Selected Topics in Hadron Spectroscopy

Mathematical Techniques

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Perliminaries

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  'The Helicity-coupling amplitudes in tensor formalism'

  A Practical Guide—Examples with $\ell \leq 4$ (BNL-QGS94-21)

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  Rank-J Tensor for $|Jm\rangle$

  General $\gamma = E/m$ (the Lorentz factor) dependence

  http://cern.ch/suchung/

- S. Huang, T. Ruan, N. Wu, and Z. Zheng, Eur. Phys. J. C 26, 609 (2003)

  Rank-J Tensor for $|Jm\rangle$

  (independently derived)

Applications

We are now ready to apply results of the previous section to a few physical problems of practical importance. As a first application, we shall write down the invariant transition amplitude for two-body reactions and derive the partial-wave expansion formula. We do this in the helicity basis, following the derivation given in the “classic” paper by Jacob and Wick. Our main purpose in this exercise is to show how the particular normalization of single-particle states influences the precise definition of the invariant amplitudes and the corresponding cross-section formula.

Next, we shall discuss the general two-body decays of resonances and give the symmetry relations satisfied by the decay amplitude, as well as the coupling formula which connects the helicity decay amplitude to the partial-wave amplitudes. Finally, we take up the discussion of the spin density matrices, introduce the multipole parameters, and then expand the angular distribution for two-body decays in terms of the multipole parameters.

Cross Section:

Consider a reaction

(1) $$a + b \to 1 + 2 + \cdots + n \ .$$

In the over-all c.m. system, let w_0 be the c.m. energy and $p_i (p_f)$ the over-all four-momentum in the initial (final) state. In terms of the invariant amplitude $\mathcal{M}_{fi}$,

(2) $$\mathrm{d}\sigma = \frac{1}{(2E_a)(2E_b)\,\beta_{\mathrm{rel}}} |\mathcal{M}_{fi}|^2 \mathrm{d}\phi_n(1, 2, \cdots, n) = \frac{1}{4\mathcal{F}} |\mathcal{M}_{fi}|^2 \mathrm{d}\phi_n(1, 2, \cdots, n) \ ,$$

where $\mathcal{F}$ is the flux factor, which in the over-all c.m. is given by

(3) $$\mathcal{F} = E_a\, E_b\, \beta_{\mathrm{rel}} = \left[(p_a \cdot p_b)^2 - (w_a\, w_b)^2 \right]^{1/2}$$
$$= p_a w_b \text{ (stationary target)} = p_i \sqrt{s} \text{ (CM system)} = p_L \sqrt{s} \cos(\theta_L/2) \text{ (pp Collider)}$$

and $d\phi_n$ is the n-body phase space:

(4) $$\mathrm{d}\phi_n(1, 2, \cdots, n) = (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{k=1}^{n} \tilde{\mathrm{d}}p_k, \qquad \tilde{\mathrm{d}}p_k = \frac{\mathrm{d}^3 \vec{p}_k}{(2\pi)^3 (2E_k)} \ .$$

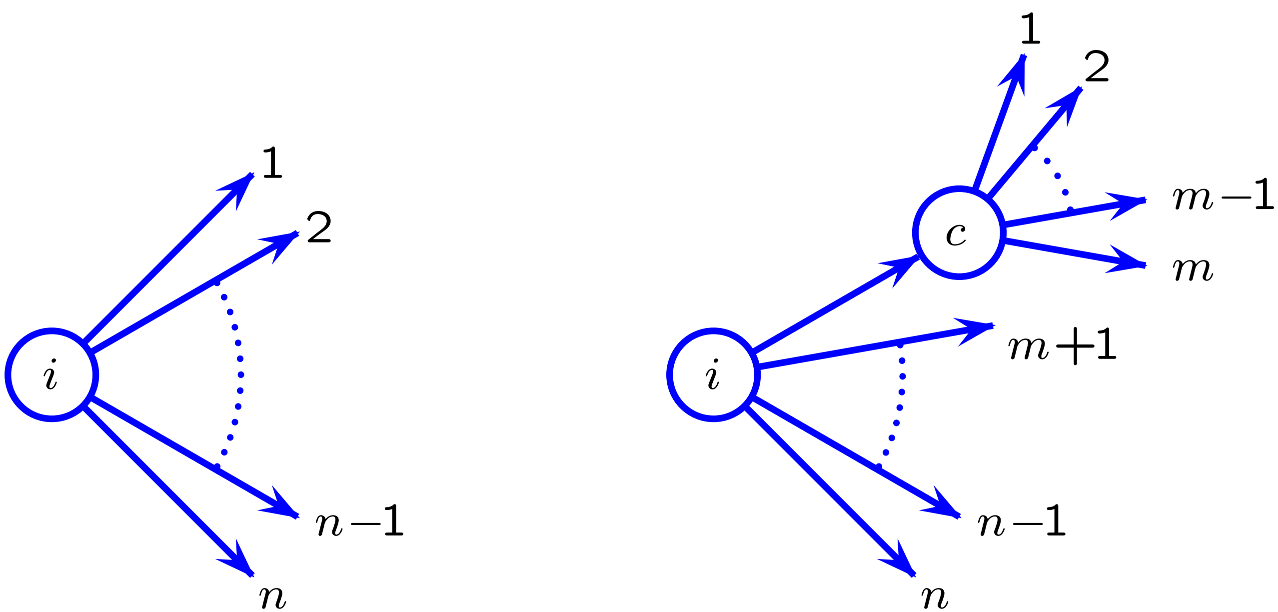
$\tilde{d}p_k$ is the invariant volume element of the k^{th} particle.

Phase Space:

The phase-space formula may be broken up into two factors as follows:

$$\mathrm{d}\phi_n = \mathrm{d}\phi_\ell(i \to c, m+1, \cdots, n) \left(\frac{\mathrm{d}w_c^2}{2\pi}\right) \mathrm{d}\phi_m(c \to 1, 2, \cdots, m) \ , \tag{5}$$

where $\ell + m = n + 1$ and c denotes a system consisting of particles 1 to m, its effective mass being w_c. The expression above may be termed the Cluster Decomposition Formula. Refer to the following figure:



Here the figure on the left side refer to $d\phi_n(1, 2, \cdots, n)$, while that on the right side refer to the cluster decomposition formula above.

After repeated application of the formula above and using the explicit expression for the two-body phase space, we may express the n-body phase-space succinctly as follows:

(6) $$\mathrm{d}\phi_n = \frac{1}{2^n} \cdot \frac{(2\pi)^4}{(2\pi)^{3n}} \cdot \frac{p_0}{w_0} \mathrm{d}\Omega_0 \prod_{k=0}^{n-2} \{p\,\mathrm{d}w\,\mathrm{d}\Omega\}_k \,, \qquad \mathrm{d}\phi_2(1,2) = \frac{1}{(4\pi)^2} \frac{p}{w} \mathrm{d}\Omega \,.$$

where w is the effective mass of the particles 1 and 2; and P and Ω denote the magnitude and direction of the relative momentum in the (1,2) rest frame. Note that we must set $n \geq 2$ and $\{\cdots\}_0 = 1$. Modify the formula above to include 3-body decays:

(7) $$\mathrm{d}\phi_n = \frac{1}{2^n} \cdot \frac{(2\pi)^4}{(2\pi)^{3n}} \cdot \frac{p_0}{w_0} \mathrm{d}\Omega_0 \prod_{k=0}^{n_2} \{p\,\mathrm{d}w\,\mathrm{d}\Omega\}_k \prod_{\ell=0}^{n_3} \{w'\,\mathrm{d}w'\,\mathrm{d}R\,\mathrm{d}E\,\mathrm{d}E'\,\}_\ell$$

where $n_2 \geq 0$, $n_3 \geq 0$ and $n = n_2 + 2n_3 + 2 \geq 2$. Again, note $n \geq 2$ and $\{\cdots\}_0 = 1$.

The 3-body phase-space formula is

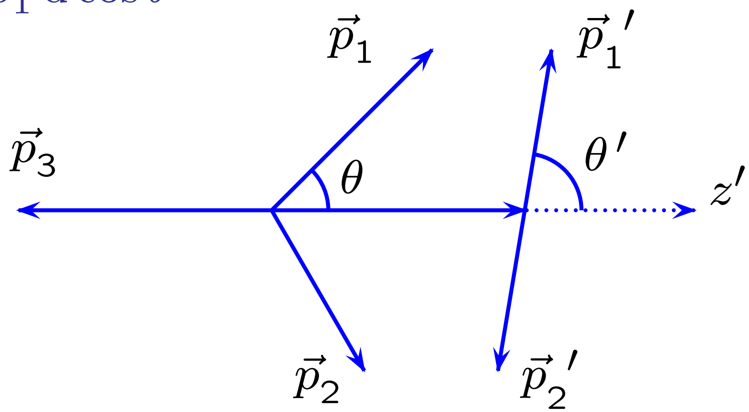
(8)
$$d\phi_3(1,2,3) = \frac{1}{2^3} \cdot \frac{1}{(2\pi)^5} \cdot \frac{p_0}{w_0}\, d\Omega_0\, p_1' dw_{12}\, d\Omega'$$
$$\rightarrow \frac{4}{(4\pi)^5}\, dR(\alpha,\beta,\gamma)\, \frac{p_3}{w_0}\, p_1'\, d\cos\theta'\, \frac{dw_{12}^2}{2w_{12}}$$

where $p = p_1 + p_2 + p_3$ and $p^2 = w_0^2$; and the primes in p_1' and $\cos\theta'$ indicate that they are evaluated in the (12)-RF. But one sees that, in the (12)-RF,

(9)
$$w_{13}^2 = (p_1 + p_3)^2 = w_1^2 + w_3^2 + 2E_1'\, E_3' + 2p_3'\, p_1' \cos\theta'$$

so that, for a fixed w_{12},

(10)
$$dw_{13}^2 = 2p_3'\, p_1'\, d\cos\theta'$$



One needs to relate p'_3 to p_3. For the purpose, write $p = p_{12} + p_3$ evaluate the 4-momentum square in the (12)-RF

(11) $$2w_{12}\, E'_3 = (w_0^2 - w_{12}^2) - w_3^2$$

and square it again

(12) $$4w_{12}^2\, p_3'^2 = (w_0^2 - w_{12}^2)^2 + w_3^4 - 2(w_0^2 + w_{12}^2)w_3^2$$

Now, start with $p - p_3 = p_{12}$ and evaluate it in the overall RF

(13) $$2w_0\, E_3 = (w_0^2 - w_{12}^2) + w_3^2$$

and square it

(14) $$4w_0^2\, p_3^2 = (w_0^2 - w_{12}^2)^2 + w_3^4 - 2(w_0^2 + w_{12}^2)w_3^2$$

So one concludes

(15) $$p'_3 = \left(\frac{w_0}{w_{12}}\right) p_3$$

So the 3-body phase-space formula becomes

(16) $$d\phi_3(1,2,3) = \frac{4}{(4\pi)^5}\, dR(\alpha,\beta,\gamma) \left(\frac{dw_{13}^2\, dw_{12}^2}{4w_0^2}\right) \quad \rightarrow \quad \text{Daltix-plot variables !}$$

This can be recast into a simple form involving energies in the overal RF. For the purpose, note

(17) $$\begin{aligned} (p_{13} = p - p_2) \quad &\rightarrow \quad w_{13}^2 = w_0^2 + w_2^2 - 2w_0\, E_2 \\ (p_{12} = p - p_3) \quad &\rightarrow \quad w_{12}^2 = w_0^2 + w_3^2 - 2w_0\, E_3 \end{aligned}$$

where E_2 and E_3 are evaluated in the overall RF. Finally, one obtains a simple formula for the 3-body phase space

(18) $$d\phi_3(1,2,3) = \frac{4}{(4\pi)^5}\, dR(\alpha,\beta,\gamma)\, dE_2\, dE_3$$

Two-body Kinematics:

Start with $p = p_1 + p_2$ where $p^2 = w^2$, $p_1^2 = w_1^2$ and $p_2^2 = w_2^2$. Calculate energies for 1 and 2 in the (12)-RF:

$$(p - p_1 = p_2) \quad \rightarrow \quad 2w\,E_1 = w^2 + w_1^2 - w_2^2$$
$$(p - p_2 = p_1) \quad \rightarrow \quad 2w\,E_2 = w^2 + w_2^2 - w_1^2 \qquad (19)$$

Note this gives $w = E_1 + E_2$. Now take the squares and solve for p^2

$$\begin{aligned} 4w^2\,p^2 &= w^4 + w_1^4 + w_2^4 - 2\left[(w\,w_1)^2 + (w\,w_2)^2 + (w_1\,w_2)^2\right] \\ &= w^4 + (w_1 + w_2)^2 (w_1 - w_2)^2 - w^2\left[(w_1 + w_2)^2 + (w_1 - w_2)^2\right] \\ &= \left[w^2 - (w_1 + w_2)^2\right]\left[w^2 - (w_1 - w_2)^2\right] \\ &= (w + w_1 + w_2)(w - w_1 - w_2)(w + w_1 - w_2)(w - w_1 + w_2) \end{aligned} \qquad (20)$$

S-matrix for two-body reactions:

Let us denote a two-body reaction by

(21) $$a + b \to c + d$$

with $\vec{p}_a$, s_a, λ_a, and η_a standing for the momentum, spin, helicity, and the intrinsic parity of the particle a, etc. Let w_0 denote the centre-of-mass (c.m.) energy and let $\vec{p}_i(\vec{p}_f)$ be the c.m. momentum of the particle $a(c)$. The invariant S-matrix element for the reaction may be written, in the over-all c.m. system,

(22) $$\begin{aligned} \langle \vec{p}_c\lambda_c; \vec{p}_d\lambda_d|S|\vec{p}_a\lambda_a; \vec{p}_b\lambda_b\rangle &= \langle \vec{p}_f\lambda_c; -\vec{p}_f\lambda_d|S|\vec{p}_i\lambda_a; -\vec{p}_i\lambda_b\rangle \\ &= (4\pi)^2 \frac{w_0}{\sqrt{p_f p_i}} \langle \Omega\lambda_c\lambda_d|S|00\lambda_a\lambda_b\rangle \ , \end{aligned}$$

where we have used with the normalization constant given previously, and we have fixed the direction $\vec{p}_i$ at the spherical angles $(0, 0)$ and $\vec{p}_f$ at $\Omega = (\theta, \phi)$. Because of the invariant normalization of the one-particle states, the absolute square of the amplitude summed over the helicities λ_a, λ_b, etc., is a Lorentz invariant quantity. It is in this sense that formula above is referred to as the “invariant S matrix”. Due to the energy-momentum conservation, one may write

(23) $$\langle \Omega\lambda_c\lambda_d|S|00\lambda_a\lambda_b\rangle = (2\pi)^4\delta^{(4)}(p_c + p_d - p_a - p_b)\ \langle \Omega\lambda_c\lambda_d|S(w_0)|00\lambda_a\lambda_b\rangle \ .$$

If we define the T operator via $S = 1 + iT$, it is clear that we may write down the T-matrix in the same way, simply replacing S by T. Now, the invariant transition amplitude $\mathcal{M}_{fi}$ is defined from the T matrix by

(24) $$(2\pi)^4\delta^{(4)}(p_c + p_d - p_a - p_b)\,\mathcal{M}_{fi} = \langle p_c\lambda_c;\, p_d\lambda_d|T|p_a\lambda_a; p_b\lambda_b\rangle$$

or

(25) $$\mathcal{M}_{fi} = (4\pi)^2\frac{w_0}{\sqrt{p_f p_i}}\langle\Omega\lambda_c\lambda_d|T(w_0)|00\lambda_a\lambda_b\rangle\ .$$

The cross section for $a + b \to c + d$ can be cast into

(26) $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{p_f}{p_i}\left|\frac{\mathcal{M}_{fi}}{8\pi w_0}\right|^2$$

Let us now expand the transition amplitude in terms of the partial-wave amplitudes:

$$\begin{aligned}\langle\Omega\lambda_c\lambda_d|T(w_0)|00\lambda_a\lambda_b\rangle &= \sum_{JM}\langle\Omega\lambda_c\lambda_d|JM\lambda_c\lambda_d\rangle\langle JM\lambda_c\lambda_d|T(w_0)|JM\lambda_a\lambda_b\rangle \\ &\quad\times\langle JM\lambda_a\lambda_b|00\lambda_a\lambda_b\rangle \\ &= \frac{1}{4\pi}\sum_J(2J+1)\langle\lambda_c\lambda_d|T^J(w_0)|\lambda_a\lambda_b\rangle D^{J\,*}_{\lambda\lambda'}(\phi_0,\theta_0,0)\ ,\end{aligned} \qquad (27)$$

where $\lambda = \lambda_a - \lambda_b$ and $\lambda' = \lambda_c - \lambda_d$.

If we define the "scattering amplitude" $f(\Omega)$ via

(28) $$\frac{d\sigma}{d\Omega} = \frac{p_f}{p_i}\left|\frac{\mathcal{M}_{fi}}{8\pi w_0}\right|^2 \rightarrow \qquad \frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

we obtain

(29) $$f(\Omega) = \frac{(p_f/p_i)^{\frac{1}{2}}}{8\pi w_0}\mathcal{M}_{fi}\ .$$

This formula then relates to the "non-relativistic" scattering amplitude $f(\Omega)$ to the Lorentz-invariant transition amplitude $\mathcal{M}_{fi}$. One sees immediately that

(30) $$f(\Omega) = \frac{1}{p_i}\sum_J\left(J+\frac{1}{2}\right)\langle\lambda_c\lambda_d|T^J(w_0)|\lambda_a\lambda_b\rangle D^{J\,*}_{\lambda\lambda'}(\phi,\theta,0)\ .$$

The partial-wave T-matrix appearing is related to the partial-wave S-matrix by

(31) $$\langle\lambda_c\lambda_d|S^J(w_0)|\lambda_a\lambda_b\rangle = \delta_{fi}\delta_{\lambda_c\lambda_a}\delta_{\lambda_d\lambda_b} + i\langle\lambda_c\lambda_d|T^J(w)|\lambda_a\lambda_b\rangle\ ,$$

where $\delta_{fi} = 1$ for elastic scattering and zero, otherwise.

If parity is conserved in the process, it follows that the partial-wave amplitude should satisfy the following symmetry relation:

(32) $$\langle -\lambda_c - \lambda_d | S^J(w_0) | -\lambda_a - \lambda_b \rangle = \eta \, \langle \lambda_c \lambda_d | S^J(w_0) | \lambda_a \lambda_b \rangle \ ,$$

where

$$\eta = \frac{\eta_c \eta_d}{\eta_a \eta_b} (-)^{s_c + s_d - s_a - s_b} \ .$$

Next, we examine the consequences of time-reversal invariance. Let us denote by $|i\rangle$ and $|f\rangle$ the initial and final system in a scattering process. Then, the time-reversed process takes the initial state $|\boldsymbol{T} f\rangle$ into the final state $|\boldsymbol{T} i\rangle$, so that time-reversal invariance implies the following relation for the S-matrix:

(33) $$\langle f | S | i \rangle = \langle \boldsymbol{T} i | S | \boldsymbol{T} f \rangle \ .$$

One finds immediately

(34) $$\langle \lambda_c \lambda_d | S^J(w_0) | \lambda_a \lambda_b \rangle = \langle \lambda_a \lambda_b | S^J(w_0) | \lambda_c \lambda_d \rangle \ ,$$

where the right-hand side refers to the process $c + d \rightarrow a + b$.

Decay Width:

Decay rates for a state a with a 4-momentum p_a and mass w decaying into n particles of 4-momenta $p_1 \ldots p_n$, i.e. for a process $p_a \to p_1 + \cdots + p_n$ can be written

$$\mathrm{d}\Gamma = \frac{1}{2\,w}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\phi_n(1,2,\ldots,n) \tag{35}$$

For a two-body decay, i.e. $p_a \to p_1 + p_2$, one has

$$\mathrm{d}\phi_2(1,2) = \frac{1}{(4\pi)^2}\,\frac{p}{w}\mathrm{d}\Omega \;\to\; \frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{1}{2(4\pi)^2}\,|\mathcal{M}_{fi}|^2\left(\frac{p}{w^2}\right) \;\to\; w^{-2}\text{ dependence} \tag{36}$$

Consider a $\pi\pi$ elastic scattering in a partial wave ℓ. The invariant amplitude takes on the form

$$\mathcal{M}_{fi} \propto \frac{w}{p}(2\ell+1)\,P_\ell(\cos\theta)\,e^{i\delta_\ell}\,\sin\delta_\ell \tag{37}$$

$$\cot\delta_\ell = \frac{w_0^2 - w^2}{w_0\Gamma(w)} \tag{38}$$

so that

$$\mathcal{M}_{fi} \propto \frac{w}{p}\,(2\ell+1)\,P_\ell(\cos\theta)\,\frac{w_0\Gamma(w)}{w_0^2 - w^2 - i w_0\Gamma(w)} \qquad \text{from unitarity} \tag{39}$$

Let $\vec{p}_i$ and $\vec{p}_f$ be the initial and final CM momenta, i.e $|\vec{p}_i| = |\vec{p}_f| = p$ and $(\vec{p}_i \cdot \vec{p}_f) = p^2 \cos\theta$. And so, if one sets

(40) $$\Gamma \propto \frac{1}{w} p^{2\ell+1}$$

then one finds the simplest invariant amplitude

(41) $$\mathcal{M}_{fi} \propto (\vec{p}_i \cdot \vec{p}_f)^\ell$$

The width formula can be written

(42) $$\Gamma(w)|_{w=w_0} = \Gamma_0, \ \Gamma(w) = \Gamma_0 \left(\frac{w_0}{w}\right) \left(\frac{p}{p_0}\right) \left[\frac{F_\ell(p)}{F_\ell(p_0)}\right]^2 \to w^{-1} \text{ dependence !}$$

where $F_\ell(p)$ is the Blatt-Weisskopf barrier factor (BW Barrier Factor), given by

(43) $$F_0(p) = 1, \quad F_1(p) = \sqrt{\frac{z}{z+1}}, \quad F_2(p) = \sqrt{\frac{z^2}{(z-3)^2 + 9z}}$$

where $z = (p/p_r)^2$ and p_r is an additional 'scale' parameter in the problem which is presumably close to 0.1973 GeV/c corresponding to the length of 1 fermi. Note that one has adopted a normalization such that $F_\ell(p) = 1$ for $z \to \infty$.

BW barrier factor: F. von Hippel and C. Quigg, Phys. Rev. 5, 624 (1972)

Two-body decays:

Let us consider a resonance of spin-parity J^η and mass w (to be called the resonance J), decaying into a two-particle system with particles 1 and 2:

(44) $$J \to 1 + 2 \,,$$

and let $s_1(s_2)$ and $\eta_1(\eta_2)$ denote the spin and intrinsic parity of the particle 1(2). In the rest frame of the resonance J (JRF), let $\vec{p}$ be the momentum of the particle 1 with the spherical angles given by $\Omega = (\theta, \phi)$. Then, the amplitude A describing the decay of spin J with the z-component M into two particles with helicities λ_1 and λ_2 may be written

(45) $$\begin{aligned} A^J_{\lambda_1\lambda_2}(M;\Omega) =& \langle \vec{p}\lambda_1; -\vec{p}\lambda_2|\mathcal{M}|JM\rangle \\ =& 4\pi \left(\frac{w}{p}\right)^{\frac{1}{2}} \langle \phi\theta\lambda_1\lambda_2|JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle \\ A^J_{\lambda_1\lambda_2}(M;\Omega) =& N_J F^J_{\lambda_1\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0), \qquad \lambda = \lambda_1 - \lambda_2 \,, \end{aligned}$$

The “helicity decay amplitude” F is given by

(46) $$F^J_{\lambda_1\lambda_2} = 4\pi \left(\frac{w}{p}\right)^{\frac{1}{2}} \langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle \,.$$

Since $\mathcal{M}$ is a rotational invariant, the helicity amplitude F can depend only on the rotationally invariant quantities, namely, J, λ_1, and λ_2.

It is easy to expand the helicity decay amplitude F in terms of the partial-wave amplitudes. Using the recoupling coefficient, we may write

$$\begin{aligned}\langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle &= \sum_{\ell s}\langle JM\lambda_1\lambda_2|JM\ell s\rangle\langle JM\ell s|\mathcal{M}|JM\rangle \\ &= \sum_{\ell s}\left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}}(\ell 0\ s\lambda|J\lambda)(s_1\lambda_1\ s_2\ {-\lambda_2}|s\lambda)\langle JM\ell s|\mathcal{M}|JM\rangle\end{aligned}$$

so that F may be expressed

(47) $$F^J_{\lambda_1\lambda_2} = \sum_{\ell s}\left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a^J_{\ell s}\,(\ell 0\ s\lambda|J\lambda)\,(s_1\lambda_1\ s_2\ {-\lambda_2}|s\lambda)\ ,$$

where the partial-wave amplitude $a^J_{\ell s}$ is defined by

(48) $$a^J_{\ell s} = 4\pi\left(\frac{w}{p}\right)^{\frac{1}{2}}\langle JM\ell s|\mathcal{M}|JM\rangle\ .$$

The normalizations have a simple relationship

(49) $$\sum_{\lambda_1\ \lambda_2}|F^J_{\lambda_1\lambda_2}|^2 = \sum_{\ell\ s}|a^J_{\ell s}|^2$$

If parity is conserved in the decay, we have

$$F^J_{\lambda_1\lambda_2} = \eta\eta_1\eta_2(-)^{J-s_1-s_2} F^J_{-\lambda_1-\lambda_2} \, , \tag{50}$$

where η_1 and η_2 are the intrinsic parities of the particles 1 and 2. If the particles 1 and 2 are identical, we have to replace the state $|JM\lambda_1\lambda_2\rangle$ by its symmetrized state, so that we obtain the following symmetry relation:

$$F^J_{\lambda_1\lambda_2} = (-)^J F^J_{\lambda_2\lambda_1} \; . \tag{51}$$

Note that J= integer always!

It is possible to obtain a further symmetry relation on F by considering the time-reversal operations. For the purpose, let us consider the elastic scattering of particles 1 and 2 in the angular momentum state $|JM\lambda_1\lambda_2\rangle$, i.e.

$$\langle JM\lambda'_1\lambda'_2|T(w)|JM\lambda_1\lambda_2\rangle \equiv \langle\lambda'_1\lambda'_2|T^J(w)|\lambda_1\lambda_2\rangle \ , \tag{52}$$

where w is the c.m. energy and coincides with the effective mass of the resonance J. Now, we make the assumption that the $J^{\rm th}$ partial wave for the elastic scattering of particles 1 and 2 is completely dominated by the resonance at the c.m. energy w

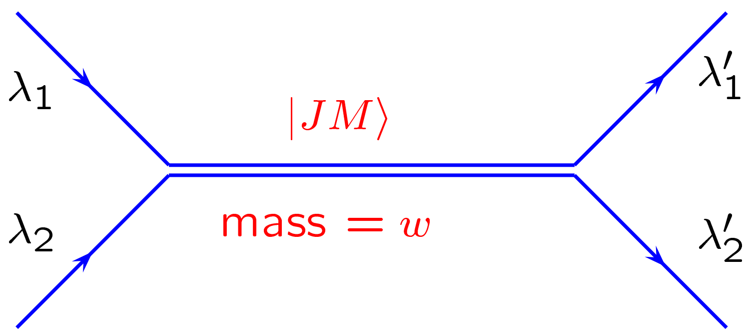


Fig. 2: Elastic scattering of particles 1 and 2, mediated by a resonance J in the s-channel.

Then, we may write

$$T(w) \sim \sum_M \mathcal{M}|JM\rangle D(w)\langle JM|\mathcal{M}^\dagger \ ,$$

where $D(w)$ is the Breit-Wigner function for the resonance and $\mathcal{M}$ is an appropriate "decay operator." We obtain

$$\langle\lambda_1'\lambda_2'|T^J(w)|\lambda_1\lambda_2\rangle \sim D(w) F^J_{\lambda_1'\lambda_2'} F^{J\,*}_{\lambda_1\lambda_2} \ ,$$

so that time-reversal invariance for elastic scattering implies

$$\langle\lambda_1'\lambda_2'|T^J(w)|\lambda_1\lambda_2\rangle = \langle\lambda_1\lambda_2|T^J(w)|\lambda_1'\lambda_2'\rangle \ \to F^J_{\lambda_1'\lambda_2'} F^{J\,*}_{\lambda_1\lambda_2} = F^J_{\lambda_1\lambda_2} F^{J\,*}_{\lambda_1'\lambda_2'} \ . \tag{53}$$

This means that the phase of the complex amplitude F does not depend on the helicities λ_1 and λ_2. Therefore, we can consider F a real quantity without loss of generality:

$$F^J_{\lambda_1\lambda_2} = \text{real} \ . \tag{54}$$

We emphasize that this result follows only from the assumption that the $J^{\rm th}$ partial wave is dominated by the resonance J at the energy w. This condition is fulfilled, for example, in the P-wave amplitudes of the $\pi^+\pi^-$ or $p\pi^+$ elastic scattering at the c.m. energies corresponding to ρ^0 and $\Delta(1232)$ masses, where it is known that these resonances saturate the unitarity limit. It is clear, however, that this condition may not be satisfied for all resonances. In this sense, the reality constraint may be considered only an "approximate" symmetry. We will show later in the discussion of the sequential decay modes that this symmetry can actually be tested experimentally.

Three-Particle Systems:

A system consisting of three particles may be treated most elegantly in the helicity basis, as was done by

M. Berman and M. Jacob, Phys. Rev. 139 B, 1023 (1965).

In this section, we shall first construct a three-particle system in a definite angular momentum state and then apply the formalism to a case of a resonance decaying into three particles. We will give the decay angular distribution in terms of the spin density matrix and discuss the implications of parity conservation. Finally, we will show that in a Dalitz-plot analysis different spin-parity states of the three-particle system do not interfere with one another.

Consider a system of three particles 1, 2, and 3. Let us use the notations s_i, η_i, λ_i, and w_i for the spin, intrinsic parity, helicity, and mass of the particle i. In the rest frame (RF) of the three particles, the momentum and energy of the particle i will be denoted by $\vec{p}_i$ and E_i. In the RF, we define the "standard orientation" of the three-particle system, as shown in Fig. 3. this coordinate system is then the "body-fixed" coordinate system, which may be rotated by the Euler angles α, β, and γ to obtain a system with arbitrary orientation.

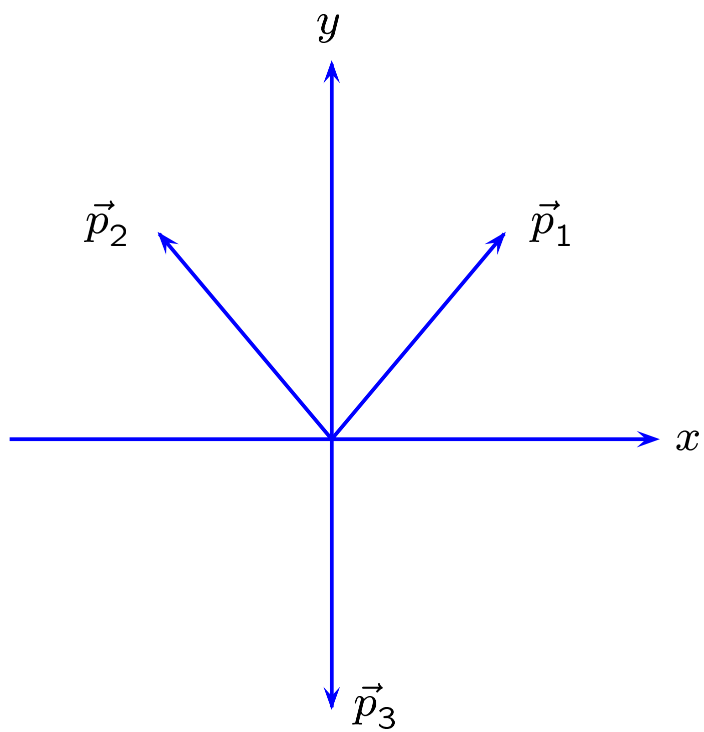


Fig. 3: Standard orientation of the three-particle rest system. Note that the y-axis is defined along the negative direction of $\vec{p}_3$, and the z-axis along $\vec{p}_1 \times \vec{p}_2$.

A system with the standard orientation can be written

$$|000, E_i\lambda_i\rangle = b\prod_{i=1}^{3}|\vec{p}_i s_i \lambda_i\rangle \ , \tag{55}$$

where b is a normalization constant and the helicity basis vectors for each individual particle are given in the usual way:

$$|\vec{p}s_i\lambda_i\rangle = U[R_i L_z(p_i)]\ s_i\lambda_i\rangle, \qquad R_i = R(\phi_i, \pi/2, 0)\ . \tag{56}$$

A three-particle system with an arbitrary orientation in the RF can now be obtained by applying a rotation $R(\alpha,\beta,\gamma)$ to the state:

$$|\alpha\beta\gamma, E_i\lambda_i\rangle = U[R(\alpha,\beta,\gamma)]\,|000, E_i\lambda_i\rangle\ . \tag{57}$$

If we impose the normalization of the above states via

$$\langle\alpha'\beta'\gamma', E_i'\lambda_i'|\alpha\beta\gamma, E_i\lambda_i\rangle = \delta^{(3)}(R'-R)\,\delta(E_1'-E_1)\,\delta(E_2'-E_2)\prod_i \delta_{\lambda_i\lambda_i'} \tag{58}$$

we obtain easily (see Appendix C) that the normalization constant b should be chosen as follows:

$$b^{-1} = 8\pi^2\sqrt{4\pi}\ . \tag{59}$$

Let us now define a state of definite angular momentum:

(60) $$|JM\mu, E_i\lambda_i\rangle = \frac{N_J}{\sqrt{2\pi}} \int \mathrm{d}R\, D^{J\,*}_{M\mu}(\alpha,\beta,\gamma)|\alpha\beta\gamma, E_i\lambda_i\rangle\ ,$$

where N_J is the normalization constant given before. That this state represents a state of definite angular momentum is easy to show following steps identical to those for two-body decays. Therefore, the states above transform under a rotation R' according to

(61) $$U[R']\,|JM\mu, E_i\lambda_i\rangle = \sum_{M'} D^J_{M'M}(R')|JM'\mu, E_i\lambda_i\rangle\ .$$

This relation also shows that, in addition to the obvious invariants E_i and λ_i, the quantity μ is also a rotational invariant.

Let us examine the transformation property of the states under parity operations

$$\Pi|000, E_i\lambda_i\rangle = b\prod_i \Pi|R_i, p_i, s_i\lambda_i\rangle$$
$$= b\prod_i \eta_i e^{-i\pi s_i}|\bar{R}_i, p_i, s_i - \lambda_i\rangle$$
(62)
$$= \left\{\prod_i \eta_i e^{-i\pi s_i}\right\} U[R(\pi, 0, 0)]\,|000, E_i - \lambda_i\rangle \ ,$$

where $\bar{R}_i = R(\pi + \phi_i, \pi/2, 0) = R(\pi, 0, 0)R_i$, so that

(63)
$$\Pi|\alpha\beta\gamma, E_i\lambda_i\rangle = \left\{\prod_i \eta_i e^{-i\pi s_i}\right\} U[R(\alpha, \beta, \gamma + \pi)]\,|000, E_i - \lambda_i\rangle \ .$$

Changing the integration over γ into one over $\gamma' = \gamma + \pi$, we obtain finally

(64)
$$\Pi|JM\mu, E_i\lambda_i\rangle = \eta_1\eta_2\eta_3(-)^{s_1+s_2+s_3+\mu}|JM\mu, E_i - \lambda_i\rangle \ .$$

We note that this formula is not the same as that given in Berman and Jacob. The reason for this is that their definition of one-particle helicity states involves a rotation $R(\phi, \theta, -\phi)$, instead of our convention $R(\phi, \theta, 0)$.

In order to treat the case when two of the three particles are identical, we shall work out a transformation formula for exchanging the particles 1 and 2. The exchange operator $\mathbb{P}_{12}$ is equivalent to performing a rotation by π around the body-fixed y-axis (see Fig. 3). Using an identity

(65) $$R(\pi+\alpha,\pi-\beta,\pi-\gamma)=R(\alpha,\beta,\gamma)R(0,\pi,0)$$

we obtain

(66) $$\mathbb{P}_{12}|\alpha\beta\gamma,E_1\lambda_1,E_2\lambda_2,E_3\lambda_3\rangle=|\pi+\alpha,\pi-\beta,\pi-\gamma,E_2\lambda_2,E_1\lambda_1,E_3\lambda_3\rangle\ ,$$

Combining this formula with the defining formula for $|JM\mu,E_i\lambda_i\rangle$, and using an identity

(67) $$D^j_{m'm}(\pi+\alpha,\pi-\beta,\pi-\gamma)=(-)^{j-m}D^j_{m'-m}(\alpha,\beta,\gamma)$$

we find

(68) $$\mathbb{P}_{12}|JM\mu,E_1\lambda_1,E_2\lambda_2,E_3\lambda_3\rangle=(-)^{J+\mu}|JM\ -\mu,E_2\lambda_2,E_1\lambda_1,E_3\lambda_3\rangle\ .$$

Again, this formula is not the same as that given in Berman and Jacob. This arises because their standard orientation for the three-particle system has been defined differently from our convention; their coordinate system has been set up with the negative x-axis along the momentum $\vec{p}_3$.

Our angular momentum states are normalized according to

(69) $$\langle J'M'mu'E_i'\lambda_i'|JM\mu E_i\lambda_i\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\mu\mu'}\delta(E_1 - E_1)\delta(E_2 - E_2)\prod_i \delta_{\lambda_i\lambda_i'} \ .$$

The completeness relation is given by

(70) $$\sum_{\substack{JM \\ \mu\lambda_i}} \int |JM\mu E_i\lambda_i\rangle \mathrm{d}E_1 \mathrm{d}E_2 \langle JM\mu E_i\lambda_i| = I \ .$$

The recoupling matrix element

(71) $$\langle \alpha\beta\gamma, E_i'\lambda_i'|JM\mu, E_i\lambda_i\rangle = \frac{N_J}{\sqrt{2\pi}} D^{J\,*}_{M\mu}(\alpha,\beta,\gamma)\delta(e_1' - E_1)\delta E_2' - E_2)\prod_1 \delta_{\lambda_i\lambda_i'} \ .$$

We are now ready to discuss the process in which a resonance J with spin-parity η and mass w decays into three particles 1, 2, and 3. In the rest frame of the resonance (JRF), let the angles (α, β, γ) describe the orientation of the three-particle system. Then, the decay amplitude may be written, with $R(\alpha, \beta, \gamma)$,

(72)
$$A^J_{\mu\,\lambda_i}(M;R) = \langle R, E_i\lambda_i|\mathcal{M}|JM\rangle = \langle R, E_i\lambda_i|JM\mu E_i\lambda_i\rangle\langle JM\mu E_i\lambda_i|\mathcal{M}|JM\rangle$$
$$A^J_{\mu\,\lambda_i}(M;R) = \frac{N_J}{\sqrt{2\pi}} F^J_\mu(E_i\lambda_i) D^{J\,*}_{M\mu}(R)$$

If the "decay operator" $\mathcal{M}$ is rotationally invariant, the decay amplitude F should depend only on the rotational invariants, i.e.

(73)
$$F^J_\mu(E_i\lambda_i) = \langle JM\mu E_i\lambda_i|\mathcal{M}|JM\rangle\ .$$

If parity is conserved in the decay, we have the symmetry:

(74)
$$F^J_\mu(E_i\lambda_i) = \eta\eta_1\eta_2\eta_3(-)^{s_1+s_2+s_3+\mu} F^j_\mu(E_i - \lambda_i)\ .$$

And, if particles 1 and 2 are identical,

(75)
$$F^J_\mu(E_1\lambda_1, E_2\lambda_2, E_3\lambda_3) = \pm(-)^{J+\mu} F^J_{-\mu}(E_2\lambda_2, E_1\lambda_1, E_3\lambda_3)\ ,$$

where the plus sign holds for two identical bosons and the minus sign for fermions.

The overall amplitude for the production and decay of a three-particle system is, for $R(\alpha, \beta, \gamma)$,

$$\sum_{JM\eta} P(JM\eta)\, F^{J\eta}_{\mu}(E_i\lambda_i)\, D^{J\,*}_{M\mu}(R)$$

where $P(JM\eta)$ is the production amplitude of state $|JM\eta\rangle$ in which η is its intrinsic parity. The distribution function takes on the form

$$I(R, E_i) \propto \sum_{\lambda_i} \sum_{\substack{JM\eta\mu \\ J'M'\eta'\mu'}} \tag{76}$$

$$P(JM\eta)\, P^*(J'M'\eta')\, F^{J\eta}_{\mu}(E_i\lambda_i)\, F^{J'\eta'\,*}_{\mu'}(E_i\lambda_i)\, D^{J\,*}_{M\mu}(R)\, D^{J'}_{M'\mu'}(R)$$

Integrating over $dR(\alpha, \beta, \gamma)$, we obtain

$$\int I(R, E_i)\, \mathrm{d}R \propto \sum_{\lambda_i} \sum_{JM\mu} \sum_{\eta\,\eta'} P(JM\eta)\, P^*(JM\eta')\, F^{J\eta}_{\mu}(E_i\lambda_i)\, F^{J\eta'\,*}_{\mu}(E_i\lambda_i)$$

$$\text{(parity in decay)} \rightarrow \propto \eta\eta' \sum_{\lambda_i} \sum_{JM\mu} \sum_{\eta\,\eta'} P(JM\eta)\, P^*(JM\eta')\, F^{J\eta}_{\mu}(E_i\lambda_i)\, F^{J\eta'\,*}_{\mu}(E_i\lambda_i)$$

Conclude: No interference between different J or η in a Dalitz-plot analysis.

Decay Modes: Examples

Recapitulate:

Within the helicity formalism, we have seen

(77) $$A^J_{\lambda_1\lambda_2}(M;\Omega) = N_J F^J_{\lambda_1\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0)\ , \quad N_J = \sqrt{\frac{2J+1}{4\pi}}\ , \quad \lambda = \lambda_1 - \lambda_2\ ,$$

where

$$F^J_{\lambda_1\lambda_2} = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a^J_{\ell s}\,(\ell 0\ s\lambda|J\lambda)\,(s_1\lambda_1\ s_2\ {-\lambda_2}|s\lambda)\ ,$$

Symmetry properties

$$F^J_{\lambda_1\lambda_2} = \eta\,\eta_1\eta_2(-)^{J-s_1-s_2}F^J_{-\lambda_1\,-\lambda_2}\ \text{(parity)}\ , \qquad F^J_{\lambda_1\lambda_2} = (-)^J F^J_{\lambda_2\lambda_1}\ \text{(identical)}$$

Obtain their counterparts for the ℓs amplitude. Starting from

(78) $$a^J_{\ell s} = 4\pi\left(\frac{w}{p}\right)^{\frac{1}{2}}\langle JM\ell s|\mathcal{M}|JM\rangle$$

we see that

$$\eta = \eta_1\eta_2(-)^\ell\ \text{(parity)}, \qquad \ell + s - 2s_1 = \text{even}\ \text{(identical)}$$

Two-pion Decays:

Two famous examples are $\rho \to \pi\pi$ and $f_2(1270) \to \pi\pi$. Let ℓ be the spin. Again, we use η to denote the intrinsic parity of the resonance decaying into two pions. The decay amplitude is

$$A^{\ell}(m;\Omega) \propto F^{\ell}\, D^{\ell\,*}_{m\,0}(\phi,\theta,0) \propto F^{\ell}\, Y^{\ell}_{m}(\Omega) \tag{79}$$

From parity conservation in the decay, we must have $F^{\ell} = \eta\,(-)^{\ell}\,F^{\ell}$, so that

$$\eta\,(-)^{\ell} = +1 \tag{80}$$

If the π's are identical (i.e. the same charge), then we must $F^{\ell} = (-)^{\ell}\,F^{\ell}$. So we see that

$$\ell = \text{even} \tag{81}$$

Examples: $\rho \not\to \pi^0\pi^0$ and $f_2(1270) \to \pi^0\pi^0$.

We consider the decays $\omega \to \pi^+\pi^0\pi^-$ and $a_2(1320) \to 3\pi$. The decay amplitude is

(82) $$A^J_\mu(M;R) \propto F^J_\mu(E_i)\, D^{J\,*}_{M\,\mu}(R)$$

From parity conservation, we see that $F^J_\mu(E_i) = \eta\,(-)^{\mu+1}\, F^J_\mu(E_i)$, so that

(83) $$\eta\,(-)^\mu = -1$$

So we conclude that $\mu =$ even if $\eta = -1$ and $\mu =$ odd if $\eta = +1$. This shows that there is one amplitude ($\mu = 0$) for the ω decay

(84) $$\omega \to 3\pi; \quad A^J_0(M;R) \propto F^J_0(E_i)\, D^{J\,*}_{M\,0}(R), \quad J = 1$$
$$\text{Integrate over } \gamma; \quad A^J_0(M;\Omega) \propto F^J_0(E_i)\, D^{J\,*}_{M\,0}(\phi,\theta,0) \propto F^J_0(E_i)\, Y^J_M(\Omega)$$

So $\omega \to 3\pi$ decay is ‘formally’ equivalent to $\rho \to 2\pi$, if the analyzer is the decay normal. There are two decay amplitudes ($\mu = \pm$) for $a_2(1320) \to 3\pi$.

(85) $$a_2(1320) \to 3\pi; \quad A^J_\pm(M;R) \propto F^J_\pm(E_i)\, D^{J\,*}_{M\,\pm}(R), \quad J = 2$$

Consider now the decays of the charged a_2, i.e. $a_2^\pm(1320) \to \pi^\pm\pi^+\pi^-$. Then we must have $F^J_\pm(E_1,E_2) = -F^J_\mp(E_2,E_1)$, i.e. the π’s 1 and 2 are identical.

Decays into $\rho\pi$ and $\omega\pi$:

Let λ be the helicity of the ρ or ω. The decay amplitude takes on the form

(86)
$$A^J_\lambda(M;\Omega) \propto F^J_\lambda D^{J\,*}_{M\lambda}(\phi,\theta,0)$$
$$F^J_\lambda = \sum_\ell \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a^J_\ell\,(\ell 0\; s\lambda|J\lambda), \qquad s=1$$

(a) Consider the decay $a_2(1320) \to \rho\pi$.

Because of parity conservation, we must have $\ell = 2$ (D wave) only. For this case, we obtain

(87)
$$F^J_\lambda = \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a^J_\ell\,(\ell 0\; s\lambda|J\lambda); \qquad J=\ell=2, \quad s=1$$

There is one single complex amplitude a^J_ℓ ($J=\ell=2$) in the problem:

(88)
$$F^J_+ = -\sqrt{\frac{1}{2}}a^J_\ell, \quad F^J_0 = 0, \quad F^J_- = \sqrt{\frac{1}{2}}a^J_\ell, \qquad J=\ell=2$$

Zemach amplitudes $\to a^J_\ell \propto p^\ell \implies a^J_\ell \propto F_\ell(p/p_R)$ (Blatt-Weisskopf barrier factors). Here p is the ρ momentum in the a_2RF.

Include the decay of the $\rho \to \pi_1 + \pi_2$. Then the overall amplitude is

(89) $$A^J(M;\Omega;w,\Omega_h) \propto \sum_\lambda F^J_\lambda D^{J\,*}_{M\lambda}(\phi,\theta,0)\,\Delta(w)\,D^{J\,*}_{\lambda\,0}(\phi_h,\theta_h,0);\ J=2,\ |\lambda|\le 1$$

where $\Omega = (\theta,\phi)$ describes the direction of the ρ momentum in the JRF, and $\Omega_h = (\theta_h,\phi_h)$ describes the direction of the momentum π_1 in the ρ helicity frame. The ρ decay amplitude (a complex constant) has been abosrbed into F^J_λ. w is the effective mass for the ρ and $\Delta(w)$ is the Breit-Wigner form

(90) $$\Delta(w) = \frac{w_0\,\Gamma_0}{w_0^2 - w^2 - i\,w_0\,\Gamma(w)}, \quad \Gamma(w) = \Gamma_0\left(\frac{w_0}{w}\right)\left(\frac{q}{q_0}\right)\left(\frac{q}{q_0}\right)^2$$

where q is the breakup momentum of the ρ in the ρRF.

(b) Consider now the decay $b_1(1235) \to \omega\pi$.

Because of parity conservation, we must have $\ell = 0$ (S wave) or $\ell = 2$ (D wave). So we see that

(91) $$F^J_\lambda = \sqrt{\frac{1}{3}}\,a^J_0 + \sqrt{\frac{5}{3}}\,a^J_2\,(20\;s\lambda|J\lambda), \qquad J = s = 1$$

Evaluating the Clebsch-Gordan coefficients, we find

$$\left\{\begin{aligned} F_+^J &= \sqrt{\frac{1}{3}}\,a_0^J + \sqrt{\frac{1}{6}}\,a_2^J \\ F_0^J &= \sqrt{\frac{1}{3}}\,a_0^J - \sqrt{\frac{2}{3}}\,a_2^J \\ F_-^J &= \sqrt{\frac{1}{3}}\,a_0^J + \sqrt{\frac{1}{6}}\,a_2^J \end{aligned}\right. \tag{92}$$

Note that $\sum_\lambda |F_\lambda^J|^2 = \sum_\ell |a_\ell^J|^2$ in which the interference terms cancel out.

Include the decay of the $\omega \to \pi_1 + \pi_2 + \pi_3$. Then the overall amplitude is

$$A^J(M;\Omega,\Omega_h) \propto \sum_\lambda F_\lambda^J\, D_{M\lambda}^{J\,*}(\phi,\theta,0)\, D_{\lambda\,0}^{s\,*}(\phi_h,\theta_h,0), \qquad J = s = 1,\ |\lambda| \le 1 \tag{93}$$

where $\Omega = (\theta,\phi)$ describes the direction of the momentum ω in the JRF, and $\Omega_h = (\theta_h,\phi_h)$ describes the direction of the momentum $\pi_1 \times \pi_2$ in the ω helicity frame. The ω decay amplitude can be integrated over

$$g^2 = \int |F_0^s(E_i)|^2\, \mathrm{d}E_1\, \mathrm{d}E_2, \quad s = 1 \text{ for } \omega$$

g has been aborbed into the F_λ^J.

Decay Modes involving γ's in the final states:

(a) Consider the decays $\omega \to \gamma + \pi^0$ and $f_1(1285) \to \gamma + \rho^0$.

The relevant decay amplitude is

(94) $$A^J_{\lambda_1\lambda_2}(M;\Omega) \propto F^J_{\lambda_1\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0), \qquad \lambda = \lambda_1 - \lambda_2, \quad \lambda_1 = \pm 1, \quad |\lambda| \leq J$$

Observe

$$\omega \to \gamma + \pi^0; \quad A^J_{\pm}(M;\Omega) \propto F^J_{\pm} D^{J\,*}_{M\,\pm}(\phi,\theta,0); \quad F^J_{\pm} = -F^J_{\mp}, \quad J = 1$$

where $s_1 = 1$ and $\gamma_1 = \pm 1$ for γ. There is one non-zero helicity-coupling amplitude, i.e. F^J_{+}. And further note

$$f_1(1285) \to \gamma + \rho^0; \quad A^J_{\pm,\,\lambda_2}(M;\Omega) \propto F^J_{\pm,\,\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0); \quad J = 1,\ \lambda = \pm 1 - \lambda_2$$

where $s_1 = 1$ and $\gamma_1 = \pm 1$ for γ; and $s_2 = 1$ and $\gamma_2 = \{-1,\,0,\,+1\}$ for the ρ^0. From parity conservation in the decay, one must have

(95) $$F^J_{\pm,\,\lambda_2} = -F^J_{\mp,\,-\lambda_2}$$

There are two non-zero amplitudes, F^J_{++} and F^J_{+0}.

(b) Consider the decays $\pi^0 \to \gamma + \gamma$ or $a_0(980) \to \gamma + \gamma$.

Once again, we start with

(96) $$A^J_{\lambda_1\lambda_2}(M;\Omega) \propto F^J_{\lambda_1\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0), \qquad \lambda = \lambda_1 - \lambda_2, \quad |\lambda| \le J$$

where $s_1 = s_2 = 1$ and $\lambda_1 = \pm 1$ and $\lambda_2 = \pm 1$. From parity conservation, we have

(97) $$F^J_{\lambda_1\,\lambda_2} = \eta\,(-)^J\,F^J_{-\lambda_1\,-\lambda_2}$$

By Bose symmetry, we should also have

(98) $$F^J_{\lambda_1\,\lambda_2} = (-)^J\,F^J_{\lambda_2\,\lambda_1}$$

There is one non-zero element F^J_{++} for π^0 or $a_0(980)$.

Suppose $J = 1$. There is again one non-zero element F^J_{++} but $F^J_{++} = 0$ by Bose symmetry. So, spin-one particles cannot decay into two photons—Landau-Yang Theorem.

Consider now $J = 2$. There are two non-zero elements F^J_{++} and F^J_{+-} if $\eta = +1$, but there exist only one element F^J_{++} if $\eta = -1$.

Density Matrix

Here we derive the symmetry properties of the general spin-density matrix for a system c produced in

$$a + b \rightarrow c + d \tag{99}$$

where the participating particles are arbitrary and include mesons, baryons as well as photons. We assume that c is an intermediate state, mesonic or baryonic, and it couples to any number of allowed decay channels.

The density matrices in the reflectivity basis were given by

S. U. Chung and T. L. Trueman, Phys. Rev. D 11, 633 (1975).

For a deeper understanding of the quantum treatment of one-particle states, the reader is referred to the book by

Steven Weinberg, '*The Quantum Theory of Fields*,' Volume I (Cambridge University Press, Cambridge, 1995), Chapter 2.

General Angular Distributions in Reflectivity Basis:

Let $|jm\rangle$ be the spin state[a] for c where the quantization axis is defined in the production plane, i.e. one takes either the helicity or the Jackson frame for c. The amplitude for production and decay of the c is

$$A \propto \sum_{\chi m} \langle \vec{p}_c\, \chi m,\, \vec{p}_d \lambda_d \,|T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle D^{\chi}_{m}(\tau) \tag{100}$$

where χ specifies the complete quantum state for c, which includes its spin j, its parity, its C-parity, its isotopic spin, and its decay products with the phase-space element given by τ; λ's refer to the helicities; T is the transition operator of the process $ab \to cd$; and D is the decay amplitude for c, which may consist of a product of the 'rotation functions' as well as the Breit-Wigner forms. The distribution function follows immediately

$$\begin{aligned} I(\tau) \propto & \sum_{\lambda_a \lambda_b \lambda_d} \quad \sum_{\chi m\ \chi' m'} \\ & \times \langle \vec{p}_c\, \chi m,\, \vec{p}_d \lambda_d \,|T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle \langle \vec{p}_c\, \chi' m',\, \vec{p}_d \lambda_d \,|T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle^* \\ & \times D^{\chi}_{m}(\tau)\, D^{\chi' *}_{m'}(\tau) \end{aligned} \tag{101}$$

[a] We use j to denote the spin and NOT J as used in the previous sections.

Note that the helicities of a, b and d are the 'external' unobserved variables and therefore summed over outside of the absolute square of the amplitude A. In terms of the generalized spin-density matrix

(102) $$\rho^{\chi\,\chi'}_{mm'} \propto \sum_{\lambda_a \lambda_b \lambda_d} \langle \vec{p}_c\,\chi m,\, \vec{p}_d \lambda_d\,|T|\,\vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle \langle \vec{p}_c\,\chi' m',\, \vec{p}_d \lambda_d\,|T|\,\vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle^*$$

the distribution function assumes an elegant form

(103) $$I(\tau) \propto \sum_{\substack{\chi m \\ \chi' m'}} \rho^{\chi\,\chi'}_{mm'}\, D^{\chi}_{m}(\tau)\, D^{\chi'\,*}_{m'}(\tau)$$

We are now ready to introduce the reflection operator through the production plane for $ab \to cd$. Let this plane be defined to be the x-z plane, i.e. the production normal is along the y-axis. Then the reflection operator defined by

$$\Pi_y = U[R_y(\pi)]\,\Pi = \Pi\,U[R_y(\pi)] \tag{104}$$

where $U[R_y(\pi)]$ is a unitary operator representing a rotation by π around the y-axis, i.e.

$$U[R_y(\pi)] = \exp(-i\pi\,J_y) \tag{105}$$

which is simply given by the standard d-function

$$U_{m'\,m}[R_y(\pi)] = d^j_{m'\,m}(\pi) = (-)^{j-m}\,\delta_{m',\,-m} \tag{106}$$

Note also that

$$U_{m'\,m}[R_y(-\pi)] = d^j_{m'\,m}(-\pi) = (-)^{j+m}\,\delta_{m',\,-m} \tag{107}$$

Let Λ is a general operator in the xz-plane. Then, we see that

$$\Big[\Pi_y, U[\Lambda(\vec{p}_c)]\Big] = 0, \quad \Pi_y\,|\vec{p}_c\,\chi\,m\rangle = \eta_c\,(-)^{j-m}\,|\vec{p}_c\,\chi\,-m\rangle, \qquad \Pi_y^2 = (-)^{2j}\,I \tag{108}$$

where η_c is the intrinsic parity of the c. Also true for helicity states ($|\vec{p}_i\,\lambda_i\rangle$, $i = a, b, d$). Also true for massless particles ($m \to \lambda = \pm j$).

We move over to the reflection-basis states for c, i.e.

(109) $$|\vec{p}_c\,\epsilon\,\chi\,m\rangle = \theta(m)\left\{|\vec{p}_c\,\chi\,m\rangle + \epsilon\,\eta_c\,(-)^{j-m}\,|\vec{p}_c\,\chi\,-m\rangle\right\}$$

where

(110) $$\theta(m) = \frac{1}{\sqrt{2}},\; m > 0; \qquad \theta(m) = \frac{1}{2},\; m = 0; \qquad \theta(m) = 0,\; m < 0$$

These basis states constitute eigenvectors of the reflection operator

(111) $$\epsilon^2 = (-)^{2j} \quad \rightarrow \quad \Pi_y\,|\vec{p}_c\,\epsilon\,\chi\,m\rangle = \epsilon\,(-)^{2j}\,|\vec{p}_c\,\epsilon\,\chi\,m\rangle$$

so that we have $\epsilon = \pm 1$ for bosons and $\epsilon = \pm i$ for fermions. Note, in addition, that $\epsilon\epsilon^* = |\epsilon|^2 = 1$ for both bosons and fermions.

The generalized density matrix in the reflectivity basis is, with $m \geq 0$ and $m' \geq 0$,

(112) $$\begin{aligned} &{}^{\epsilon\,\epsilon'}\rho^{\chi\,\chi'}_{mm'} \propto \\ &\sum_{\lambda_a\lambda_b\lambda_d} \langle\epsilon\,\vec{p}_c\,\chi m,\,\vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\,\vec{p}_b\lambda_b\rangle\langle\epsilon'\,\vec{p}_c\,\chi' m',\,\vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\,\vec{p}_b\lambda_b\rangle^* \end{aligned}$$

We shall explore the consequences of a reflection operation applied to the transition matrices above. The key observation is that Π_y leaves the transition operator T unperturbed, i.e. $[\Pi_y, T] = 0$. The three-vectors $\vec{p}_i$ ($i = a, b, c, d$) are left unchanged under Π_y, i.e. the boost operators and/or the rotations about the y-axis, which enter in the definitions of the helicity or the Jackson frames, remain invariant under Π_y. Therefore, the parity conservation is relegated to exploring the consequences of Π_y acting on the 'rest' states. Insert ${\Pi_y}^{-1}\,\Pi_y = {\Pi_y}^{\dagger}\,\Pi_y = I$ next to each T and propagate $\Pi_y^{\dagger}$ and Π_y backwards and forwards, respectively, to find

$$ {}^{\epsilon\,\epsilon'}\rho^{\chi\,\chi'}_{mm'} \propto \epsilon\epsilon'^{\,*}\,(-)^{2(j-j')} \sum_{\lambda_a\lambda_b\lambda_d} \tag{113}$$

$$\langle \epsilon\,\vec{p}_c\,\chi m,\, \vec{p}_d, -\lambda_d\,|T|\,\vec{p}_a, -\lambda_a,\, \vec{p}_b, -\lambda_b\rangle\,\langle \epsilon'\,\vec{p}_c\,\chi' m',\, \vec{p}_d, -\lambda_d\,|T|\,\vec{p}_a, -\lambda_a,\, \vec{p}_b, -\lambda_b\rangle^*$$

so that

$$ {}^{\epsilon\,\epsilon'}\rho^{\chi\,\chi'}_{mm'} = \epsilon\,\epsilon'^{\,*} \times {}^{\epsilon\,\epsilon'}\rho^{\chi\,\chi'}_{mm'} \tag{114}$$

So we see that $\epsilon\,\epsilon'^{\,*} = +1$. Multiply it by ϵ' from the right and noting that $\epsilon'\,\epsilon'^{\,*} = |\epsilon'|^2 = +1$, we find $\epsilon = \epsilon'$. Here we have carefully handled the derivation, so that the formula above applies to both bosons and fermions.

The density matrix can be written, quite generally, in a block-diagonal form

$$ {}^{\epsilon}\rho^{\chi\,\chi'}_{mm'} \propto \sum_{\lambda_a\lambda_b\lambda_d} \langle \epsilon\,\vec{p}_c\,\chi m,\,\vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\,\vec{p}_b\lambda_b\rangle\langle \epsilon\,\vec{p}_c\,\chi' m',\,\vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\,\vec{p}_b\lambda_b\rangle^* \tag{115} $$

a fundamental formula which incorporates parity conservation in the production process $ab \to cd$. The distribution function in the reflectivity basis is

$$ I(\tau) \propto \sum_{\epsilon}^{2} \sum_{\substack{\chi m \\ \chi' m'}} {}^{\epsilon}\rho^{\chi\,\chi'}_{mm'}\;{}^{\epsilon}D^{\chi}_{m}(\tau)\;{}^{\epsilon}D^{\chi'\,*}_{m'}(\tau), \qquad m \geq 0,\; m' \geq 0 \tag{116} $$

where ${}^{\epsilon}D$ is the decay amplitude in the reflectivity basis. Consider a simple decay

$$ c \to s_1(\lambda_1) + s_2(\lambda_2) $$

In the c RF, we have, with $\vec{p}_c = 0$ and $\tau = R(\phi,\theta,0)$,

$$ \begin{aligned} {}^{\epsilon}D^{\chi}_{m}(\tau) &= \langle \vec{q}\lambda_1; -\vec{q}\lambda_2|\mathcal{M}|\epsilon\, jm\rangle \\ &= N_j\,F^{j}_{\lambda_1\,\lambda_2}\,\theta(m)\left\{D^{j\,*}_{m\lambda}(R) + \epsilon\,\eta_c\,(-)^{j-m}\,D^{j\,*}_{-m\lambda}(R)\right\}, \quad \lambda = \lambda_1 - \lambda_2 \end{aligned} \tag{117} $$

The rank of the density matrix is determined by the number of independent terms in the summation on helicities. Let

$$n_i = 2s_i + 1 \quad \text{or} \quad n_i = 2\,(\text{photons}) \quad \text{for} \quad i = a,\ b,\ d$$

depending on whether a particle is massive or massless. The total number in the sum is

$$N = n_a\, n_b\, n_d \tag{118}$$

So the rank of the density matrix is $(N+1)/2$ if N is odd, and it is $N/2$ if N is even. Note that the reduction in the rank comes from parity conservation in the production process (to show this, apply Π_y again to the amplitudes).

Table I. Rank of Spin-Density Matrix for X

Reaction	Rank
$\pi^- p \to \pi^- X^+$	1
$\pi^- p \to X^- p$	2
$\pi^+ n \to X^- \Delta^{++}$	4
$\bar{p} p \to X^- p$	4
$\bar{n} p \to X^- \Delta^{++}$	8
$\gamma p \to X^0 p$	4
$\gamma p \to \pi^+ X^0$	2
$\nu p \to e^- X^{++}$	1 †
$e^- p \to e^- X^+$	1 †
$\phi p \to X^0 p$	6
$\phi p \to \pi^+ X^0$	3
$\pi^- \eta \to \pi^- X^0$	1
$\pi^- \phi \to \pi^- X^0$	2

† The electrons are assumed to come with one helicity.
True in general for odd-half-integer spins in the limit of zero mass.

Consider an $N_\epsilon \times N_\epsilon$ density matrix, with $i = \{\chi\, m\}$ and $j = \{\chi'\, m'\}$,

(119) $$ {}^\epsilon\rho_{ij} = \sum_{k=1}^{K_\epsilon} {}^\epsilon V_{ik}\, {}^\epsilon V^*_{jk}, \quad \Longrightarrow \quad {}^\epsilon\rho = {}^\epsilon V\, {}^\epsilon V^\dagger, \quad \Longrightarrow \quad {}^\epsilon\rho = {}^\epsilon\rho^\dagger $$

where $i, j = 1, \cdots, N_\epsilon$; $k = 1, \cdots, K_\epsilon$, and $K_\epsilon\ (= 1, \cdots, \infty)$ is the rank of the density matrix. Note that ${}^\epsilon\rho$ is an $N_\epsilon \times N_\epsilon$ square matrix, whereas ${}^\epsilon V$ is, in general, a retangular matrix $N_\epsilon \times K_\epsilon$. The 'Cholesky' decomposition of ${}^\epsilon V$ is, e.g. for $N_\epsilon = 6$ or for $6 \times 6\ {}^\epsilon\rho$,

(120) $$ {}^\epsilon V = \{{}^\epsilon V_{ik}\} = \begin{pmatrix} {}^\epsilon V_{11} & 0 & 0 & 0 & 0 & 0 \\ {}^\epsilon V_{21} & {}^\epsilon V_{22} & 0 & 0 & 0 & 0 \\ {}^\epsilon V_{31} & {}^\epsilon V_{32} & {}^\epsilon V_{33} & 0 & 0 & 0 \\ {}^\epsilon V_{41} & {}^\epsilon V_{42} & {}^\epsilon V_{43} & {}^\epsilon V_{44} & 0 & 0 \\ {}^\epsilon V_{51} & {}^\epsilon V_{52} & {}^\epsilon V_{53} & {}^\epsilon V_{54} & {}^\epsilon V_{55} & 0 \\ {}^\epsilon V_{61} & {}^\epsilon V_{62} & {}^\epsilon V_{63} & {}^\epsilon V_{64} & {}^\epsilon V_{65} & {}^\epsilon V_{66} \end{pmatrix} $$

where ${}^\epsilon V_{ik}$ is complex in general but ${}^\epsilon V_{ii} =$ real ≥ 0. There are 6 real diagonal elements and 15 complex off-diagonal elements of ${}^\epsilon V$, for a total of 36 parameters required to describe a $6 \times 6\ {}^\epsilon\rho$. The rank is given by the number of columns counting from the left, with the rest being zero. For example, if the rank=2, then we must have ${}^\epsilon V_{ik} = 0$, $i \geq k \geq 3$. In this case, there are 2 real diagonal elements and 9 complex off-diagonal elements, for a total of 20 parameters in the problem.

Now go back to the notation $\{\chi\, m\}$ and $\{\chi'\, m'\}$

(121) $$^{\epsilon}\rho^{\chi\,\chi'}_{mm'} = \sum_{k=1}^{K_\epsilon} {}^{\epsilon}V^{\chi}_{m\,k}\, {}^{\epsilon}V^{\chi'}_{m'\,k}{}^{*} \quad , \qquad m \geq 0,\ m' \geq 0$$

and define

(122) $$^{\epsilon}U_k(\tau) = \sum_{\chi\, m}^{N_\epsilon} {}^{\epsilon}V^{\chi}_{m\,k}\, {}^{\epsilon}D^{\chi}_{m}(\tau), \quad m \geq 0$$

The distribution function

$$I(\tau) \propto \sum_{\epsilon}^{2} \sum_{\substack{\chi m \\ \chi' m'}}^{N_\epsilon} {}^{\epsilon}\rho^{\chi\,\chi'}_{mm'}\, {}^{\epsilon}D^{\chi}_{m}(\tau)\, {}^{\epsilon}D^{\chi'}_{m'}{}^{*}(\tau)$$

becomes

(123) $$I(\tau) \propto \sum_{\epsilon}^{2} \sum_{k=1}^{K_\epsilon} \left| {}^{\epsilon}U_k(\tau) \right|^2$$

Maximum-Likelihood Method

Introduce the so-called extended likelihood function for finding ‘n’ events in a given mass bin

(124) $$\mathcal{L} \propto \left[\frac{\bar{n}^n}{n!} \, \mathrm{e}^{-\bar{n}} \right] \prod_i^n \left[\frac{I(\tau_i)}{\int I(\tau)\, \eta(\tau)\, \phi(\tau)\, \mathrm{d}\tau} \right]$$

where $\eta(\tau)$ is the experimental finite acceptance at τ and the invariant phase-space element given by

(125) $$\mathrm{d}\phi = \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \right) \mathrm{d}\tau = \phi(\tau)\, \mathrm{d}\tau$$

The first bracket in $\mathcal{L}$ represents the Poisson probability for finding ‘n’ events in the mass bin, and the expectation value $\bar{n}$ is

(126) $$\bar{n} \propto \int I(\tau)\, \eta(\tau)\, \phi(\tau)\, \mathrm{d}\tau$$

The likelihood function $\mathcal{L}$ can now be written, dropping the factors depending on n alone,

$$\mathcal{L} \propto \left[\prod_i^n I(\tau_i) \right] \exp \left[- \int I(\tau)\, \eta(\tau)\, \phi(\tau)\, \mathrm{d}\tau \right]$$

The ‘log’ of the likelihood function now has the form,

(127) $$\ln \mathcal{L} = \sum_i^n \ln I(\tau_i) \; - \int I(\tau)\,\eta(\tau)\,\phi(\tau)\,\mathrm{d}\tau$$

We shall adopt the following shorthand notation

(128) $$\alpha = \{\epsilon k; \chi\, m\} \quad \text{and} \quad \alpha' = \{\epsilon k; \chi'\, m'\}$$

and write

(129) $$I(\tau) = \sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\, D_\alpha(\tau)\, D^*_{\alpha'}(\tau)$$

The so-called ‘experimental’ normalization integral is given by

(130) $$\Psi^x_{\alpha\,\alpha'} = \int \left[D_\alpha(\tau)\, D^*_{\alpha'}(\tau) \right] \eta(\tau)\, \phi(\tau)\, \mathrm{d}\tau$$

so that

(131) $$\ln \mathcal{L} = \sum_i^n \ln \left[\sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\, D_\alpha(\tau_i)\, D^*_{\alpha'}(\tau_i) \right] - \sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\, \Psi^x_{\alpha\,\alpha'}$$

We explore the normalization for V's by setting $V = x\,W$, where x is independent of α,

(132) $$\ln \mathcal{L} = \sum_i^n \ln \left[x^2 \sum_{\alpha\alpha'} W_\alpha\, W^*_{\alpha'}\, D_\alpha(\tau_i)\, D^*_{\alpha'}(\tau_i) \right] - x^2 \sum_{\alpha\alpha'} W_\alpha\, W^*_{\alpha'}\, \Psi^x_{\alpha\,\alpha'}$$

At the maximum, we should have

(133) $$0 = \frac{\partial \ln \mathcal{L}}{\partial x^2} = \sum_i^n \left[\frac{1}{x^2} \right] - \sum_{\alpha\alpha'} W_\alpha\, W^*_{\alpha'}\, \Psi^x_{\alpha\,\alpha'}$$

so that

(134) $$\sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\, \Psi^x_{\alpha\,\alpha'} = n$$

We can define the theoretical normalization integral, with $\eta(\tau) = 1$,

(135) $$\Psi_{\alpha\,\alpha'} = \int \left[D_\alpha(\tau)\, D^*_{\alpha'}(\tau) \right] \phi(\tau)\, \mathrm{d}\tau$$

The predicted number of events is

$$N = \sum_{\alpha\alpha'} V_\alpha \, V^*_{\alpha'} \, \Psi_{\alpha\,\alpha'}$$

(136)

$$\equiv \sum_{\alpha\alpha'} N_{\alpha\alpha'} \;, \qquad N_{\alpha\alpha'} = V_\alpha \, V^*_{\alpha'} \, \Psi_{\alpha\,\alpha'}$$

So the predicted number of events for a partial wave α is

(137) $$N_{\alpha\alpha} = |V_\alpha|^2 \, \Psi_{\alpha\,\alpha} \;, \qquad \Psi_{\alpha\,\alpha} = \int \left| D_\alpha(\tau) \right|^2 \phi(\tau) \, \mathrm{d}\tau$$

The predicted number of events for the interference between the partial waves α and α' is

(138) $$N_{\alpha\alpha'} + N_{\alpha'\alpha} = 2\Re\left\{ V_\alpha \, V^*_{\alpha'} \, \Psi_{\alpha\,\alpha'} \right\} \;, \qquad \alpha \neq \alpha'$$

To Be Continuned. . .