Selected Topics in Hadron Spectroscopy

Mathematical Techniques

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Covariant Helicity-coupling Amplitudes

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A general formulation is given for constructing covariant helicity-coupling amplitudes involving two-body decays with arbitrary integer spins. The decay amplitudes are given exclusively in terms of both definite orbital angular momentum and total intrinsic spin. A systematic method is developed for calculating the energy and momentum dependence of daughter particles in the decay amplitudes, and a general formula for arbitrary integer spins is given.

If a daughter particle has spin 1 or higher, the helicity-coupling amplitudes depend in general on the Lorentz factor $\gamma = E/m$, where m is the mass of the daughter and E is its energy in the parent rest frame. A significant simplification results with the exclusive use of spin tensors and momenta defined along the helicity axis for the daughter states, which is defined to be along the original z-axis. This technique separates out the angular distribution contained in the D function from the problem of finding a proper energy and momentum dependence of the helicity-coupling amplitudes.

Two-body decay: $J \to s + \sigma$

	Parent	Daughter 1	Daughter 2
Spin	J	s	σ
Parity	η_J	η_s	η_σ
4-momentum	$p = (w; 0, 0, 0)$	$q = (q_0; 0, 0, q)$	$k = (k_0; 0, 0, -k)$
Energy	$p_0 = w$	q_0	k_0
Mass	w	m	μ
Energy/Mass	1	$\gamma_s = q_0/m$	$\gamma_\sigma = k_0/\mu$
Velocity	0	β_s	β_σ
Helicity	$\lambda - \nu$	λ	ν
$\gamma\,\beta$	0	$\gamma_s\beta_s = q/m$	$\gamma_\sigma\beta_\sigma = k/\mu$
Wave function	$\phi^*(\lambda - \nu)$	$\omega(\lambda)$	$\varepsilon(-\nu)$

where

$$\delta = \lambda - \nu \quad \text{and} \quad p = q + k, \quad r = q - k = (q_0 - k_0; 0, 0, 2q) \text{ in } J\text{RF}$$

Consider a state with spin(parity)=$J(\eta_J)$ decaying into two states with $s(\eta_s)$ and $\sigma(\eta_\sigma)$. The decay amplitudes (helicity formalism) are given, in the rest frame of J,

(1) $$\mathcal{M}^J_{\lambda\,\nu}(\theta,\phi,M)=\sqrt{\frac{2J+1}{4\pi}}\,D^{J\,*}_{M\,\delta}(\phi,\theta,0)F^J_{\lambda\,\nu}$$

If the angles are zero, then we have

(2) $$\mathcal{M}^J_{\lambda\,\nu}(0,0;\delta)=\sqrt{\frac{2J+1}{4\pi}}\,F^J_{\lambda\,\nu}$$

Equivalently, the decay amplitude (canonical formalism) can be written

(3) $$\mathcal{M}^J_{\ell\,S}(\theta,\phi;m_1,\,m_2;\,M)=G^J_{\ell\,S}\,(sm_1\,\sigma m_2|Sm_s)\sum_m(\ell m\,Sm_s|JM)\,Y^\ell_m(\theta,\phi)$$

Again, if the angles are zero, we find

(4) $$\mathcal{M}^J_{\ell\,S}(0,0;m_1,\,m_2;\,m_s)=\sqrt{\frac{2\ell+1}{4\pi}}\,G^J_{\ell\,S}\,(sm_1\,\sigma m_2|Sm_s)\,(\ell 0\,Sm_s|Jm_s)$$

Here we must have $m_1=\lambda$, $m_2=-\nu$ and $m_s=\lambda-\nu=\delta$, so that

(5) $$\mathcal{M}^J_{\ell\,S}(0,0;\lambda,\,-\nu;\delta)=\sqrt{\frac{2\ell+1}{4\pi}}\,G^J_{\ell\,S}\,(\ell 0\,S\delta|J\delta)\,(s\lambda\,\sigma\,-\nu|S\delta)$$

Comparing the two amplitudes (helicity and canonical), we see that

(6) $$\left(\mathcal{M}^J_{\lambda\,\nu} = \sum_{\ell\,S} \mathcal{M}^J_{\ell\,S} \right) \rightarrow \boxed{F^J_{\lambda\,\nu} = \sum_{\ell\,S} \left(\frac{2\ell+1}{2J+1}\right)^{1/2} (\ell 0\, S\delta|J\delta)\,(s\lambda\,\sigma\,-\nu|S\delta)\, G^J_{\ell\,S}}$$

where

(7) $$\sum_{\lambda\,\nu} \left| F^J_{\lambda\,\nu} \right|^2 = \sum_{\ell\,S} \left| G^J_{\ell\,S} \right|^2$$

Generalize to relativistic cases: $\boxed{(\ell 0\, S\delta|J\delta)\, G^J_{\ell\,S} \quad \rightarrow \quad g^J_{\ell\,S}\, A^J_{\ell\,S}(\delta)}$

(8) $$F^J_{\lambda\,\nu} = \sum_{\ell\,S} g^J_{\ell\,S}\,(s\lambda\,\sigma\,-\nu|S\delta)\, A^J_{\ell\,S}(\delta)$$

and compute $A^J_{\ell\,S}(\delta)$ in tensor formalism

(9) $$A^J_{\ell\,S}(\delta) = \left[p^n, \psi(S\delta), t^\ell(r), \phi^*(J\delta) \right]_w \quad \text{or} \quad |J\delta\rangle \rightarrow |S\delta\rangle + |\ell 0\rangle$$

where $n = 0$ or 1 and $[\cdots]_w$ indicates that a Lorentz invariant amplitude is to be contructed out of the four variables indicated, using the metric $\tilde{g}_{\alpha\beta}(w)$ or the totally antisymmetric rank-4 tensor $\epsilon_{\alpha\beta\gamma\delta}$, evaluated in the JRF.

The wave functions in a total intrinsic spin S is given by

$$\psi(Sm_s) = \sum_{m_a\, m_b} (sm_a\ \sigma m_b|Sm_s)\,\omega(sm_a)\,\varepsilon(\sigma m_b), \quad \psi(S\ -m_s) = (-)^{S-m_s}\,\psi^*(Sm_s)$$

(10)

where $\omega(sm_a)$ is a rank-s tensor, while $\varepsilon(\sigma m_b)$ is a rank-σ tensor. So $\psi(Sm_s)$ is a rank-($s+\sigma$) tensor. Note that $t^\ell(r)$ is a rank-ℓ tensor and $\phi(JM)$ is a rank-J tensor.

Take the example in which both ω and ε refer to spin-1 states. Then, $\psi(Sm_s)$ is a rank-2 tensor given by

(11)
$$\begin{cases} \psi^{\alpha\beta}(22) = \omega^\alpha(+)\,\varepsilon^\beta(+) \\ \psi^{\alpha\beta}(21) = \dfrac{1}{\sqrt{2}}\left[\omega^\alpha(+)\,\varepsilon^\beta(0) + \omega^\alpha(0)\,\varepsilon^\beta(+)\right] \\ \psi^{\alpha\beta}(20) = \dfrac{1}{\sqrt{6}}\left[\omega^\alpha(+)\,\varepsilon^\beta(-) + \omega^\alpha(-)\,\varepsilon^\beta(+) + 2\omega^\alpha(0)\,\varepsilon^\beta(0)\right] \\ \psi^{\alpha\beta}(11) = \dfrac{1}{\sqrt{2}}\left[\omega^\alpha(+)\,\varepsilon^\beta(0) - \omega^\alpha(0)\,\varepsilon^\beta(+)\right] \\ \psi^{\alpha\beta}(10) = \dfrac{1}{\sqrt{2}}\left[\omega^\alpha(+)\,\varepsilon^\beta(-) - \omega^\alpha(-)\,\varepsilon^\beta(+)\right] \\ \psi^{\alpha\beta}(00) = \dfrac{1}{\sqrt{3}}\left[\omega^\alpha(+)\,\varepsilon^\beta(-) + \omega^\alpha(-)\,\varepsilon^\beta(+) - \omega^\alpha(0)\,\varepsilon^\beta(0)\right] \end{cases}$$

The wave functions in the JRF are given by

$$
\left\{
\begin{array}{lclccccc}
\phi^\alpha(\pm) & = & \mp\frac{1}{\sqrt{2}} & (& 0; & 1, & \pm i, & 0 &) \\
\phi^\alpha(0) & = & & (& 0; & 0, & 0, & 1 &) \\
\\
\chi^\alpha(\pm) & = & \mp\frac{1}{\sqrt{2}} & (& 0; & 1, & \pm i, & 0 &) \\
\chi^\alpha(0) & = & & (& 0; & 0, & 0, & 1 &) \\
\\
\omega^\alpha(\pm) & = & \mp\frac{1}{\sqrt{2}} & (& 0; & 1, & \pm i, & 0 &) \\
\omega^\alpha(0) & = & & (& \gamma_s\beta_s; & 0, & 0, & \gamma_s &) \\
\\
\varepsilon^\alpha(\pm) & = & \mp\frac{1}{\sqrt{2}} & (& 0; & 1, & \pm i, & 0 &) \\
\varepsilon^\alpha(0) & = & & (& -\gamma_\sigma\beta_\sigma; & 0, & 0, & \gamma_\sigma &)
\end{array}
\right. \tag{12}
$$

Note that

$$p_\alpha\phi^\alpha(\lambda) = p_\alpha\chi^\alpha(\lambda) = q_\alpha\omega^\alpha(\lambda) = k_\alpha\varepsilon^\alpha(\lambda) = 0 \tag{13}$$

for any λ and

$$
\begin{aligned}
\phi(-\lambda) = (-)^\lambda\,\phi^*(\lambda), \quad \chi(-\lambda) = (-)^\lambda\,\chi^*(\lambda), \\
\omega(-\lambda) = (-)^\lambda\,\omega^*(\lambda), \quad \varepsilon(-\lambda) = (-)^\lambda\,\varepsilon^*(\lambda)
\end{aligned} \tag{14}
$$

These polarization four-vectors satisfy

(15)
$$\begin{cases} p_\alpha \phi^\alpha(m) = 0 \\ \phi^*_\alpha(m)\phi^\alpha(m') = -\delta_{mm'} \\ \sum_m \phi_\alpha(m)\phi^*_\beta(m) = \tilde{g}_{\alpha\beta}(w) \end{cases}$$

where

(16)
$$\tilde{g}_{\alpha\beta}(w) = -g_{\alpha\beta} + \frac{p_\alpha p_\beta}{w^2}$$

(17)
$$\tilde{g}_{\alpha\beta}(w) = \tilde{g}^{\alpha\beta}(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The argument w will be dropped whenever there is no ambiguity; however, one must remember that there exist in the problem two additional $\tilde{g}$'s, i.e. $\tilde{g}(m)$ and $\tilde{g}(\mu)$ for the states s and σ, respectively. The wave function for orbital angular momenta, in the JRF,

(18)
$$\tilde{g}_{\alpha\beta}(w)\, r \to \qquad \tilde{r}(0) \equiv \tilde{r} = (0; 0, 0, r)$$

Wave Functions for Arbtrary Integer Spin:

The general spin-J wave function can be written

(19)
$$\phi^{\delta_1\cdots\delta_J}(Jm) = [a^J(m)]^{\frac{1}{2}} \sum_{m_0} 2^{m_0/2} \sum_{P} \phi^{\alpha_1}(+)\cdots\phi^{\beta_1}(0)\cdots\phi^{\gamma_1}(-)\cdots$$
$$a^J(m) = \frac{(J+m)!(J-m)!}{(2J)!}, \quad \phi(J\ -m) = (-)^m\phi^*(Jm)$$

where the indices $\{\delta_1\cdots\delta_J\}$ have been broken up into three distinct sets in the second summation, i.e., $\{\alpha_i\}$ with $(i=1,m_+)$, $\{\beta_i\}$ with $(i=1,m_0)$ and $\{\gamma_i\}$ with $(i=1,m_-)$, where $m_\pm$ stands for the numbers of $\phi(\pm)$'s and m_0 for $\phi(0)$'s. Note that

(20)
$$J = m_+ + m_0 + m_-, \quad m = m_+ - m_- \quad \text{and} \quad 2m_\pm = J \pm m - m_0$$

m_0 ranges from $0(1)$, $2(3)$,$\cdots$, to $J-m$ =even(odd).
The second sum in $\phi(Jm)$ represents a summation on the permutations

$$\{(+)(+)\cdots(0)(0)\cdots(-)(-)\cdots\}$$

It is seen readily that the number of terms in the summation is given by

(21)
$$b^j(m,m_0) = \frac{j!}{m_+!m_0!m_-!}$$

For spin-2 wave functions, we have

$$\begin{cases} \phi_{\alpha\beta}(2,+2) = \phi_\alpha(+)\,\phi_\beta(+) \\ \phi_{\alpha\beta}(2,+1) = \frac{1}{\sqrt{2}}\left[\phi_\alpha(+)\,\phi_\beta(0) + \phi_\alpha(0)\,\phi_\beta(+)\right] \\ \phi_{\alpha\beta}(2,0) = \frac{1}{\sqrt{6}}\left\{\left[\phi_\alpha(+)\,\phi_\beta(-) + \phi_\alpha(-)\,\phi_\beta(+)\right] + 2\phi_\alpha(0)\,\phi_\beta(0)\right\} \end{cases} \tag{22}$$

and for spin 3,

$$\phi_{\alpha\beta\gamma}(3,+3) = \phi_\alpha(+)\,\phi_\beta(+)\,\phi_\gamma(+)$$

$$\phi_{\alpha\beta\gamma}(3,+2) = \frac{1}{\sqrt{3}}\left[\phi_\alpha(+)\,\phi_\beta(+)\,\phi_\gamma(0) + \phi_\alpha(+)\,\phi_\beta(0)\,\phi_\gamma(+) + \phi_\alpha(0)\,\phi_\beta(+)\,\phi_\gamma(+)\right]$$

$$\phi_{\alpha\beta\gamma}(3,+1) = \frac{1}{\sqrt{15}}\left[\phi_\alpha(+)\,\phi_\beta(+)\,\phi_\gamma(-) + \phi_\alpha(+)\,\phi_\beta(-)\,\phi_\gamma(+) + \phi_\alpha(-)\,\phi_\beta(+)\,\phi_\gamma(+)\right]$$
$$+ \frac{2}{\sqrt{15}}\left[\phi_\alpha(+)\,\phi_\beta(0)\,\phi_\gamma(0) + \phi_\alpha(0)\,\phi_\beta(+)\,\phi_\gamma(0) + \phi_\alpha(0)\,\phi_\beta(0)\,\phi_\gamma(+)\right]$$

$$\phi_{\alpha\beta\gamma}(3,0) = \frac{1}{\sqrt{10}}\left[\phi_\alpha(0)\,\phi_\beta(+)\,\phi_\gamma(-) + \phi_\alpha(0)\,\phi_\beta(-)\,\phi_\gamma(+) + \phi_\alpha(-)\,\phi_\beta(+)\,\phi_\gamma(0)\right.$$
$$\left. + \phi_\alpha(+)\,\phi_\beta(-)\,\phi_\gamma(0) + \phi_\alpha(+)\,\phi_\beta(0)\,\phi_\gamma(-) + \phi_\alpha(-)\,\phi_\beta(0)\,\phi_\gamma(+)\right] + \frac{2}{\sqrt{10}}\phi_\alpha(0)\,\phi_\beta(0)\,\phi_\gamma(0)$$

Orbital Angular Momenta in Tensor Representation:

The orbital angular momenta in tensor representation can be defined through the projection operator

(23)
$$P^\ell = \sum_m \tau^\ell(m)\,\tau^{\ell\,*}(m)$$

In the JRF where $\vec{r} = (0, 0, r)$, we have

(24)
$$[\tau^{\ell\,*}(m) \otimes rrr\cdots] = c_\ell\, r^\ell\, \delta_{m\,0}, \quad c_\ell = \ell! \left[\frac{2^\ell}{(2\ell)!}\right]^{1/2}$$

So we see that

(25)
$$[P^\ell \otimes rrr\cdots] = \chi(0)\, r^\ell$$

where we have introduced a new wave function

(26)
$$\chi(\ell 0) = c_\ell\, \tau^\ell(0)$$

Here $\tau^\ell(m)$ is the normalized wave function for spin ℓ defined previously.

(27) $$\chi_{ijk\cdots}(\ell 0) = \frac{(\ell!)^2}{(2\ell)!} \sum_{m_0} 2^{(\ell+m_0)/2} \sum_P \left[\chi(+)\cdots\chi(0)\cdots\chi(-)\right]_{ijk\cdots}$$

and $m_0 = 0(1),\ 2(3),\ 4(5), \ldots$ for $\ell =$even (odd). It is clear that $\chi(\ell 0)$ is now devoid of the coefficients with $\sqrt{(\cdots)}$. Note the identity

(28) $$t^\ell_{ijk\cdots}(r) = \chi_{ijk\cdots}(\ell 0)\, r^\ell$$

Indeed, we find, with $\chi(0) = (0,0,1), \quad \chi(\pm) = \mp\frac{1}{\sqrt{2}}(1, \pm i, 0)$,

(29)
$$\chi_i(10) = \chi_i(0)$$
$$\chi_{ij}(20) = \frac{1}{3}\left[\chi_i(+)\,\chi_j(-) + \chi_i(-)\,\chi_j(+) + 2\,\chi_i(0)\,\chi_j(0)\right]$$
$$\chi_{ijk}(30) = \frac{1}{5}\left[\chi_i(0)\,\chi_j(+)\,\chi_k(-) + \chi_i(0)\,\chi_j(-)\chi_k(+)\right.$$
$$+ \chi_i(+)\,\chi_j(0)\,\chi_j(-) + \chi_i(-)\,\chi_j(0)\chi_k(+)$$
$$\left. + \chi_i(+)\,\chi_j(-)\,\chi_j(0) + \chi_i(-)\,\chi_j(+)\chi_k(0) + 2\,\chi_i(0)\,\chi_j(0)\,\chi_k(0)\right]$$

For completeness, we work out the wave function for ℓ =3 and 5 as well

$$\chi_{ijkl}(40) = \frac{2}{35}\left\{ \sum_P^6 \left[\chi_i(+)\,\chi_j(-)\,\chi_k(+)\,\chi_l(-)\right] \right.$$

$$\left. + 2\sum_P^{12}\left[\chi_i(0)\,\chi_j(0)\,\chi_k(+)\,\chi_l(-)\right] + 4\,\chi_i(0)\,\chi_j(0)\,\chi_k(0)\,\chi_l(0)\right\}$$

$$\chi_{ijklm}(50) = \frac{2}{63}\left\{ \sum_P^{30}\left[\chi_i(0)\,\chi_j(+)\,\chi_k(-)\,\chi_l(+)\,\chi_m(-)\right] \right.$$

$$\left. + 2\sum_P^{20}\left[\chi_i(0)\,\chi_j(0)\,\chi_k(0)\,\chi_l(+)\,\chi_m(-)\right] + 4\,\chi_i(0)\,\chi_j(0)\,\chi_k(0)\,\chi_l(0)\,\chi_m(0)\right\}$$

It can be shown that the use of $\chi_{ijk\cdots}(\ell 0)$ tensors for the orbital angular momenta leads to a relatively simpler algebra than that of $t^{\ell}_{ijk\cdots}(r)$.

Examples:

i) $\underline{1^- \to 0^- + 0^-}$

The Lorentz invariant amplitude is, with $\ell = 1$ and $S = 0$,

(30)
$$A_{10}^{(1)} = r^\alpha\, \phi^*_\alpha(0)$$
$$(\text{JRF}) \to = [r \cdot \phi^*(0)] = [\chi(0) \cdot \phi^*(0)]\, r = r, \quad F_0^J = g_1\, r, \quad J = 1$$

ii) $\underline{2^+ \to 0^- + 0^-}$

The Lorentz invariant amplitude is, with $\ell = 2$ and $S = 0$,

(31)
$$A_{20}^{(2)} = r^\alpha\, r^\beta\, \phi^*_{\alpha\beta}(2,0) = [rr : \phi^*(2,0)] = [\chi(20) : \phi^*(2,0)]\, r^2$$

$$\phi_{\alpha\beta}(2,m) = \sum_{m_1 m_2} (1m_1 1m_2|2m)\, \phi_\alpha(m_1)\, \phi_\beta(m_2), \quad \phi_{\alpha\beta}(2,-m) = (-)^m\, \phi^*_{\alpha\beta}(2,m)$$

and we find

(32)
$$A_{20}^{(2)} = \sqrt{\frac{2}{3}}\, r^2, \quad F_0^J = \sqrt{\frac{2}{3}}\, g_2\, r^2, \quad J = 2$$

iii) $\underline{3^- \to 0^- + 0^-}$

The Lorentz invariant amplitude is, with $\ell = 3$ and $S = 0$,

(33) $$A^{(3)}_{30} = r^\alpha\, r^\beta\, r^\gamma\, \phi^*_{\alpha\beta\gamma}(3,0) = [rrr\ \therefore\ \phi^*(3,0)] = [\chi(30)\ \therefore\ \phi^*(3,0)]\, r^3$$

where, with $\phi_{\alpha\beta\gamma}(3,-m) = (-)^m\, \phi^*_{\alpha\beta\gamma}(3,m)$, and so

(34) $$A^{(3)}_{30} = \sqrt{\frac{2}{5}}\, r^3, \quad F^J_0 = \sqrt{\frac{2}{5}}\, g_3\, r^3, \quad J = 3$$

iv) $\underline{1^- \to 1^- + 0^-}$

The Lorentz invariant amplitude is, with $\ell = 1$ and $S = 1$,

(35)
$$
\begin{aligned}
A^{(1)}_{11}(\lambda) &= \varepsilon^{\alpha\beta\gamma\delta}\, p_\alpha\, \omega_\beta(\lambda)\, r_\gamma\, \phi^*_\delta(\lambda) = [p\,\omega(\lambda)\, r\, \phi^*(\lambda)] = [p\,\omega(\lambda)\,\chi(0)\,\phi^*(\lambda)]\, r \\
(JRF) \to &= w\left(\vec{\omega}(\lambda) \times \vec{r} \cdot \vec{\phi}^*(\lambda)\right) = w[\vec{\omega}(\lambda)\, \vec{r}\, \vec{\phi}^*(\lambda)] \\
&= \pm(i\, w)\, r, \qquad \lambda = \pm 1 \\
&= 0, \qquad \lambda = 0
\end{aligned}
$$

and

(36)
$$F^J_\pm = \pm(i\, w)\, g_1\, r, \quad F^J_0 = 0, \qquad J = 1$$

(37)
$$(\text{parity}) \to F^J_\lambda = -F^J_{-\lambda}$$

v) $\underline{2^+ \to 1^- + 0^-}$

The Lorentz invariant amplitude is, with $\ell = 2$ and $S = 1$,

(38) $$A^{(2)}_{21}(\lambda) = \varepsilon^{\alpha\beta\gamma\delta}\, p_\alpha\, \omega_\beta(\lambda)\, r_\gamma\, r^\upsilon\, \phi^*_{\upsilon\delta}(2,\lambda) = [p,\, \omega(\lambda),\, \chi(20)\cdot\phi^*(\lambda)]\, r^2$$

Define

(39) $$\boxed{r\,\xi_\delta(2,\lambda) = r^\upsilon\,\phi_{\upsilon\delta}(2,\lambda)} \quad \xrightarrow{\text{JRF}} \quad r\,\xi_i(2,\lambda) = r_k\,\phi_{ki}(2,\lambda)$$

and we find

(40) $$\begin{cases} \xi_\delta(2,+2) = 0 \\ \xi_\delta(2,+1) = \dfrac{1}{\sqrt{2}}\,\phi_\delta(+) \\ \xi_\delta(2,0) = \sqrt{\dfrac{2}{3}}\,\phi_\delta(0) \end{cases}$$

so that, in JRF,

(41) $$\begin{aligned} A^{(2)}_{21}(\lambda) &= w\left(\vec\omega(\lambda)\times\vec r\cdot\vec\xi^{\,*}(2,\lambda)\right) r = w[\vec\omega(\lambda)\,\vec r\,\vec\xi^{\,*}(2,\lambda)]\, r \\ &= \pm\frac{1}{\sqrt{2}}\,(i\,w)\,r^2, \qquad \lambda = \pm 1 \\ &= 0, \qquad \lambda = 0 \end{aligned}$$

The general decay amplitude is

(42) $$F^J_{\pm} = \pm \frac{1}{\sqrt{2}}\,(i\,w)\,g_2\,r^2, \quad F^J_0 = 0, \quad J = 2$$

vi) $\underline{1^+ \to 1^- + 0^-}$

The helicity-coupling amplitudes are, with $\ell = 0$ or 2 and $S = 1$,

(43) $$\begin{cases} F^J_{\pm} = \sqrt{\frac{1}{3}} G^J_0 + \sqrt{\frac{1}{6}} G^J_2, & J = 1 \\ F^J_0 = \sqrt{\frac{1}{3}} G^J_0 - \sqrt{\frac{2}{3}} G^J_2, & J = 1 \end{cases}$$

The Lorentz invariant amplitudes are

(44)

$$\begin{aligned} A^{(1)}_{01}(\lambda) &= [\omega(\lambda) \cdot \phi^*(\lambda)], \\ &= \omega^\alpha(\lambda)\phi^*_\alpha(\lambda), \\ (JRF) \to &= \omega_i(\lambda)\,\phi^*_i(\lambda), \end{aligned}$$

$$\begin{aligned} A^{(1)}_{21}(\lambda) &= [\omega \cdot \chi(20) \cdot \phi^*(\lambda)]\,r^2 \\ &= \omega^\alpha(\lambda)\,t^{(2)}_{\alpha\beta}(\tilde{r})\,\phi^{*\,\beta}(\lambda) \\ (JRF) \to &= \omega_i(\lambda)\,t^{(2)}_{ij}(r)\,\phi^*_j(\lambda) \end{aligned}$$

where, in the JRF,

(45) $$t^{(2)}_{\alpha\beta}(\tilde{r}) \to \qquad t^{(2)}_{ij}(r) = r_i\, r_j - \frac{1}{3} r^2\, \delta_{ij}$$

so that, again in the JRF,

(46) $$A^{(1)}_{21}(\lambda) = [\vec{r}\cdot\vec{\omega}(\lambda)][\vec{r}\cdot\vec{\phi}^*(\lambda)] - \frac{1}{3} r^2 [\vec{\omega}(\lambda)\cdot\vec{\phi}^*(\lambda)]$$

We see that

(47) $$A^{(1)}_{01}(\pm) = 1, \qquad A^{(1)}_{01}(0) = \gamma_s$$
$$A^{(1)}_{21}(\pm) = -\tfrac{1}{3} r^2, \qquad A^{(1)}_{21}(0) = \tfrac{2}{3}\gamma_s\, r^2$$

(48) $$\begin{cases} F^J_{\pm} = g_0 - \dfrac{1}{3} g_2 r^2, & J = 1 \\ F^J_0 = \left(g_0 + \dfrac{2}{3} g_2 r^2\right)\gamma_s, & J = 1 \end{cases}$$

Only in the limit $\gamma_s = q_0/m \to 1$, these expressions reduce to the previous ones, with the replacement

(49) $$G^J_0 = \sqrt{3}\, g_0\,, \qquad G^J_2 = -\sqrt{\frac{2}{3}}\, g_2\, r^2$$

Now explore the meaning of the Lorentz factor $\gamma_s = q_0/m$. We first examine the Taylor series expansion of γ_s

$$\gamma_s = \frac{q_0}{m} = \left[1 + \left(\frac{q}{m}\right)^2\right]^{1/2}$$

(50)
$$\simeq 1 + \frac{1}{2}\left(\frac{q}{m}\right)^2 - \frac{1}{2\cdot 4}\left(\frac{q}{m}\right)^4 + \frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{q}{m}\right)^6$$

$$- \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\left(\frac{q}{m}\right)^8 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}\left(\frac{q}{m}\right)^{10} - \cdots$$

where $q = r/2$ in the JRF. So the orbital angular momentum is no longer quantized.

The decay amplitude is

(51)
$$\mathcal{M}^J_\lambda(\theta,\phi,M) \propto D^{J\,*}_{M\lambda}(\phi,\theta,0)F^J_\lambda$$

Assume that $\omega(\lambda)$ is a 'stable' particle. Then λ becomes an external variable which is unobserved; there should be no interference between different helicities, and the angular distribution must include a sum over the helicities. Assume further that the only non-zero density-matrix element is ρ_{00}.

Then, the angular distribution in the JRF becomes, after integrating over ϕ,

$$I(\theta) \propto \sum_{\lambda} [d^J_{0\lambda}(\theta)]^2 \, |F^J_\lambda|^2, \qquad J = 1 \tag{52}$$

This leads to a distribution

$$I(\theta) \propto |F^J_0|^2 \cos^2(\theta) + |F^J_+|^2 \sin^2(\theta) \tag{53}$$

so that, assuming g_0 and g_2 are relatively real,

$$\begin{aligned} I(\theta) \propto\ & g_0^2[(\gamma_s^2 - 1)\cos^2(\theta) + 1] + \frac{1}{9} g_2^2\, r^4[(4\gamma_s^2 - 1)\cos^2(\theta) + 1] \\ & \pm \frac{2}{3} g_0\, g_2\, r^2[(2\gamma_s^2 + 1)\cos^2(\theta) - 1] \end{aligned} \tag{54}$$

Note

$$r = 2q \quad \text{and} \quad \gamma_s^2 = 1 + \left(\frac{q}{m}\right)^2 \tag{55}$$

in the JRF.

vii) $\underline{2^- \to 2^+ + 0^-}$

This problem involves three orbital angular momenta, $\ell = 0$, 2 and 4. Correspondingly, the invariant amplitudes are

$$
\left\{
\begin{array}{l}
A^{(2)}_{02}(\lambda) = [\omega(\lambda) : \phi^*(\lambda)] \\
A^{(2)}_{22}(\lambda) = [\cdot\omega(\lambda) \cdot t^{(2)}(r) \cdot \phi^*(\lambda)\cdot] = [\,\cdot\,\omega(\lambda) \cdot \chi(20) \cdot \phi^*(\lambda) \cdot\,]\, r^2 \\
A^{(2)}_{42}(\lambda) = [\omega(\lambda) : t^{(4)}(r) : \phi^*(\lambda)] = [\omega(\lambda) : \chi(40) : \phi^*(\lambda)]\, r^4
\end{array}
\right. \tag{56}
$$

where

$$
\begin{aligned}
t^{(4)}_{ijkl}(r) &= r_i\, r_j\, r_k\, r_l - \frac{1}{7} r^2 \sum_P^6 (\delta_{ij}\, r_k\, r_l) + \frac{1}{35} r^4 \sum_P^3 (\delta_{ij}\, \delta_{kl}) \\
&= \chi_{ijkl}(40)\, r^4
\end{aligned} \tag{57}
$$

and so

$$
\left\{
\begin{array}{l}
F^J_2 = g_0 - \dfrac{1}{3}\, g_2\, r^2 + \dfrac{2}{35}\, g_4\, r^4 \\
F^J_1 = \left(g_0 + \dfrac{1}{6}\, g_2\, r^2 - \dfrac{8}{35}\, g_4\, r^4 \right) \gamma_s \\
F^J_0 = g_0 \left(\dfrac{2}{3}\gamma_s^2 + \dfrac{1}{3} \right) + \dfrac{1}{3}\, g_2\, r^2 \left(\dfrac{4}{3}\gamma_s^2 - \dfrac{1}{3} \right) + \dfrac{12}{35}\, g_4\, r^4 \left(\dfrac{2}{3}\gamma_s^2 + \dfrac{1}{3} \right)
\end{array}
\right. \tag{58}
$$

viii) $\underline{3^+ \to 2^+ + 0^-}$

This problem involves $\ell = 1$, 3 or 5. So we need rank-3 and rank-5 tensors corresponding to $\ell = 3$ and 5. The relevant orbital-momentum tensors are

$$t^{(3)}_{ijk}(r) = r_i\, r_j\, r_k - \frac{1}{5} r^2 (\delta_{ij}\, r_k + \delta_{jk}\, r_i + \delta_{ki}\, r_j)$$
$$= \chi_{ijk}(30)\, r^3$$

(59)

$$t^{(5)}_{ijklm}(r) = r_i\, r_j\, r_k\, r_l\, r_m - \frac{1}{9} r^2 \sum_P^{10} (\delta_{ij}\, r_k\, r_l\, r_m) + \frac{1}{63} r^4 \sum_P^{15} (\delta_{ij}\, \delta_{kl}\, r_m)$$
$$= \chi_{ijklm}(50)\, r^5$$

The decay amplitudes can be written

$$A^{(3)}_{12}(\lambda) \propto [\omega(\lambda) : \phi^*(\lambda) \cdot \chi(0)]\, r$$

(60)

$$A^{(3)}_{32}(\lambda) \propto [\cdot\, \omega(\lambda) \cdot \chi(30) : \phi^*(\lambda) \cdot]\, r^3$$

$$A^{(3)}_{52}(\lambda) \propto [\omega(\lambda) : \chi(50) \therefore \phi^*(\lambda)]\, r^5$$

It can be shown that (the required algebra tedious and difficult)

$$F_2^J = \frac{1}{\sqrt{3}} \left(g_1 - \frac{2}{5} g_3 r^2 + \frac{2}{21} g_5 r^4 \right) r$$

$$F_1^J = \sqrt{\frac{2}{15}}\, \gamma_s \left(2g_1 + \frac{1}{5} g_3 r^2 - \frac{10}{21} g_5 r^4 \right) r$$

$$F_0^J = \sqrt{\frac{3}{5}} \left\{ g_1 \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3} \right) + \frac{4}{15} g_3 \left(\frac{3}{2}\gamma_s^2 - \frac{1}{2} \right) r^2 + \frac{20}{63} g_5 \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3} \right) r^4 \right\} r$$

We need to find a different techinique which results in a relatively simpler algebra. Or, better still, find a general formula which does not require writing down invariant amplitudes.

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ix) $0^+ \to 1^- + 1^-$

The allowed orbital angular momenta are $\ell = 0$ and 2. Since $J = 0$, we must have $S = \ell = 0$ or 2 and $\lambda = \nu = 0$ or 1. So we find in the JRF

(61)
$$\begin{cases} A^{(0)}_{00}(0) = \tilde{g}_{\alpha\beta}(w)\,\psi^{\alpha\beta}(00) = -\dfrac{1}{\sqrt{3}}(2+\gamma_s\gamma_\sigma) \\ A^{(0)}_{22}(0) = [\chi(20):\psi(20)]_w\, r^2 \\ \qquad\qquad = \dfrac{1}{3}\sqrt{\dfrac{2}{3}}\,(1+2\gamma_s\gamma_\sigma) \end{cases}$$

The helicity-coupling amplitudes are

(62)
$$\begin{cases} F^{(0)}_{\pm\pm} = -g^{(0)}_{00}\left(\dfrac{1}{3}\gamma_s\gamma_\sigma + \dfrac{2}{3}\right) + \dfrac{1}{3}\,g^{(0)}_{22}\left(\dfrac{2}{3}\gamma_s\gamma_\sigma + \dfrac{1}{3}\right) r^2 \\ F^{(0)}_{00} = g^{(0)}_{00}\left(\dfrac{1}{3}\gamma_s\gamma_\sigma + \dfrac{2}{3}\right) + \dfrac{2}{3}\,g^{(0)}_{22}\left(\dfrac{2}{3}\gamma_s\gamma_\sigma + \dfrac{1}{3}\right) r^2 \end{cases}$$

In the nonrelativistic limit, we have $\gamma_s = \gamma_\sigma = 1$, so that

$$\begin{cases} F^{(0)}_{++} = -g^{(0)}_{00} + \dfrac{1}{3}\,g^{(0)}_{22}\,r^2 \\ F^{(0)}_{00} = g^{(0)}_{00} + \dfrac{2}{3}\,g^{(0)}_{22}\,r^2 \end{cases} \qquad \begin{cases} F^J_{\pm\pm} = \sqrt{\dfrac{1}{3}}G^J_{00} + \sqrt{\dfrac{1}{6}}G^J_{22} \\ F^J_{00} = -\sqrt{\dfrac{1}{3}}G^J_{00} + \sqrt{\dfrac{2}{3}}G^J_{22} \end{cases}$$

x) $\underline{1^+ \to 1^+ + 1^-}$ [also for $1^- \to 1^- + 1^-$ or $\pi_1(2000) \to \omega\rho$]

The allowed orbital angular momenta are $\ell = 1$ and 3. Since $J = 1$, we can have $S = 0$, 1 or 2. The invariant amplitudes are, with $\delta = \lambda - \nu$,

(63)
$$\begin{cases} A^{(1)}_{1S}(\delta) = [\chi(0) \cdot \psi(S\delta) \cdot \phi^*(\delta)]_w\, r, \quad S = 0,\ 1,\ 2 \\ B^{(1)}_{1S}(\delta) = [\phi^*(\delta) \cdot \psi(S\delta) \cdot \chi(0)]_w\, r, \quad S = 0,\ 1,\ 2 \\ A^{(1)}_{3S}(\delta) = [\psi(S\delta)\ :\ \chi(30) \cdot \phi^*(\delta)]_w\, r^3, \quad S = 2 \end{cases}$$

So we see that

(64)
$$\begin{cases} F^{(1)}_{++} = -\frac{1}{3} g^{(1)}_{10}\, \gamma_s \gamma_\sigma\, r + \frac{1}{3}(g^{(1)}_{12} + f^{(1)}_{12})\, \gamma_s \gamma_\sigma\, r + \frac{1}{5} g^{(1)}_{32} \left(\frac{1}{3} + \frac{2}{3}\gamma_s\gamma_\sigma\right) r^3 \\ F^{(1)}_{0+} = \frac{1}{2}(g^{(1)}_{11}\, \gamma_s - f^{(1)}_{11}\, \gamma_\sigma)\, r + \frac{1}{2}(g^{(1)}_{12}\, \gamma_s + f^{(1)}_{12}\, \gamma_\sigma)\, r \\ \qquad\qquad - \frac{1}{5} g^{(1)}_{32} \left(\frac{1}{2}\gamma_s + \frac{1}{2}\gamma_\sigma\right) r^3 \\ F^{(1)}_{+0} = -\frac{1}{2}(g^{(1)}_{11}\, \gamma_s - f^{(1)}_{11}\, \gamma_\sigma)\, r + \frac{1}{2}(g^{(1)}_{12}\, \gamma_s + f^{(1)}_{12}\, \gamma_\sigma)\, r \\ \qquad\qquad - \frac{1}{5} g^{(1)}_{32} \left(\frac{1}{2}\gamma_s + \frac{1}{2}\gamma_\sigma\right) r^3 \\ F^{(1)}_{00} = \frac{1}{3} g^{(1)}_{10}\, \gamma_s \gamma_\sigma\, r + \frac{2}{3}(g^{(1)}_{12} + f^{(1)}_{12})\, \gamma_s \gamma_\sigma\, r + \frac{2}{5} g^{(1)}_{32} \left(\frac{1}{3} + \frac{2}{3}\gamma_s\gamma_\sigma\right) r^3 \end{cases}$$

x) $1^+ \to 1^+ + 1^-$ (continued)

So the amplitudes depend on six parameters — $g_{10}^{(1)}$, $g_{11}^{(1)}$, $g_{12}^{(1)}$, $g_{32}^{(1)}$, $f_{11}^{(1)}$ and $f_{12}^{(1)}$.
A potential seventh parameter, $f_{10}^{(1)}$, occurs with the same functional dependence on γ's as $g_{10}^{(1)}$, and hence it can be absorbed into $g_{10}^{(1)}$.

The helicity-coupling amplitudes are, in the previous formulation,

(65)
$$\begin{cases} \sqrt{2}F_{++}^{(1)} = \sqrt{\frac{2}{3}}g_{10}^{(1)}\,r - \sqrt{\frac{2}{15}}g_{12}^{(1)}\,r + \frac{1}{\sqrt{5}}g_{32}^{(1)}\,r^3 \\ \sqrt{2}F_{0+}^{(1)} = \left[\frac{1}{\sqrt{2}}g_{11}^{(1)}\,r - \sqrt{\frac{3}{10}}g_{12}^{(1)}\,r - \frac{1}{\sqrt{5}}g_{32}^{(1)}\,r^3\right]\gamma_s \\ \sqrt{2}F_{+0}^{(1)} = \left[-\frac{1}{\sqrt{2}}g_{11}^{(1)}\,r - \sqrt{\frac{3}{10}}g_{12}^{(1)}\,r - \frac{1}{\sqrt{5}}g_{32}^{(1)}\,r^3\right]\gamma_\sigma \\ F_{00}^{(1)} = \left[-\frac{1}{\sqrt{3}}g_{10}^{(1)}\,r - \frac{2}{\sqrt{15}}g_{12}^{(1)}\,r + \sqrt{\frac{2}{5}}g_{32}^{(1)}\,r^3\right]\gamma_s\,\gamma_\sigma \end{cases}$$

Consider now the decay $1^- \to 1^- + 1^-$. Carrying out the Bose symmetrization requirement, we find

$$
\left\{
\begin{array}{l}
F^{(1)}_{++} = 0 \\
F^{(1)}_{0+} = \frac{1}{2}(g^{(1)}_{11} - f^{(1)}_{11}) \left(\frac{1}{2}\gamma_s + \frac{1}{2}\gamma_\sigma\right) r + \frac{1}{4}(g^{(1)}_{12} - f^{(1)}_{12}) (\gamma_s - \gamma_\sigma) r \\
F^{(1)}_{+0} = -\frac{1}{2}(g^{(1)}_{11} - f^{(1)}_{11}) \left(\frac{1}{2}\gamma_s + \frac{1}{2}\gamma_\sigma\right) r + \frac{1}{4}(g^{(1)}_{12} - f^{(1)}_{12}) (\gamma_s - \gamma_\sigma) r \\
F^{(1)}_{00} = 0
\end{array}
\right. \tag{66}
$$

The amplitudes depend on two parameters $(g^{(1)}_{11} - f^{(1)}_{11})$ and $(g^{(1)}_{12} - f^{(1)}_{12})$. The terms corresponding to the second parameter, with $\ell + S =$ odd, vanish in the nonrelativistic limit $\gamma_s = \gamma_\sigma = 1$.

xi) $\underline{1^- \to 1^+ + 1^-}$

The allowed orbital angular momenta are $\ell = 0$ and 2. Since $J = 1$, we must have $S = 1$ or 2.

$$\begin{cases} A_{01}^{(1)}(\delta) = [p,\ \psi(1\delta),\ \phi^*(\delta)]_w \\ A_{2S}^{(1)}(\delta) = [p,\ \psi(S\delta)\cdot\chi(20),\ \phi^*(\delta)]_w\, r^2, \quad S = 1,2 \\ B_{2S}^{(1)}(\delta) = [p,\ \chi(20)\cdot\ \psi(S\delta), \phi^*(\delta)]_w\, r^2, \quad S = 1,2 \\ C_{2S}^{(1)}(\delta) = [p,\ \psi(S\delta),\ \chi(20)\cdot\phi^*(\delta)]_w\, r^2, \quad S = 1,2 \end{cases} \tag{67}$$

The helicity-coupling amplitudes are

$$
\begin{cases}
F^{(1)}_{++} = (i\,w)\left[g^{(1)}_{01} - \frac{1}{3}\,g^{(1)}_{21}\,r^2 - \frac{1}{3}f^{(1)}_{21}\,r^2 + \frac{2}{3}h^{(1)}_{21}\,r^2\right] \\
F^{(1)}_{0+} = (i\,w)\left[g^{(1)}_{01}\left(\frac{1}{2}\gamma_\sigma + \frac{1}{2}\gamma_s\right) - \frac{1}{3}h^{(1)}_{21}\left(\frac{1}{2}\gamma_\sigma + \frac{1}{2}\gamma_s\right)r^2\right. \\
\qquad + \frac{1}{6}\,g^{(1)}_{21}\,(2\gamma_\sigma - \gamma_s)\,r^2 + \frac{1}{6}\,f^{(1)}_{21}\,(2\gamma_s - \gamma_\sigma)\,r^2 + \frac{1}{6}h^{(1)}_{22}(\gamma_\sigma - \gamma_s)\,r^2 \\
\qquad \left. - \frac{1}{2}\,g^{(1)}_{22}\left(\frac{2}{3}\gamma_\sigma + \frac{1}{3}\gamma_s\right)r^2 + \frac{1}{2}\,f^{(1)}_{22}\left(\frac{2}{3}\gamma_s + \frac{1}{3}\gamma_\sigma\right)r^2\right] \\
F^{(1)}_{+0} = (i\,w)\left[g^{(1)}_{01}\left(\frac{1}{2}\gamma_\sigma + \frac{1}{2}\gamma_s\right) - \frac{1}{3}h^{(1)}_{21}\left(\frac{1}{2}\gamma_\sigma + \frac{1}{2}\gamma_s\right)r^2\right. \\
\qquad + \frac{1}{6}\,g^{(1)}_{21}\,(2\gamma_\sigma - \gamma_s)\,r^2 + \frac{1}{6}\,f^{(1)}_{21}\,(2\gamma_s - \gamma_\sigma)\,r^2 - \frac{1}{6}h^{(1)}_{22}(\gamma_\sigma - \gamma_s)\,r^2 \\
\qquad \left. + \frac{1}{2}\,g^{(1)}_{22}\left(\frac{2}{3}\gamma_\sigma + \frac{1}{3}\gamma_s\right)r^2 - \frac{1}{2}\,f^{(1)}_{22}\left(\frac{2}{3}\gamma_s + \frac{1}{3}\gamma_\sigma\right)r^2\right]
\end{cases}
\tag{68}
$$

There are a total of seven parameters for the amplitudes.

This example applies equally well to the case: $1^+ \to 1^- + 1^-$, where the two vector states are identical. The resulting helicity-coupling amplitudes are

$$(69)\begin{cases} F^{(1)}_{++} = 0 \\ F^{(1)}_{0+} = (i\,w)\left[\frac{1}{4}\,(g^{(1)}_{21} - f^{(1)}_{21})\,(\gamma_\sigma - \gamma_s) - \frac{1}{2}\,(g^{(1)}_{22} - f^{(1)}_{22})\,\left(\frac{1}{2}\gamma_\sigma + \frac{1}{2}\gamma_s\right)\right] r^2 \\ F^{(1)}_{+0} = (i\,w)\left[\frac{1}{4}\,(g^{(1)}_{21} - f^{(1)}_{21})\,(\gamma_\sigma - \gamma_s) + \frac{1}{2}\,(g^{(1)}_{22} - f^{(1)}_{22})\,\left(\frac{1}{2}\gamma_\sigma + \frac{1}{2}\gamma_s\right)\right] r^2 \end{cases}$$

and the amplitudes depend only on two parameters $(g^{(1)}_{21} - f^{(1)}_{21})$ and $(g^{(1)}_{22} - f^{(1)}_{22})$. The first term is *purely* relativistic and so it vanishes in the limit $\gamma_s = \gamma_\sigma = 1$. In addition, one may expect that the first term should remain small and insignificant, since γ_s should be nearly equal to γ_σ independent of the size of w compared to m or μ.

The helicity-coupling amplitudes are, in the nonrelativistic limit,

$$
\left\{
\begin{aligned}
\sqrt{2}F^{(1)}_{++} &= \frac{1}{\sqrt{3}}G^{(1)}_{01} - \sqrt{\frac{2}{3}}\,G^{(1)}_{21} \\
\sqrt{2}F^{(1)}_{0+} &= \frac{1}{\sqrt{3}}G^{(1)}_{01} + \frac{1}{\sqrt{6}}G^{(1)}_{21} - \frac{1}{\sqrt{2}}G^{(1)}_{22} \\
\sqrt{2}F^{(1)}_{+0} &= \frac{1}{\sqrt{3}}G^{(1)}_{01} + \frac{1}{\sqrt{6}}G^{(1)}_{21} + \frac{1}{\sqrt{2}}G^{(1)}_{22}
\end{aligned}
\right.
\tag{70}
$$

There are three parameters for three amplitudes.

To Be Continuned. . .