[L(p)]\, W^\mu(p)\, U^{-1}[L(p)] \\ &= U^{-1}[L^{-1}(p)]\, W^\mu(p)\, U[L^{-1}(p)] \\ &= L^{-1}(p)^\mu{}_\nu\, W^\nu(p) \end{aligned}$$

so that

(68)
$$\left\{ \begin{aligned} wS^i(p) &= W^i(p) - \frac{1}{E+w} p^i\, W^0(p) \\ wS^i &= W^i - \frac{1}{P^0+w} P^i\, W^0 : \end{aligned} \right.$$
Original Definition

Note

(69)
$$wS^i(p)\,|\vec{p}, jm\rangle = U[L(p)]\, W^i\,|k, jm\rangle = wU[L(p)]\, J^i\,|k, jm\rangle$$

Or

(70)
$$\left\{ \begin{aligned} S_z(p)\,|\vec{p}, jm\rangle &= U[L(p)]\, J_z|k, jm\rangle = m\,|\vec{p}, jm\rangle \\ S_\pm(p)\,|\vec{p}, jm\rangle &= U[L(p)]\, J_\pm|k, jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|\vec{p}, jm \pm 1\rangle \end{aligned} \right.$$

One- and Two-Particle States

Define

(71) $$U^S(\mathcal{R}) = U^S[R(\alpha\beta\gamma)] = \exp[-i\,\alpha\,S_z]\,\exp[-i\,\beta\,S_y]\,\exp[-i\,\gamma\,S_z]$$

It is clear that

(72) $$U^S(\mathcal{R})\,|\vec{p}, jm\rangle = \sum_{m'} |\vec{p}, jm'\rangle\, D^j_{m'm}(\mathcal{R})$$

So the operator $U^S(\mathcal{R})$ acts on the angular momentum part of the canonical states $|\vec{p}, jm\rangle$ and leaves the momentum $\vec{p}$ unpertubed.

One- and Two-Particle States

Define

(73) $$U^L(\mathcal{R}) = U^L[R(\alpha\beta\gamma)] = \exp[-i\,\alpha\,L_z]\,\exp[-i\,\beta\,L_y]\,\exp[-i\,\gamma\,L_z]$$

Because $\vec{S}$ and $\vec{L}$ commute, it is clear that

(74) $$U(\mathcal{R}) = U^S(\mathcal{R})\,U^L(\mathcal{R}) = U^L(\mathcal{R})\,U^S(\mathcal{R})$$

Noting that

$$U^L(\mathcal{R}) = [U^S(\mathcal{R})]^{-1}\,U(\mathcal{R}) = U^S(\mathcal{R}^{-1})\,U(\mathcal{R})$$

and using the group property of the D^j functions, one finds

(75) $$U^L(\mathcal{R})\,|\vec{p}, jm\rangle = |\mathcal{R}\vec{p}, jm\rangle$$

So, for the canonical states, $U^S(\mathcal{R})$ acts on the spin while leaving the momentum invariant, whereas $U^L(\mathcal{R})$ rotates the momentum but leaves the spin unchanged. This is to be contrasted with the actions of $U(\mathcal{R})$, which act on both the spin and the momentum.

Two-Particle States

Consider a system of two spinless particles with momenta p_1 and p_2 and masses w_1 and w_2. Let w be the effective mass of the two-particle system and let $p = p_1 + p_2$ be the total 4-momentum and let q be the breakup momentum in the rest frame of the two-particle system. $\Omega = (\theta, \phi)$ describes to the direction of $\vec{p}_1$ in the rest frame. We work out the normalization of the product of the two ket states

(76)
$$\int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \langle \vec{p}_1 | \vec{p}_1^{\,\prime} \rangle \, \langle \vec{p}_2 | \vec{p}_2^{\,\prime} \rangle = \int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \tilde{\delta}(\vec{p}_1 - \vec{p}_1^{\,\prime}) \, \tilde{\delta}(\vec{p}_2 - \vec{p}_2^{\,\prime}) = 1$$

provided $\vec{p}_1^{\,\prime} = \vec{p}_1$ and $\vec{p}_2^{\,\prime} = \vec{p}_2$. Adopt a normalization for the two-particle system

(77)
$$|p, \Omega\rangle = a \, |\vec{p}_1\rangle \, |\vec{p}_2\rangle, \quad a = \text{a normalization constant}$$

by requiring

(78)
$$\langle p, \Omega \, | \, p', \Omega' \rangle = (2\pi)^4 \, \delta^{(4)}(p - p') \, \delta^{(2)}(\Omega - \Omega')$$

One- and Two-Particle States

It is seen that six variables contained in $\vec{p}_1$ and $\vec{p}_2$ have been transformed into the 4-momentum p and Ω. To find a, perform a change of variables

$$\int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \langle p, \Omega \,|\, p', \Omega' \rangle = a^2 \int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \langle \vec{p}_1 | \vec{p}_1' \rangle \, \langle \vec{p}_2 | \vec{p}_2' \rangle = a^2$$

$$= (2\pi)^4 \int \tilde{\mathrm{d}}p_1 \tilde{\mathrm{d}}p_2 \, \delta^{(4)}(p - p') \, \delta^{(2)}(\Omega - \Omega')$$

(79)

$$\text{in RF} \rightarrow = \frac{1}{(4\pi)^2} \int \left(\frac{q^2 \mathrm{d}q \, \mathrm{d}^2\Omega}{E_1 E_2} \right) \delta(w - E_1 - E_2) \, \delta^{(2)}(\Omega - \Omega')$$

$$E_i = \sqrt{w_i^2 + q^2} \rightarrow = \frac{1}{(4\pi)^2} \left(\frac{q}{w} \right)$$

where RF stands the two-particle rest frame, and $i = 1$ or 2. So we see that

(80)

$$a = \frac{1}{4\pi} \sqrt{\frac{q}{w}}$$

One- and Two-Particle States

Define, in the rest frame of the two-particle system of *arbitrary spin*,

(81)
$$\begin{cases} \mathring{q} = (E_1, \vec{q}\,) = (E_1, q\sin\theta\cos\phi, q\sin\theta\sin\phi, q\cos\theta) \\ \breve{q} = (E_2, -\vec{q}\,) = (E_2, -q\sin\theta\cos\phi, -q\sin\theta\sin\phi, -q\cos\theta) \end{cases}$$

Using the boost operators $U[B_z(q)]$ and $U[B_{-z}(q)]$ along the positive and negative z-axis and with $R(\hat{q}) = R_z(\phi)R_y(\theta)$,

(82)
$$\begin{cases} U[L(\mathring{q})] = U[R(\hat{q})]\,U[B_z(q)]\,U^{-1}[R(\hat{q})] \\ U[L(\breve{q})] = U[R(\hat{q})]\,U[B_{-z}(q)]\,U^{-1}[R(\hat{q})] \end{cases}$$

one sets

(83)
$$\begin{aligned} |\Omega\, m_1 m_2\rangle &= \frac{1}{4\pi}\sqrt{\frac{q}{w}}\; U[L(\mathring{q})]|s_1 m_1\rangle\, U[L(\breve{q})]|s_2 m_2\rangle \\ &= \frac{1}{4\pi}\sqrt{\frac{q}{w}}\; |\vec{q}, s_1 m_1\rangle\, |-\vec{q}, s_2 m_2\rangle \end{aligned}$$

Adopt a normalization

(84)
$$\langle\Omega\, m_1 m_2 \,|\, \Omega'\, m_1' m_2'\rangle = \delta^{(2)}(\Omega - \Omega')\,\delta_{m_1 m_1'}\,\delta_{m_2 m_2'}$$

One- and Two-Particle States

Define

(85) $$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}, \quad \vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}, \quad \vec{L} = \vec{L}^{(1)} + \vec{L}^{(2)}$$

and

(86) $$\begin{cases} U[R(\Omega)] = \exp(-i\,\phi\,J_z)\,\exp(-i\,\theta\,J_y) \\ U^S[R(\Omega)] = \exp(-i\,\phi\,S_z)\,\exp(-i\,\theta\,S_y) \\ U^L[R(\Omega)] = \exp(-i\,\phi\,L_z)\,\exp(-i\,\theta\,L_y) \end{cases}$$

Introduce a new ket state of a given S

(87) $$|\Omega\, S m_s\rangle = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | S m_s)\, |\Omega\, m_1 m_2\rangle$$

It follows, from

$$U^S(\mathcal{R})\,|\vec{p}, jm\rangle = \sum_{m'} |\vec{p}, jm'\rangle\, D^j_{m'm}(\mathcal{R})$$

that

(88) $$U^S[R(\Omega')]\,|\Omega\, S m_s\rangle = \sum_{m'_s} |\Omega\, S m'_s\rangle\, D^S_{m'_s m_s}(\phi', \theta', 0)$$

One- and Two-Particle States

Now construct a ket state of a given ℓ and S

(89)
$$|\ell m\, S m_s\rangle = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | S m_s) \int \mathrm{d}\Omega\, Y^\ell_m(\Omega) |\Omega\, m_1 m_2\rangle$$

Apply U^L on the states above and find

$$U^L[R(\Omega')]\, |\ell m\, S m_s\rangle = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | S m_s) \int \mathrm{d}\Omega\, Y^\ell_m(\Omega) |R(\Omega')\, \Omega\, m_1 m_2\rangle$$

$$(\Omega'' = R\,\Omega) \rightarrow = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | S m_s) \int \mathrm{d}\Omega''\, Y^\ell_m(R^{-1}\Omega'') |\Omega''\, m_1 m_2\rangle$$

$$\text{(group property)} \rightarrow = \sum_{m'} D^{\ell\,*}_{mm'}(R^{-1}) \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | S m_s) \int \mathrm{d}\Omega''\, Y^\ell_{m'}(\Omega'') |\Omega''\, m_1 m_2\rangle$$

$$\text{(unitary property)} \rightarrow = \sum_{m'} |\ell m'\, S m_s\rangle\, D^\ell_{m'm}(\phi, \theta, 0)$$

Under a general rotation then, the ket states transform according to

(90)
$$U[R(\Omega')]\, |\ell m\, S m_s\rangle = U^L[R(\Omega')]\, U^S[R(\Omega')]\, |\ell m\, S m_s\rangle$$
$$= \sum_{m'\, m'_s} |\ell m'\, S m'_s\rangle\, D^\ell_{m'm}(\phi', \theta', 0)\, D^S_{m'_s m_s}(\phi', \theta', 0)$$

One- and Two-Particle States

Construct next a ket state

(91)
$$
\begin{aligned}
|JM\ell S\rangle &= \sum_{m m_s} (\ell m\, S m_s | JM)\, |\ell m\, S m_s\rangle \\
&= \sum_{\substack{m m_s \\ m_1 m_2}} (\ell m\, S m_s | JM)(s_1 m_1\, s_2 m_2 | S m_s) \int \mathrm{d}\Omega\, Y_m^\ell(\Omega) |\Omega\, m_1 m_2\rangle
\end{aligned}
$$

normalized

(92)
$$\langle JM\ell S | J'M'\ell'S'\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\ell\ell'}\delta_{SS'}$$

It is apparent that

(93)
$$U[R(\Omega')]\,|JM\ell S\rangle = \sum_{M'} |JM'\ell S\rangle\, D^J_{M'M}(\phi', \theta', 0)$$

so that

(94)
$$
\begin{cases}
\vec{J}^{\,2}\,|JM\ell S\rangle = J(J+1)\,|JM\ell S\rangle \\
\vec{S}^{\,2}\,|JM\ell S\rangle = S(S+1)\,|JM\ell S\rangle \\
\vec{L}^{\,2}\,|JM\ell S\rangle = \ell(\ell+1)\,|JM\ell S\rangle
\end{cases}
$$

One- and Two-Particle States

Consider two-particle states in the helicity basis. Define

$$U[\mathcal{L}(\mathring{q})] = U[R(\hat{q})]\,U[B_z(q)], \quad U[\mathcal{L}(\breve{q})] = U[R(\hat{q})]\,U[B_{-z}(q)]$$

$$|\Omega\,\lambda_1\lambda_2\rangle = \frac{1}{4\pi}\sqrt{\frac{q}{w}}\,U[\mathcal{L}(\mathring{q})]|s_1\lambda_1\rangle\,U[\mathcal{L}(\breve{q})]|s_2\,{-\lambda_2}\rangle = \frac{1}{4\pi}\sqrt{\frac{q}{w}}\,|\vec{q},\lambda_1\rangle\,|-\vec{q},-\lambda_2\rangle$$

Construct, with $\boxed{\lambda = \lambda_1 - \lambda_2}$,

$$|JM\lambda_1\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}}\int \mathrm{d}\Omega\,D^{J*}_{M\lambda}(\phi,\theta,0)\,|\Omega\,\lambda_1\lambda_2\rangle$$

$$\langle JM\lambda_1\lambda_2|J'M'\lambda_1'\lambda_2'\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}$$

$$U[R(\Omega)]\,|JM\lambda_1\lambda_2\rangle = \sum_{M'}|JM'\lambda_1\lambda_2\rangle\,D^J_{M'M}(\phi,\theta,0)$$

Recoupling coefficient:

(95) $$\langle J'M'\ell'S'|JM\lambda_1\lambda_2\rangle = \sqrt{\frac{2\ell+1}{2J+1}}(\ell\,0\,S\lambda|J\lambda)(s_1\lambda_1\,s_2\,{-\lambda_2}|S\lambda)\,\delta_{JJ'}\delta_{MM'}$$

Two-body Decays

In the two-body rest frame, the decay amplitude is

(96)
$$
\begin{aligned}
A^J_M(\Omega) &= \sum_{\lambda_1\lambda_2} \Big\{\langle\vec{q},\lambda_1|\,\langle-\vec{q},-\lambda_2|\Big\}\mathcal{M}|JM\rangle \\
&= 4\pi\sqrt{\frac{w}{q}}\sum_{\lambda_1\lambda_2}\langle\Omega\,\lambda_1\lambda_2|JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle \\
&= \sqrt{\frac{2J+1}{4\pi}}\sum_{\lambda_1\lambda_2} D^{J*}_{M\lambda}(\phi,\theta,0)\,F^J_{\lambda_1\lambda_2}, \quad \lambda=\lambda_1-\lambda_2
\end{aligned}
$$

where

(97)
$$F^J_{\lambda_1\lambda_2} = 4\pi\sqrt{\frac{w}{q}}\,\langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle$$

and, from parity conservation in the decay,

(98)
$$F^J_{\lambda_1\lambda_2} = \eta_J\,\eta_1\,\eta_2\,(-)^{J-s_1-s_2}\,F^J_{-\lambda_1,-\lambda_2}$$

One- and Two-Particle States

Alternatively, the decay amplitude may be written

(99)
$$
\begin{aligned}
A^J_M(\Omega) &= \sum_{m_1 m_2} \Big\{ \langle \vec{q}, s_1 m_1 |\, \langle -\vec{q}, s_2 m_2 | \Big\} \mathcal{M} | JM \rangle \\
&= \sum_{\ell S} \sum_{m_1 m_2} \Big\{ \langle \vec{q}, s_1 m_1 |\, \langle -\vec{q}, s_2 m_2 | \Big\} JM\ell S \rangle \langle JM\ell S | \mathcal{M} | JM \rangle \\
&= 4\pi \sqrt{\frac{w}{q}} \sum_{\ell S} \sum_{m_1 m_2} \langle \Omega\, m_1 m_2\, | JM\ell S \rangle \langle JM\ell S | \mathcal{M} | JM \rangle \\
&= \sum_{\ell S} \sum_{\substack{m m_s \\ m_1 m_2}} (\ell m\, S m_s | JM)(s_1 m_1\; s_2 m_2 | S m_s)\, Y^\ell_m(\Omega)\, G^J_{\ell S}
\end{aligned}
$$

where

(100)
$$
G^J_{\ell S} = 4\pi \sqrt{\frac{w}{q}}\, \langle JM\ell S | \mathcal{M} | JM \rangle, \quad \eta_J = \eta_1\, \eta_2\, (-)^\ell
$$
(Parity)

One- and Two-Particle States

Recoupling coefficient:

(101)
$$
\begin{aligned}
F^J_{\lambda_1\lambda_2} &= 4\pi\sqrt{\frac{w}{q}}\,\langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle \\
&= 4\pi\sqrt{\frac{w}{q}}\,\sum_{\ell S}\langle JM\lambda_1\lambda_2|JM\ell S\rangle\langle JM\ell S|\mathcal{M}|JM\rangle \\
&= \sum_{\ell S}\langle JM\lambda_1\lambda_2|JM\ell S\rangle\, G^J_{\ell S} \\
&= \sum_{\ell S}\sqrt{\frac{2\ell+1}{2J+1}}\,(\ell\, 0\, S\lambda|J\lambda)(s_1\lambda_1\, s_2\, {-\lambda_2}|S\lambda)\, G^J_{\ell S}
\end{aligned}
$$

where

(102)
$$G^J_{\ell S} \propto q^\ell \quad \text{for} \quad q \to 0$$

Spin-1 States in Momentum Space

Homogeneous Lorentz transformations are given by

(103) $$\Lambda = \exp[-\frac{i}{2}\,\omega_{\mu\nu}\,J^{\mu\nu}\,]$$

so that $J^{\mu\nu}$ is now an operator imbedded in the momentum space. For infinitesimal transformations, we have

(104) $$\Lambda^{\rho}{}_{\sigma} = \delta^{\rho}{}_{\sigma} - \frac{i}{2}\,\omega_{\mu\nu}\,(J^{\mu\nu})^{\rho}{}_{\sigma}$$

As this must be equal to

$$\Lambda^{\rho}{}_{\sigma} = \delta^{\rho}{}_{\sigma} + \omega^{\,\rho}{}_{\sigma}$$

we can deduce that

(105) $$(J^{\mu\nu})^{\rho}{}_{\sigma} = i\,(g^{\mu\rho}\,\delta^{\nu}{}_{\sigma} - g^{\nu\rho}\,\delta^{\mu}{}_{\sigma})\,;$$ see M. Maggiore, Capter 2

Spin-1 States in Momentum Space

Explicitly, we find

(106)
$$(J_x)^\rho{}_\sigma \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix}, \quad (K_x)^\rho{}_\sigma \Rightarrow \begin{pmatrix} 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(J_y)^\rho{}_\sigma \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad (K_y)^\rho{}_\sigma \Rightarrow \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(J_z)^\rho{}_\sigma \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (K_z)^\rho{}_\sigma \Rightarrow \begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}$$

and

(107)
$$[J^i, J^j] = i\,\varepsilon^{ijk}\,J^k, \quad [J^i, K^j] = i\,\varepsilon^{ijk}\,K^k, \quad [K^i, K^j] = -i\,\varepsilon^{ijk}\,J^k$$

Spin-1 States in Momentum Space

The spin-1 wave functions at rest are

(108) $$e^{\mu}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad e^{\mu}(\pm 1) = \mp \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}$$

It can be shown that

(109) $$\begin{cases} J^2\, e(m) = j(j+1)\, e(m), \quad j = 1, \quad m = -1,\ 0,\ +1 \\ J_z\, e(m) = m\, e(m), \quad m = -1,\ 0,\ +1 \\ J_{\pm}\, e(0) = \sqrt{2}\, e(\pm 1), \quad J_{\pm}\, e(\mp 1) = \sqrt{2}\, e(0), \quad J_{\pm}\, e(\pm 1) = 0 \end{cases}$$

Define

(110) $$\theta^i = \frac{1}{2}\, \varepsilon^{ijk}\, \omega^{jk} = \frac{1}{2}\, \varepsilon^{ijk}\, \omega_{jk}, \quad \alpha^i = \omega_{0\,i}$$

so that the homogeneous Lorentz transformations take on the form

(111) $$\Lambda = \exp[-i\, \vec{\theta} \cdot \vec{J} - i\, \vec{\alpha} \cdot \vec{K}\,]$$

Spin-1 States in Momentum Space

for a boost along the z-axis, we know that

(112) $$[B_z(p)]^{\rho}{}_{\sigma} = \exp[-i\,\alpha\,(K_z)^{\rho}{}_{\sigma}\,] \Rightarrow \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}$$

Here $\cosh\alpha = E/w$ and $\sinh\alpha = p/w$. After a boost along the z-axis, the wave functions become

(113) $$e^{\mu}(\vec{p}, m) = [B_z(p)]^{\mu}{}_{\nu}\, e^{\nu}(m)$$

or, writing out the components explicitly,

(114) $$e^{\mu}(\vec{p}, 0) = \begin{pmatrix} p/w \\ 0 \\ 0 \\ E/w \end{pmatrix}, \qquad e^{\mu}(\vec{p}, \pm 1) = \mp\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}$$

Note that $p_{\mu}\, e^{\mu}(\vec{p}, m) = 0$ in any Lorentz frame.

Spin-1 States in Momentum Space

Define 'relativistic spin' via

(115) $$W^{\mu}(p) = \frac{1}{2}\,\varepsilon^{\,\mu\,\alpha\,\beta\,\gamma}\,p_{\alpha}\,J_{\beta\gamma} = \frac{1}{2}\,\varepsilon^{\,\mu\,\alpha\,\beta\,\gamma}\,J_{\alpha\beta}\,p_{\gamma}$$

where p^{μ}'s are c-numbers, i.e. *not* operators. Given a momentum $p^{\mu} = (E, \vec{p})$ and $p^{\mu}\,p_{\mu} = w^2$, we can define the modfied form of the relativistic spin W^{μ} and their derivatives S^i and L^i

(116) $$\begin{cases} W^0(p) = \vec{J}\cdot\vec{p} \\ \vec{W}(p) = E\,\vec{J} + \vec{K}\times\vec{p} \\ w\vec{S}(p) = E\,\vec{J} + \vec{K}\times\vec{p} - \dfrac{\vec{p}\,(\vec{J}\cdot\vec{p}\,)}{E+w} \\ w\vec{L}(p) = -(E-w)\,\vec{J} - \vec{K}\times\vec{p} + \dfrac{\vec{p}\,(\vec{J}\cdot\vec{p}\,)}{E+w} \end{cases}$$

Here $(E, \vec{p})$ are *not* operators but merely c-numbers. These are the operators acting on ket states with 4-momenta $(E, \vec{p}\,)$.

Spin-1 States in Momentum Space

Consider a special case of $\vec{p}$ along the z-direction, i.e. $p_x = p_y = 0$ and $p_z = p$. Then, we see that

$$-W^{\mu}(p)\,W_{\mu}(p) = w^2\,J^2 + p^2\,(J_x^2 + J_y^2 + K_x^2 + K_y^2) + 2E\,p\,(J_x\,K_y - J_y\,K_x - i\,K_z) \tag{117}$$

We find

$$-W^{\mu}(p)\,W_{\mu}(p) = 2\begin{pmatrix} -p^2 & 0 & 0 & E\,p \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & w^2 & 0 \\ -E\,p & 0 & 0 & E^2 \end{pmatrix} \quad \text{(observe: } p \to 0\text{)} \tag{118}$$

so that

$$\begin{cases} -W^{\mu}(p)\,W_{\mu}(p)\,e^{\mu}(\vec{p},0) = 2w^2\,e^{\mu}(\vec{p},0) \\ -W^{\mu}(p)\,W_{\mu}(p)\,e^{\mu}(\vec{p},\pm 1) = 2w^2\,e^{\mu}(\vec{p},\pm 1) \end{cases} \tag{119}$$

So, from considering a special case, we have proven a general formula

$$-W^{\mu}(p)\,W_{\mu}(p) \quad : \quad j(j+1)\,w^2\,I = 2w^2\,I, \quad j = 1 \tag{120}$$

where, again, I is the 4×4 identity matrix.

Spin-1 States in Momentum Space

Consider now the spin operator $\vec{S}$. Once again we confine ourselves to $\vec{p}$ along the z-direction, i.e. $p_x = p_y = 0$ and $p_z = p$. Its z-component is

(121)
$$
\begin{aligned}
w\, S_z(p) &= E\, J_z - \frac{p^2}{E+w}\, J_z = w\, J_z, \quad \text{setting } p^2 = E^2 - w^2 \\
&= w \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Applying it to the wave functions, we find

(122)
$$
\begin{cases} w\, S_z(p)\, e^\mu(\vec{p}, 0) = 0 \\ w\, S_z(p)\, e^\mu(\vec{p}, \pm 1) = \pm w\, e^\mu(\vec{p}, \pm 1) \end{cases}
$$

Spin-1 States in Momentum Space

Next, we evaluate

$$w\,S_x(p) = E\,J_x + p\,K_y = i\begin{pmatrix} 0 & 0 & +p & 0 \\ 0 & 0 & 0 & 0 \\ +p & 0 & 0 & -E \\ 0 & 0 & +E & 0 \end{pmatrix}$$

(123)

$$w\,S_y(p) = E\,J_y - p\,K_x = i\begin{pmatrix} 0 & -p & 0 & 0 \\ -p & 0 & 0 & +E \\ 0 & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix}$$

Setting $S_\pm(p) = S_x(p) \pm i\,S_y(p)$, we obtain the desired result

(124)

$$\begin{cases} w\,S_\pm(p)\,e^\mu(\vec{p},0) = \sqrt{2}\,w\,e^\mu(\vec{p},\pm 1) \\ w\,S_\pm(p)\,e^\mu(\vec{p},\mp) = \sqrt{2}\,w\,e^\mu(\vec{p},0) \\ w\,S_\pm(p)\,e^\mu(\vec{p},\pm) = 0 \end{cases}$$

Spin-1 States in Momentum Space

Under a finite rotation $R^S(\alpha,\beta,\gamma)$, one has

(125) $$R^S(\alpha,\beta,\gamma)\, e^\mu(\vec{p},m) = \sum_{m'} e^\mu(\vec{p},m')\, D^{(1)}_{m'm}(\alpha,\beta,\gamma)$$

Consider a rank-2 tensor

(126) $$e^{\mu\nu}(\vec{p},Sm) = \sum_{m_1\, m_2} (1m_1\, 1m_2|Sm)\, e^\mu(\vec{p}_1,m_1)\, e^\nu(\vec{p}_2,m_2)$$

where $\vec{p} = \vec{p}_1 + \vec{p}_2$ and $S = 0, 1$ or 2. It is clear that

(127) $$R^S(\alpha,\beta,\gamma)\, e^{\mu\nu}(\vec{p},Sm) = \sum_{m'} e^{\mu\nu}(\vec{p},Sm')D^S_{m'm}(\alpha,\beta,\gamma)$$

General Formulation of Covariant Helicity-Coupling Amplitudes

S. U. Chung, PR D57, 431 (1998)

Spin-1 States in Momentum Space

To Be Continuned. . .

Selected Topics in Hadron Spectroscopy

Mathematical Techniques

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http://cern.ch/suchung/
http://www.phy.bnl.gov/~e852/reviews.html

One-Particle States at Rest

States of a single particle at rest (mass $w > 0$) may be denoted by $|jm\rangle$, where j is the spin and m the z-component of the spin. The states $|jm\rangle$ are the canonical basis vectors by which the angular momentum operators are represented in the standard way. Since the angular momentum operators are the infinitesimal generators of the rotation operator, the spin of a particle characterizes how the particle at rest transforms under spatial rotations.

Let us denote the three components of the angular momentum operator by J_x, J_y, and J_z (or J_1, J_2, and J_3). They are Hermitian operators satisfying the following commutation relations:

$$[J_i, J_j] = i\,\epsilon_{ijk}\,J_k\ , \tag{1}$$

where i, j, and k run from 1 to 3. The operators J_i act on the canonical basis vectors $|jm\rangle$ as follows:

$$\begin{aligned} J^2|jm\rangle &= j(j+1)|jm\rangle \\ J_z|jm\rangle &= m|jm\rangle \\ J_\pm|jm\rangle &= [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}}\,|jm \pm 1\rangle\ , \end{aligned} \tag{2}$$

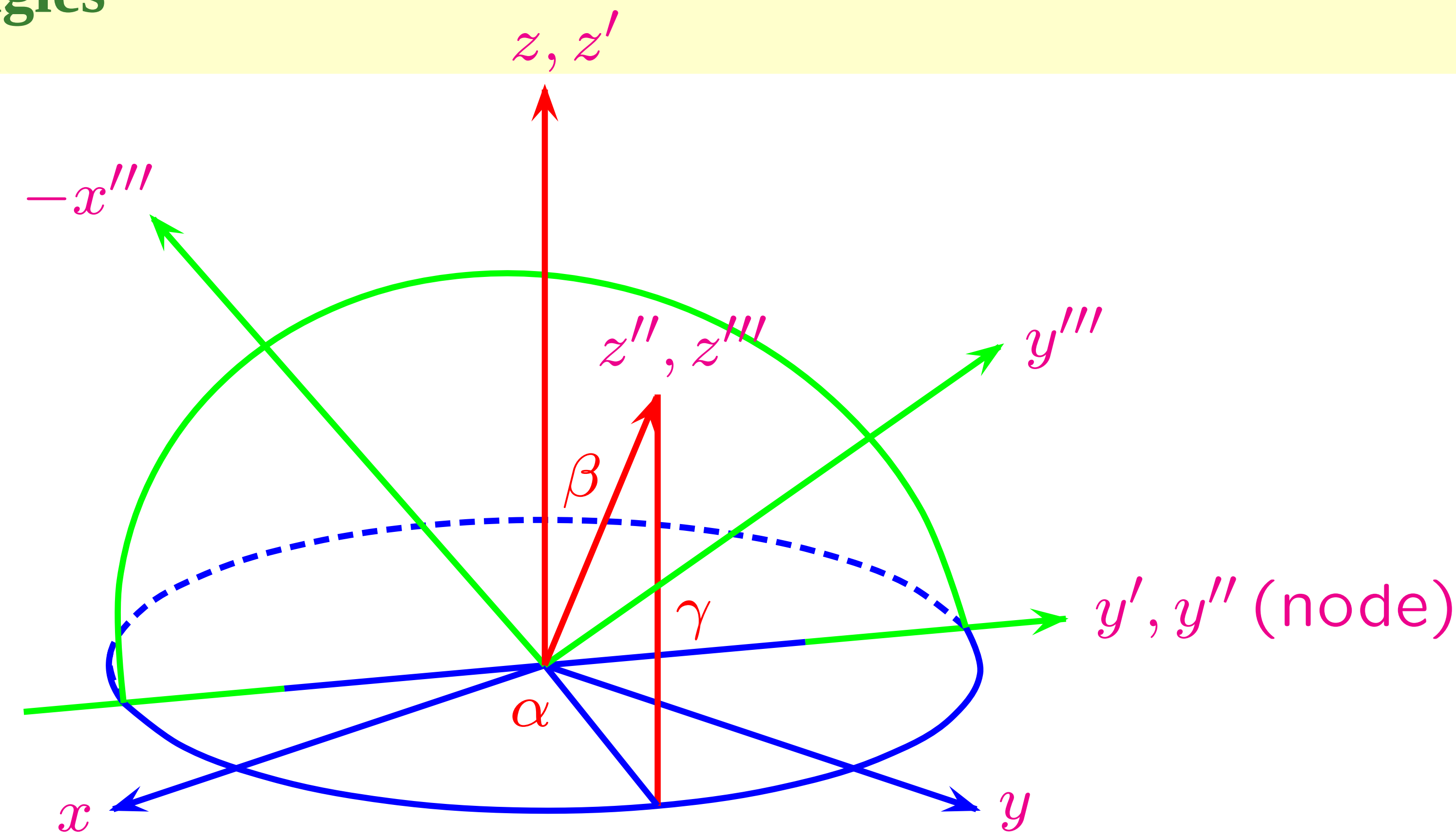
where $J^2 = J_x^2 + J_y^2 + J_z^2$ and $J_\pm = J_x \pm iJ_y$.

The states $|jm\rangle$ are normalized in the standard way and satisfy the completeness relation:

(3) $$\langle j'm'|jm\rangle = \delta_{j'j}\,\delta_{m'm}, \qquad \sum_{jm} |jm\rangle\langle jm| = I\ ,$$

where I denotes the identity operator.
A finite rotation of a physical system (with respect to a fixed coordinate axis) may be denoted by $R(\alpha,\beta,\gamma)$, where (α,β,γ) are the standard Euler angles. We use the so-called active rotations.

Euler Angles



$$U[R(\alpha,\beta,\gamma)] = \exp[-i\,\alpha\,J_z]\,\exp[-i\,\beta\,J_y]\,\exp[-i\,\gamma\,J_z]$$

$$= \exp[-i\,\gamma\,J_z''\,]\,\exp[-i\,\beta\,J_y'\,]\,\exp[-i\,\alpha\,J_z]$$

$$D^j_{m'm}(\alpha,\beta,\gamma) = \langle jm'|U[R(\alpha,\beta,\gamma)]|jm\rangle$$

M. E. Rose,
'Elementary Theory of Angular Momentum,'
John Wiley & Sons, Inc. (see Chapter IV)

To each R, there corresponds a unitary operator $U[R]$, which acts on the states $|jm\rangle$, and preserves the multiplication law:

$$U[R_2\,R_1] = U[R_2]\,U[R_1]\ .$$

Now the unitary operator representing a rotation $R(\alpha, \beta, \gamma)$ may be written

$$U[R(\alpha, \beta, \gamma)] = e^{-i\alpha J_z}\, e^{-i\beta J_y}\, e^{-i\gamma J_z} \tag{4}$$

corresponding to the rotation of a physical system (active rotation!) by γ around the z-axis, β around the y-axis, and finally by α around the z-axis, with respect to a fixed (x, y, z) coordinate system. Then rotation of a state $|jm\rangle$ is given by

$$U[R(\alpha, \beta, \gamma)]\,|jm\rangle = \sum_{m'} |jm'\rangle\, D^{j}_{m'm}(\alpha, \beta, \gamma)\ , \tag{5}$$

$D^{j}_{m'm}(\alpha, \beta, \gamma)$ is the matrix representation of the rotation group.

$D^j_{m' m}$ is the standard rotation matrix as given by M. E. Rose:

$$
\begin{aligned}
D^j_{m' m}(R) &= D^j_{m' m}(\alpha, \beta, \gamma) = \langle jm' | \, U[R(\alpha,\beta,\gamma)] \, | jm \rangle \\
&= e^{-i\, m' \alpha} \, d^j_{m' m}(\beta) \, e^{-i\, m \gamma}
\end{aligned}
\tag{6}
$$

and

$$
d^j_{m', m}(\beta) = \langle jm' | e^{-i\,\beta J_y} | jm \rangle \ . \tag{7}
$$

In Appendix A of the “Spin Formalisms” (S. U. Chung), some useful formulae involving $D^j_{m', m}$ and $d^j_{m', m}$ are listed.

Relativistic One-Particle States

Relativistic one-particle states with momentum $\vec{p}$ may be obtained by applying on the states $|jm\rangle$ a unitary operator which represents a Lorentz transformation that takes a particle at rest to a particle of momentum $\vec{p}$. There are two distinct ways of doing this, leading to canonical and helicity descriptions of relativistic free particle states.

Let us first consider an arbitrary four-momentum p^μ defined by

(8) $$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p}\,) \; .$$

With the metric tensor given by

(9) $$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

we can also define a four-momentum with lower indices:

(10) $$p_\mu = g_{\mu\nu}\, p^\nu = (E, -\vec{p}\,) \; .$$

My convention:

p is used to denote both the four-momentum and the magnitude of the three-momentum.

The proper homogeneous orthochronous Lorentz transformation takes the four-momentum p^μ into p'^μ as follows:

$$p'^{\,\mu} = \Lambda^\mu{}_\nu \, p^\nu \ , \tag{11}$$

where $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix defined by

$$g_{\alpha\beta}\,\Lambda^\alpha{}_\mu\,\Lambda^\beta{}_\nu = g_{\mu\nu}, \qquad \det\Lambda = 1, \qquad \Lambda^0{}_0 > 0 \ . \tag{12}$$

The Lorentz transformation given by $\Lambda^\mu{}_\nu$ includes, in general, rotations as well as the pure Lorentz transformations. Let us denote by $L^\mu{}_\nu(\vec\beta\,)$ a pure time-like Lorentz transformation, where $\vec\beta$ is the velocity of the transformation. Of particular importance is the pure Lorentz transformation along the z-axis, denoted by $L_z(\beta)$:

$$L_z(\beta) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \tag{13}$$

where $\beta = \tanh\alpha$.

In terms of $L_z(\beta)$, it is easy to define a pure Lorentz transformation along an arbitrary direction $\vec{\beta}$:

(14) $$L(\vec{\beta}\,) = R(\phi,\theta,0)L_z(\beta)R^{-1}(\phi,\theta,0)\ ,$$

where $R(\phi,\theta,0)$ is the rotation which takes the z-axis into the direction of $\vec{\beta}$ with spherical angles (θ,ϕ):

(15) $$\hat{\beta} = R(\phi,\theta,0)\hat{z}\ .$$

The relation $L(\vec{\beta}\,)$ is an obvious one, but the reader can easily check for a special case with $\phi = 0$:

(16) $$R(\phi,\theta,0) = \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix}, \quad R_{ij} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Now the action of an arbitrary Lorentz transformation Λ on relativistic particle states may be represented by a unitary operator $U[\Lambda]$. The operator preserves the multiplication law, called the group property:

$$U[\Lambda_2 \Lambda_1] = U[\Lambda_2]\, U[\Lambda_1] \; . \tag{17}$$

Let us denote by $L(\vec{p}\,)$ the "boost" which takes a particle with mass $w > 0$ from rest to momentum $\vec{p}$ and the corresponding unitary operator acting on the particle states by $U[L(\vec{p}\,)]$:

$$U[L(\vec{p}\,)] = e^{-i\alpha \hat{p}\cdot\vec{K}} \; , \tag{18}$$

where $\tanh\alpha = p/E$, $\sinh\alpha = p/w$, and $\cosh\alpha = E/w$.

A boost operator defines a Hermitian vector operator $\vec{K}$, and the components K_i are then the infinitesimal generators of "boosts". The three components K_i together with J_i form the six infinitesimal generators of the homogeneous Lorentz group, and they satisfy definite commutation relations among them. We do not list the relations here, for they are not needed for our purposes.

From the relation $L(\vec{\beta}\ )$ and the group property, one obtains

$$U[L(\vec{p}\ )] = U[\overset{\circ}{R}(\phi,\theta,0)]U[L_z(p)]U^{-1}[\overset{\circ}{R}(\phi,\theta,0)]\ , \tag{19}$$

where the rotation $\overset{\circ}{R}$ takes the z-axis into the direction of $\vec{p}$ with spherical angles (θ,ϕ):

$$\hat{p} = \overset{\circ}{R}(\phi,\theta,0)\hat{z}\ . \tag{20}$$

We are now ready to define the “standard” or canonical state describing a single particle with spin j and momentum $\vec{p}$:

$$\begin{aligned} |\vec{p}, jm\rangle &= |\phi, \theta, p, jm\rangle = U[L(\vec{p}\,)]\,|jm\rangle \\ &= U[\overset{\circ}{R}(\phi,\theta,0)]\,U[L_z(p)]\,U^{-1}[\overset{\circ}{R}(\phi,\theta,0)]\,|jm\rangle\ , \end{aligned} \tag{21}$$

where $|jm\rangle$ is the particle state at rest as defined in the previous section. We emphasize that the z-component of spin m is measured in the rest frame of the particle and *not* in the frame where the particle has momentum $\vec{p}$.

The advantage of the canonical state is that the state transforms formally under rotation in the same way as the “rest-state” $|jm\rangle$:

$$\begin{aligned} U[R]\,|\vec{p}, jm\rangle &= U[R\overset{\circ}{R}]\,U[L_z(p)]\,U^{-1}[R\overset{\circ}{R}]\,U[R]\,|jm\rangle \\ &= \sum_{m'} D^j_{m'm}(R)\,|R\,\vec{p}, jm'\rangle\ , \end{aligned} \tag{22}$$

It is clear from the rottional property of the canonical states that one may take over all the non-relativistic spin formalisms and apply them to situations involving relativistic particles with spin. One ought to remember, however, that the z-component of spin is defined only in the particle rest frame obtained from the frame where the particle has momentum $\vec{p}$ via the pure Lorentz transformation $L^{-1}(\vec{p}\,)$. See Fig. 1a.

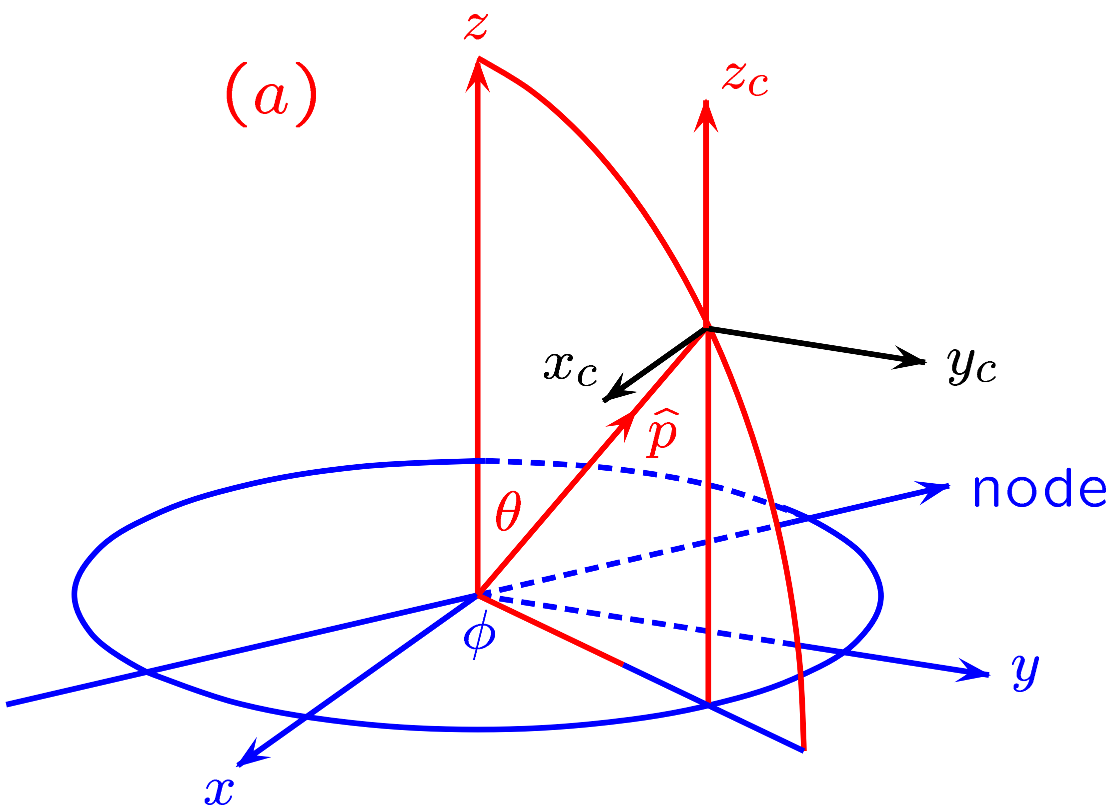


Fig. 1a: The orientation of the coordinate systems associated with a particle at rest in the canonical description $(\hat{x}_c,\ \ \hat{y}_c,\ \ \hat{z}_c)$.

Next, we shall define the helicity state describing a single particle with spin j and momentum $\vec{p}$ [see Fig. 1b]:

(23)
$$
\begin{aligned}
|\vec{p}, j\lambda\rangle &= |\phi, \theta, p, j\lambda\rangle \\
&= U[\mathring{R}(\phi,\theta,0)]\,U[L_z(p)]\,|j\lambda\rangle = U[L(\vec{p}\,)]\,U[\mathring{R}(\phi,\theta,0)]\,|j\lambda\rangle
\end{aligned}
$$

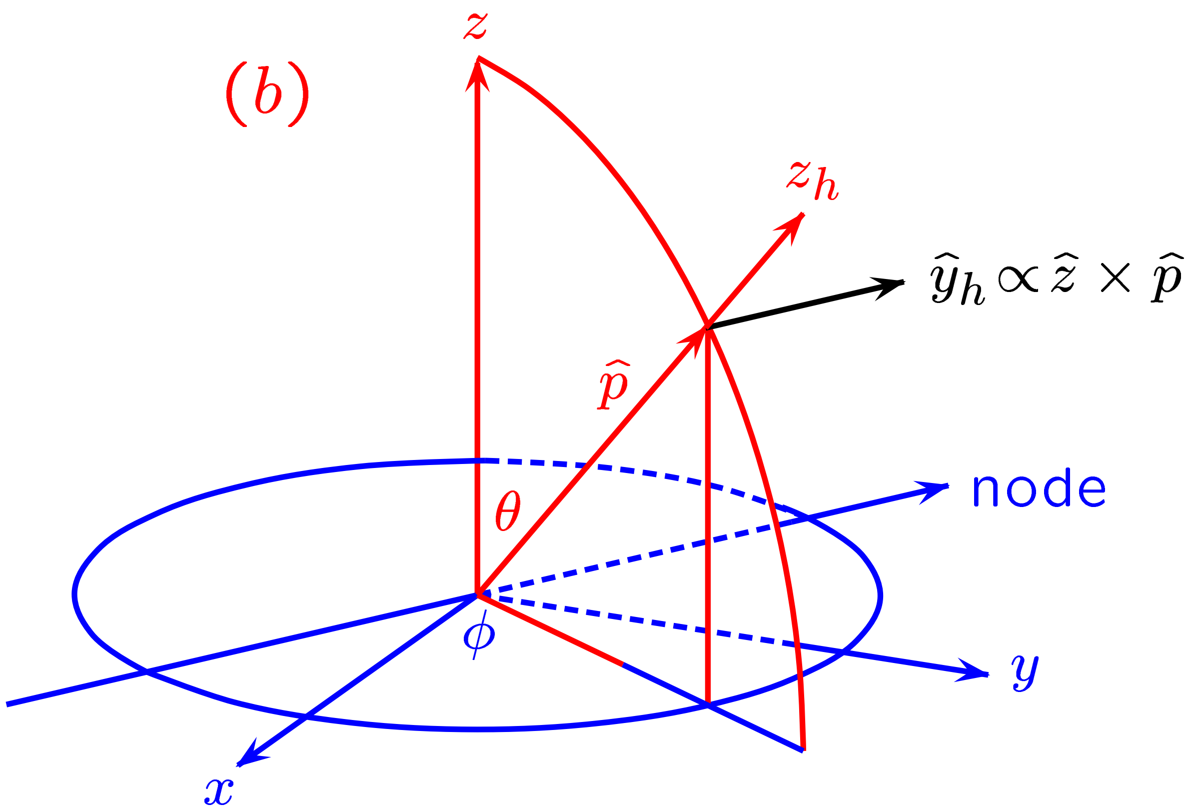


Fig. 1b: The orientation of the coordinate systems associated with a particle at rest in the helicity description $(\hat{x}_h = \hat{y}_h \times \hat{z}_h,\;\; \hat{y}_h \propto \hat{z} \times \hat{p},\;\; \hat{z}_h = \hat{p})$.

Helicity states may be defined in two different ways. One may first rotate the rest state $|j\lambda\rangle$ by $\overset{\circ}{R}$, so that the quantization axis is along the $\vec{p}$ direction and then boost the system along $\vec{p}$ to obtain the helicity state $|\vec{p}, j\lambda\rangle$. Or, equivalently, one may first boost the rest state $|j\lambda\rangle$ along the z-axis and then rotate the system to obtain the state $|\vec{p}, j\lambda\rangle$. That these two different definitions of helicity state are equivalent is obvious from the relation for $U[L(\vec{p}\,)]$ given previously.

One sees that, by definition, the helicity quantum number λ is the component of the spin along the momentum $\vec{p}$, and as such it is a rotationally invariant quantity, simply because the quantization axis itself rotates with the system under rotation. This fact may be seen easily from the definition of helicity states:

$$
\begin{aligned}
U[R]\,|\vec{p}, j\lambda\rangle &= U[R\overset{\circ}{R}]\,U[L_z]\,|j\lambda\rangle \\
&= |R\,\vec{p}, j\lambda\rangle \; .
\end{aligned}
\tag{24}
$$

In addition, the helicity λ remains invariant under pure Lorentz transformation that takes $\vec{p}$ into $\vec{p}\,'$, which is parallel to $\vec{p}$. The invariance of λ under L' may be seen by

$$U[L']\,|\vec{p}, j\lambda\rangle = U[L']U[L(\vec{p}\,)]U[\mathring{R}]\,|j\lambda\rangle = U[L(\vec{p}\,'\,)]U[\mathring{R}]\,|j\lambda\rangle = |\vec{p}\,', j\lambda\rangle \;. \tag{25}$$

There is a simple connection between the canonical and helicity descriptions. From the definitions of canonical and helicity states, on finds easily that

$$|\vec{p}, j\lambda\rangle = U[\mathring{R}]U[L_z]U^{-1}[\mathring{R}]U[\mathring{R}]\,|j\lambda\rangle = \sum_m D^j_{m\lambda}(\mathring{R})\,|\vec{p}, jm\rangle \;. \tag{26}$$

We shall adopt here the following normalizations for the one-particle states:

(27)
$$\langle \vec{p}\,'\, j' m' | \vec{p} j m \rangle = \tilde{\delta}(\vec{p}\,' - \vec{p}\,) \delta_{jj'} \delta_{mm'}$$
$$\langle \vec{p}\,'\, j' \lambda' | \vec{p} j \lambda \rangle = \tilde{\delta}(\vec{p}\,' - \vec{p}\,) \delta_{jj'} \delta_{\lambda\lambda'} \ ,$$

where $\delta(\vec{p}\,' - \vec{p}\,)$ is the Lorentz invariant δ-function given by

(28)
$$\tilde{\delta}(\vec{p}\,' - \vec{p}\,) = (2\pi)^3 (2E)\, \delta^{(3)}(\vec{p}\,' - \vec{p}\,) \ .$$

It can be shown that, with the invariant normalization given above, an arbitrary Lorentz transformation operator $U[\Lambda]$ acting on the states $|\vec{p}, jm\rangle$ or $|p, j\lambda\rangle$ is indeed a unitary operator, i.e. $U^{+}U = I$. With the invariant volume element as defined by

(29)
$$\tilde{\mathrm{d}}p = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3(2E)} \ ,$$

the completeness relations may be written as follows:

(30)
$$\sum_{jm} \int |\vec{p} j m\rangle \, \tilde{\mathrm{d}}p \, \langle \vec{p} j m| = I, \qquad \sum_{j\lambda} \int |\vec{p} j \lambda\rangle \, \tilde{\mathrm{d}}p \, \langle \vec{p} j \lambda| = I \ ,$$

where I denotes the identity operator.

Parity and Time-Reversal Operations

Classically, the action of parity and time-reversal operations, denoted $\boldsymbol{P}$ and $\boldsymbol{T}$, may be expressed as follows:

$$\begin{aligned} \boldsymbol{P}: &\quad \vec{x} \to -\vec{x}, \quad \vec{p} \to -\vec{p}, \quad \vec{J} \to \vec{J} \\ \boldsymbol{T}: &\quad \vec{x} \to \vec{x}, \quad \vec{p} \to -\vec{p}, \quad \vec{J} \to -\vec{J} \end{aligned} \tag{31}$$

where $\vec{x}$, $\vec{p}$, and $\vec{J}$ stand for the coordinate, momentum, and angular momentum, respectively. It is clear that $\boldsymbol{P}$ and $\boldsymbol{T}$ commute with rotations, i.e.

(32) $$[\boldsymbol{P}, R] = 0, \quad [\boldsymbol{T}, R] = 0\ .$$

By definition, one sees also that the pure Lorentz transformations (in particular, boosts) act under $\boldsymbol{P}$ and $\boldsymbol{T}$ according to

(33) $$\boldsymbol{P} L(\vec{p}\,) = L(-\vec{p}\,)\boldsymbol{P}, \qquad \boldsymbol{T} L(\vec{p}\,) = L(-\vec{p}\,)\boldsymbol{T}\ .$$

Let us now define operators acting on the physical states, representing the parity and time-reversal operations:

(34) $$\Pi = U[\boldsymbol{P}], \qquad \mathbb{T} = \bar{U}[\boldsymbol{T}]\ ,$$

where Π is a unitary operator and $\mathbb{T}$ is an anti-unitary (or anti-linear unitary) operator. $\boldsymbol{T}$ is represented by an anti-unitary operator due to the fact that the time-reversal operation transforms an initial state into a final state and vice versa. Operators Π, $\mathbb{T}$, $U[R]$, and $U[L(\vec{p}\,)]$ acting on the physical states should obey the same relations

(35) $$\Big[\Pi, U[R]\Big] = 0 \qquad \Big[\mathbb{T}, U[R]\Big] = 0$$

and

(36) $$\begin{aligned} \Pi\, U[L(\vec{p}\,)] &= U[L(-\vec{p}\,)]\,\Pi \\ \mathbb{T}\, U[L(\vec{p}\,)] &= U[L(-\vec{p}\,)]\,\mathbb{T}\ . \end{aligned}$$

We are now ready to express the actions of Π and $\mathbb{T}$ on the rest states $|jm\rangle$. It is clear that the quantum numbers j and m do not change under Π:

(37) $$\Pi|jm\rangle = \eta|jm\rangle \ ,$$

where η is the intrinsic parity of the particle represented by $|jm\rangle$. Let us write the action of $\mathbb{T}$ as follows:

$$\mathbb{T}\,|jm\rangle = \sum_k T_{km}|jk\rangle \ .$$

Remembering the anti-unitarity of $\mathbb{T}$,

$$\sum_k D^j_{m'k}(R)\,T_{km} = \sum_k T_{m'k}\,D^{j\,*}_{km}(R) \ .$$

The above relation may be satisfied, if $T_{m'm}$ is given by

$$T_{m'm} = d^j_{m'm}(\pi) = (-)^{j-m}\delta_{m',-m} \ , \quad D^{j\,*}_{m'm}(\alpha,\beta,\gamma) = (-)^{m'-m}D^j_{-m'-m}(\alpha\beta\gamma)$$

so that the action of $\mathbb{T}$ on the states $|jm\rangle$ may be expressed as

(38) $$\mathbb{T}\,|jm\rangle = (-)^{j-m}|j-m\rangle \ .$$

One can show that the canonical state with momentum $\vec{p}$ transforms under Π and $\mathbb{T}$ as follows:

(39)
$$\Pi\,|\vec{p}, jm\rangle = \eta|-\vec{p}, jm\rangle$$
$$\Pi\,|\phi, \theta, p, jm\rangle = \eta|\pi+\phi, \pi-\theta, p, jm\rangle$$

and

(40)
$$\mathbb{T}\,|\vec{p}, jm\rangle = (-)^{j-m}|-\vec{p}, j-m\rangle$$
$$\mathbb{T}\,|\phi, \theta, p, jm\rangle = (-)^{j-m}|\pi+\phi, \pi-\theta, p, j-m\rangle\ .$$

Next, we wish to express the consequences of Π and $\mathbb{T}$ operations on the helicity states $|p, j\lambda\rangle$. One finds

(41)
$$\Pi|\phi, \theta, p, j\lambda\rangle = \eta\, e^{-i\pi j}|\pi+\phi, \pi-\theta, p, j\,{-\lambda}\rangle$$

(42)
$$\mathbb{T}\,|\phi, \theta, p, j\lambda\rangle = e^{-i\pi\lambda}|\pi+\phi, \pi-\theta, p, j\lambda\rangle\ .$$

Now the helicity λ is an eigenvalue of $\vec{J}\cdot\hat{p}$. Note that $\vec{J}\cdot\hat{p} \to -\vec{J}\cdot\hat{p}$ under $\boldsymbol{P}$ and $\vec{J}\cdot\hat{p} \to \vec{J}\cdot\hat{p}$ under $\boldsymbol{T}$. This explains why the helicity λ changes sign under Π, while it remains invariant under $\mathbb{T}$.

Two-Particle States

A system consisting of two particles with arbitrary spins may be constructed in two different ways; one using the canonical basis vectors $|\vec{p}, jm\rangle$, and the other using the helicity basis vectors $|\vec{p}, j\lambda\rangle$. We shall construct in this section both the canonical and helicity states for a two-particle system having definite spin and z-component, and then derive the recoupling coefficient which connects the two bases. Afterwards, we investigate the transformation properties of the two-particle states under Π and $\mathbb{T}$, as well as the consequences of the symmetrization required when the two particles are identical.

Construction of two-particle states:

Consider a system of two particles 1 and 2 with intrinsic spins s_1 and s_2 and masses w_1 and w_2. In the two-particle rest frame, let $\vec{p}$ be the momentum of the particle 1, with its direction given by the spherical angles (θ, ϕ). we define the two-particle state in the canonical basis by

(43) $$|\phi\theta m_1 m_2\rangle = a\, U[L(\vec{p}\,)]\,|s_1 m_1\rangle U[L(-\vec{p}\,)]\,|s_2 m_2\rangle \;,$$

where $|s_1 m_1\rangle$ is the rest-state of particle i and a is the normalization constant to be determined later. $L(\pm\vec{p}\,)$ is the boost given by

(44) $$L(\pm\vec{p}\,) = \mathring{R}(\phi,\theta,0) L_{\pm z}(p) \mathring{R}^{-1}(\phi,\theta,0) \;,$$

where $\mathring{R}(\phi,\theta,0)$ is again the rotation which carries the z-axis into the direction of $\vec{p}$ and $L_{\pm z}(p)$ is the boost along the $\pm z$-axis.

Owing to the rotational property of canonical one-particle states, one may define a state of intrinsic total spin s by

(45) $$|\phi\theta s m_s\rangle = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | s m_s) |\phi\theta m_1 m_2\rangle \ ,$$

where $(s_1 m_1\, s_2 m_2 | s m_s)$ is the usual Clebsch-Gordan coefficient. Using the formula

(46) $$D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} = \sum_{j_3 \mu_3 m_3} (j_1 \mu_1\, j_2 \mu_2 | j_3 \mu_3)(j_1 m_1\, j_2 m_2 | j_3 m_3) D^{j_3}_{\mu_3 m_3}$$

and the orthonormality of the Clebsch-Gordan coefficients, one may easily show that, if R is a rotation which takes $\Omega = (\theta, \phi)$ into $R' = R\Omega$,

(47) $$U[R]\, |\Omega\, s m_s\rangle = \sum_{m'_s} D^s_{m'_s m_s}(R) |R'\, s m'_s\rangle \ ,$$

so that the total spin s is a rotational invariant.

The state of a fixed orbital angular momentum is constructed in the usual way:

(48) $$|\ell m s m_s\rangle = \int \mathrm{d}\Omega\, Y^\ell_m(\Omega) |\Omega s m_s\rangle \ ,$$

where $\mathrm{d}\Omega = \mathrm{d}\phi\, \mathrm{d}\cos\theta$. Let us investigate the rotational property of the formula above

(49) $$U[R]\, |\ell m s m_s\rangle = \int \mathrm{d}\Omega\, Y^\ell_m(\Omega) D^s_{m_s' m_s}(R) |R' s m_s'\rangle \ ,$$

where $R' = R'(\alpha', \beta', \gamma') = R\,\Omega, \quad \mathrm{d}\Omega = \mathrm{d}\alpha'\, \mathrm{d}\cos\beta'$, and, from

(50) $$Y^\ell_m(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell *}_{m0}(\phi, \beta, 0)$$

we see that

(51) $$\begin{aligned} Y^\ell_m(\Omega) &= \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell *}_{m0}(R^{-1} R') \\ \text{(group property of } D\text{'s)} \rightarrow &= \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'} D^{\ell\, *}_{m m'}(R^{-1}) D^{\ell *}_{m' 0}(R') \\ \text{(unitarity of } D\text{'s)} \rightarrow &= \sum_{m'} D^\ell_{m' m}(R) Y^\ell_{m'}(\beta', \alpha') \ , \end{aligned}$$

one obtains the result

$$U[R]\,|\ell m s m_s\rangle = \sum_{m' m'_s} D^{\ell}_{m' m}(R) D^{s}_{m'_s m_s}(R) |\ell m' s m'_s\rangle \; . \tag{52}$$

This shows that the states $|\ell m s m_s\rangle$ transform under rotation as a product of two "rest states" $|\ell m\rangle$ and $|s m_s\rangle$.

Now, it is easy to construct a state of total angular momentum J:

$$
\begin{aligned}
|JM\ell s\rangle &= \sum_{mm_s} (\ell m\; sm_s|JM)|\ell m s m_s\rangle \\
&= \sum_{\substack{mm_s \\ m_1 m_2}} (\ell m\; sm_s|JM)(s_1 m_1\; s_2 m_2|sm_s)\times \\
&\quad \times \int \mathrm{d}\Omega\, Y^{\ell}_{m}(\Omega)|\Omega m_1 m_2\rangle\ .
\end{aligned} \tag{53}
$$

One sees immediately

$$
U[R]\,|JM\ell s\rangle = \sum_{M'} D^{J}_{M'M}(R)|JM'\ell s\rangle\ . \tag{54}
$$

Note that, as expected, ℓ and s are rotational invariants: This is the equivalent of the non-relativistic L-S coupling.

Next, we turn to the problem of constructing two-particle states from the helicity basis vectors $|\vec{p}, j\lambda\rangle$. we write

$$|\phi\theta\lambda_1\lambda_2\rangle = a\,U[\overset{\circ}{R}]\Big\{U[L_z(p)]\,|s_1\lambda_1\rangle U[L_{-z}(p)]\,|s_2\ -\lambda_2\rangle\Big\} \tag{55}$$

$$\equiv U[\overset{\circ}{R}(\phi,\theta,0)]\,|00\,\lambda_1\lambda_2\rangle\ ,$$

where $|s_i\lambda_i\rangle$ is the rest state of particle i and a the normalization constant introduced previously. We have constructed the helicity state for the particle 2 in such a way that its helicity quantum number is $+\lambda_2$.

States of a definite total angular momentum J may be constructed as follows:

$$|JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi}\int \mathrm{d}R\, D^{J\,*}_{M\mu}(R)U[R]\,|00\lambda_1\lambda_2\rangle\ , \tag{56}$$

where N_J is a normalization constant to be determined later. Let us apply an arbitrary rotation R' on this state

$$U[R'\,]\,|JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi}\int \mathrm{d}R\, D^{J\,*}_{M\mu}(R)U[R''\,]\,|00\lambda_1\lambda_2\rangle\ ,$$

where $R'' = R'R$. But, by using the multiplication law and the unitarity of the D-functions,

$$\begin{aligned} D^{J\,*}_{M\mu}(R) &= D^{J\,*}_{M\mu}(R'^{\,-1}R''\,) \\ &= \sum_{M'} D^{J\,*}_{MM'}(R'^{\,-1})D^{J\,*}_{M'\mu}(R''\,) \\ \text{Unitarity} \rightarrow &= \sum_{M'} D^{J}_{M'M}(R')\,D^{J\,*}_{M'\mu}(R''\,)\ . \end{aligned} \tag{57}$$

Using this relation, as well as the fact that $dR = dR''$, one obtains the result

$$U[R'\,]\,|JM\lambda_1\lambda_2\rangle = \sum_{M'} D^J_{M'M}(R'\,)|JM'\lambda_1\lambda_2\rangle\ , \tag{58}$$

so that the states are indeed states of a definite angular momentum J. Note that, as expected, λ_1 and λ_2 are rotational invariants.

Now, let us specify the rotation R by writing $R = R(\phi,\theta,\gamma)$. Then,

$$\begin{aligned} U[R(\phi,\theta,\gamma)]\,|00\lambda_1\lambda_2\rangle &= U[R(\phi,\theta,0)]U[R(0,0,\gamma)]\,|00\lambda_1\lambda_2\rangle \\ &= e^{-i(\lambda_1-\lambda_2)\,\gamma}\,U[R(\phi,\theta,0)]\,|00\lambda_1\lambda_2\rangle\ . \end{aligned} \tag{59}$$

The last relation follows because of the commutation relation: $[R(0,0,\gamma), L_{\pm z}(p)] = 0$. Integrating over $d\gamma$, one obtains

$$|JM\lambda_1\lambda_2\rangle = N_J \int \mathrm{d}\Omega\, D^{J\,*}_{M\lambda}(\phi,\theta,0)\ |\phi\theta\lambda_1\lambda_2\rangle\ , \tag{60}$$

where $\lambda = \lambda_1 - \lambda_2$.

Normalization:

For simplicity of notation, we shall deal with spinless particles. Two-particle states are normalized

(61) $$\langle \vec{p}_1{}' \vec{p}_2{}' | \vec{p}_1 \vec{p}_2 \rangle = \tilde{\delta}(\vec{p}_1{}' - \vec{p}_1)\, \tilde{\delta}(\vec{p}_2{}' - \vec{p}_2) \ ,$$

where

(62) $$\tilde{\delta}(\vec{p}_i{}' - \vec{p}_i) = (2\pi)^3 (2E_i)\, \delta^{(3)}(\vec{p}_i{}' - \vec{p}_i), \qquad i = 1 \text{ or } 2 \ .$$

A system consisting of two momenta $\vec{p}_1$ and $\vec{p}_2$ may be described, in general, by one four-momentum p representing the sum of the four-momenta of particles 1 and 2 and Ω describing the orientation of the relative momentum in the (1,2) rest frame, i.e.

(63) $$|p, \Omega\rangle = a\, |\vec{p}_1 \vec{p}_2\rangle \ , \quad \text{where} \quad p = p_1 + p_2 \quad \text{and} \quad p^2 = w^2.$$

We adopt the normalization for this state as follows:

(64) $$\langle p', \Omega' | p, \Omega \rangle = (2\pi)^4 \delta^{(4)}(p' - p)\, \delta^{(2)}(\Omega' - \Omega) \ .$$

Multiply the formula above by the invariant volume element $\tilde{d}p_1\, \tilde{d}p_2$, where

(65) $$\tilde{\mathrm{d}}p = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 (2E)} \ ,$$

and integrate over these variables.

Using the notation for which $d\phi_n$ is the n-body phase space so that

(66) $$\mathrm{d}\phi_2(1,2) = (2\pi)^4 \delta^{(4)}(p' - p)\, \tilde{\mathrm{d}}p_1\, \tilde{\mathrm{d}}p_2$$

we need to evaluate an integration, in the two-particle rest frame (RF),

$$a^2 = \int \delta^{(2)}(\Omega' - \Omega)\, \mathrm{d}\phi_2(1,2)$$

(67) $$\left(\int \mathrm{d}^3\vec{p}_2;\ \vec{p}_1 \to \vec{p}\right) \to = \frac{1}{(2\pi)^2} \int \delta(E_1 + E_2 - w)\, \delta^{(2)}(\Omega' - \Omega)\, \frac{p^2\, \mathrm{d}p\, \mathrm{d}\Omega}{4E_1\, E_2}$$

$$\left(\int \mathrm{d}\Omega;\ E_i = \sqrt{p^2 + w_i^2}\right) \to = \frac{1}{(4\pi)^2}\, \frac{p}{w}$$

where p is the relative momentum in the two-particle RF and $w = E_1 + E_2$ is the effective mass of the two-particle system. We see immediately that

(68) $$a = \frac{1}{4\pi} \sqrt{\frac{p}{w}}\ ,$$

We are now ready to specify the normalizations for the two-particle states with arbitrary spin.

$$\langle \Omega' m_1' m_2' | \Omega m_1 m_2 \rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{m_1 m_1'}\delta_{m_2 m_2'} \tag{69}$$

and

$$\langle \Omega' \lambda_1' \lambda_2' | \Omega \lambda_1 \lambda_2 \rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{\lambda_1 \lambda_1'}\delta_{\lambda_2 \lambda_2'} \ . \tag{70}$$

The states $|JM\ell s\rangle$ obey the following normalizations:

$$\langle J' M' \ell' s' | JM\ell s \rangle = \delta_{JJ'}\delta_{MM'}\delta_{\ell\ell'}\delta_{ss'} \ . \tag{71}$$

Since the D-functions are normalized according to

(72) $$\int \mathrm{d}R\, D^{j_1\,*}_{\mu_1 m_1}(R) D^{j_2}_{\mu_2 m_2}(R) = \frac{8\pi^2}{2j_1+1}\delta_{j_1 j_2}\delta_{\mu_1\mu_2}\delta_{m_1 m_2}\ ,$$

where $R = R(\alpha,\beta,\gamma)$ and $\mathrm{d}R = \mathrm{d}\alpha\,\mathrm{d}\cos\beta\,\mathrm{d}\gamma$, the state $|JM\lambda_1\lambda_2\rangle$ is seen to be normalized according to

(73) $$\langle J'M'\lambda_1'\lambda_2'|JM\lambda_1\lambda_2\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\ ,$$

if the constant N_J is set equal to

(74) $$N_J = \sqrt{\frac{2J+1}{4\pi}}\ .$$

The completeness relations may now be written

(75)
$$\sum_{\substack{JM\\ \ell s}} |JM\ell s\rangle\langle JM\ell s| = I$$

and

(76)
$$\sum_{\substack{JM\\ \lambda_1\lambda_2}} |JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2| = I \ .$$

Finally, we note the relation

(77)
$$\langle\Omega\lambda_1'\lambda_2'|JM\lambda_1\lambda_2\rangle = N_J D^{J\,*}_{M\lambda}(\phi,\theta,0)\,\delta_{\lambda_1\lambda_1'}\,\delta_{\lambda_2\lambda_2'} \ .$$

Connection between canonical and helicity states:

We start from

$$|\phi\theta\lambda_1\lambda_2\rangle = a\,U[\overset{\circ}{R}]\Big\{U[L_z(p)]\,|s_1\lambda_1\rangle U[L_{-z}(p)]\,|s_2\,{-\lambda_2}\rangle\Big\}$$

(78)

$$= a\,U[L(\vec{p}\,)]U[\overset{\circ}{R}]\,|s_1\lambda_1\rangle U[L(-\vec{p}\,)]U[\overset{\circ}{R}]\,|s_2\,{-\lambda_2}\rangle$$

$$= \sum_{m_1 m_2} D^{s_1}_{m_1\lambda_1}(\phi,\theta,0)D^{s_2}_{m_2-\lambda_2}(\phi,\theta,0)|\phi\theta m_1 m_2\rangle\ ,$$

we see that

$$|JM\lambda_1\lambda_2\rangle = N_J\sum_{m_1 m_2}\int \mathrm{d}\Omega\, D^{J\,*}_{M\lambda}(\phi,\theta,0)D^{s_1}_{m_1\lambda_1}(\phi,\theta,0)D^{s_2}_{m_2-\lambda_2}(\phi,\theta,0)|\phi\theta m_1 m_2\rangle\ .$$

(79)

The product of three D-functions appearing may be reduced as follows:

(80)
$$D^{s_1}_{m_1\lambda_1}D^{s_2}_{m_2-\lambda_2} = \sum_{s m_s}(s_1 m_1 s_2 m_2|s m_s)(s_1\lambda_1 s_2-\lambda_2|s\lambda)D^{s}_{m_s\lambda}$$

and

(81) $$D^{J\,*}_{M\lambda} D^{s}_{m_s\lambda} = \sum_{\ell m} \sqrt{\frac{4\pi}{2\ell+1}} \left(\frac{2\ell+1}{2J+1}\right) (\ell m\ s m_s|JM)(\ell 0\ s\lambda|J\lambda) Y^{\ell}_{m}\ .$$

we obtain finally

(82) $$|JM\lambda_1\lambda_2\rangle = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0\ s\lambda|J\lambda)(s_1\lambda_1\ s_2\ {-\lambda_2}|s\lambda)|JM\ell s\rangle\ ,$$

so that the recoupling coefficient between canonical and helicity states is given by

(83) $$\langle J'M'\ell s|JM\lambda_1\lambda_2\rangle = \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0\ s\lambda|J\lambda)(s_1\lambda_1\ s_2\ {-\lambda_2}|s\lambda)\delta_{JJ'}\delta_{MM'}\ .$$

This relation may be inverted to give

(84) $$\begin{aligned} |JM\ell s\rangle &= \sum_{\lambda_1\lambda_2} |JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2|JM\ell s\rangle \\ &= \sum_{\lambda_1\lambda_2} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0\ s\lambda|J\lambda)(s_1\lambda_1\ s_2\ {-\lambda_2}|s\lambda)|JM\lambda_1\lambda_2\rangle\ . \end{aligned}$$

Symmetry relations:

The canonical states $|JM\ell s\rangle$ transform in a particularly simple manner under symmetry operations (e.g. parity and time-reversal), and the derivation is also much simpler than for helicity states. For this reason, we shall first investigate the consequences of symmetry operations on the canonical states, and then obtain the corresponding relations for the states $|JM\lambda_1\lambda_2\rangle$.
We shall first start with the parity operation. We find

(85) $$\Pi|\phi\theta m_1 m_2\rangle = \eta_1\eta_2|\pi+\phi,\pi-\theta,m_1 m_2\rangle \ ,$$

where $\eta_1\,(\eta_2)$ is the intrinsic parity of particle 1(2). We then obtain immediately

(86) $$Y^\ell_m(\pi-\theta,\pi+\phi) = (-)^\ell\, Y^\ell_m(\theta,\phi), \qquad \Pi|JM\ell s\rangle = \eta_1\eta_2(-)^\ell|JM\ell s\rangle \ ,$$

so that the "ℓ-s coupled" states are in an eigenstate of Π with the eigenvalue $\eta_1\eta_2(-)^\ell$, a well known result. Using the recoupling formula and the symmetry relations of Clebsch-Gordan coefficients, one finds for the helicity states

(87) $$\Pi\,|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-)^{J-s_1-s_2}|JM\,{-\lambda_1}\,{-\lambda_2}\rangle \ .$$

Again, the helicities reverse sign, as was the case for the single-particle states.

Consequences of the time-reversal operation may be explored in a similar fashion. one finds

(88) $$\mathbb{T}\,|\phi\theta m_1 m_2\rangle = (-)^{s_1-m_1}(-)^{s_2-m_2}|\pi+\phi,\pi-\theta,-m_1-m_2\rangle\ .$$

and so

(89) $$\mathbb{T}\,|JM\ell s\rangle = (-)^{J-M}|J-M\ell s\rangle$$

and

(90) $$\mathbb{T}\,|JM\lambda_1\lambda_2\rangle = (-)^{J-M}|J-M\lambda_1\lambda_2\rangle\ .$$

Now, we investigate the effects of symmetrization required when the particles 1 and 2 are identical— No field theory here—

(91) $|JM\ell s\rangle_s = a_s[1 + (-)^{2s_1}\,\mathbb{P}_{12}]\,|JM\ell s\rangle\ , \qquad \mathbb{P}_{12}|JM\ell s\rangle = (-)^{\ell+s-2s_1}|JM\ell s\rangle$

where $\mathbb{P}_{12}$ is the particle-exchange operator and a_s is the normalization constant. Again, using the defining equation, one obtains

(92) $$|JM\ell s\rangle_s = a_s[1 + (-)^{\ell+s}]\,|JM\ell s\rangle\ ,$$

so that $\boxed{\ell + s = \text{even}}$ for a system of identical particles in an eigenstate of orbital angular momentum ℓ and total spin s and $a_s = 1/2$. Now, the symmetrized helicity state may be written

(93) $$|JM\lambda_1\lambda_2\rangle_s = b_s(\lambda_1\lambda_2)[1 + (-)^{2s_1}\,\mathbb{P}_{12}]\,|JM\lambda_1\lambda_2\rangle$$

where $b_s(\lambda_1\lambda_2)$ is another normalization constant. Using $|JM\ell s\rangle_s$, one finds

(94) $$|JM\lambda_1\lambda_2\rangle_s = b_s(\lambda_1\lambda_2)\left\{|JM\lambda_1\lambda_2\rangle + (-)^J|JM\lambda_2\lambda_1\rangle\right\}\ ,$$

where $b_s(\lambda_1\lambda_2) = 1/\sqrt{2}$ for $\lambda_1 \neq \lambda_2$ and $b_s(\lambda_1\lambda_2) = 1/2$ for $\lambda_1 = \lambda_2$ (and $J =$ even). Note that, for a system of identical particles, the symmetrized states in both canonical and helicity bases have the same forms, regardless of whether the particles involved are fermions or bosons.

To Be Continued. . .