Hadron Spectroscopy

Mathematical Techniques

Suh-Urk Chung

PNU/Busan/Korea, TUM/Munich/Germany
BNL/NY/USA

http://cern.ch/suchung/
http://www.phy.bnl.gov/~e852/reviews.html
Amplitude Analysis for Two-pseudoscalar Systems

Primary References:

General Angular Distributions

Consider the following reaction

$$\pi^- + p \rightarrow \pi^- \eta + p \tag{1}$$

In the [Jackson frame](#), the amplitudes may be expanded in terms of the partial waves for the $\pi\eta$ system:

$$U_k(\Omega) = \sum_{\ell m} V_{\ell m k} A_{\ell m}(\Omega), \quad \Omega = (\theta, \phi) \tag{2}$$

where $V_{\ell m k}$ stands for the production amplitude for a state $|\ell m\rangle$ and $k$ represents the spin degrees of freedom for the initial and final nucleons ($k = 1, 2$ for spin-flip and spin-nonflip amplitudes). $A_{\ell m}(\Omega)$ is the decay amplitude given by

$$A_{\ell m}(\Omega) = \sqrt{\frac{2\ell + 1}{4\pi}} D_{m0}^\ell(\phi, \theta, 0) = Y_{\ell m}^m(\Omega) \tag{3}$$

where the angles $\Omega = (\theta, \phi)$ describe the direction of the $\eta$ in the Jackson frame. The angular distribution is given by

$$I(\Omega) = \sum_k |U_k(\Omega)|^2 \tag{4}$$
The eigenstates of this reflection operator are

\[ |\epsilon \ell m \rangle = \theta(m) \left\{ |\ell m \rangle - \epsilon(-)^m |\ell - m \rangle \right\} \]  

where

\[ \theta(m) = \begin{cases} \frac{1}{\sqrt{2}}, & m > 0 \\ \frac{1}{2}, & m = 0 \\ 0, & m < 0 \end{cases} \]  

For a positive reflectivity, the \( m = 0 \) states are not allowed, i.e.

\[ |\epsilon \ell 0 \rangle = 0, \quad \text{if} \quad \epsilon = + \]  

The reflectivity quantum number \( \epsilon \) has been defined so that it coincides with the naturality of the exchanged particle in Reaction (1).

\[ \epsilon U_k(\Omega) = \sum_{\ell m} \epsilon V_{\ell m k} \sqrt{\frac{2\ell + 1}{4\pi}} \epsilon D_{m0}^\ell(\phi, \theta, 0) \]
and the resulting angular distribution is

\[ I(\Omega) = \sum_{\epsilon k} |\epsilon U_k(\Omega)|^2 \] (9)

The angular distribution may be expanded in terms of the moments \( H(LM) \) via

\[ I(\Omega) = \sum_{LM} \left( \frac{2L+1}{4\pi} \right) H(LM) D_{M0}^{L*}(\phi, \theta, 0) \] (10)

so that

\[ H(LM) = \int d\Omega \ I(\Omega) \ D_{M0}^{L}(\phi, \theta, 0) \rightarrow \text{measurable experimentally} \] (11)

\[ H(LM) = \sum_{\epsilon k} \sum_{\ell m} \left( \frac{2\ell' + 1}{2\ell + 1} \right)^{1/2} \epsilon V_{\ell m k} \epsilon^* V_{\ell' m' k}^* \epsilon b(\ell' m' LM \ell m) (\ell'0 L0 |\ell0) \]

where a new function \( \epsilon b \) is a sum of Clebsch-Gordan coefficients:

\[ \epsilon b(\ell' m' LM \ell m) = \theta(m')\theta(m) \left[ (\ell' m' LM |\ell m) + (-)^M (\ell' m' L | -M |\ell m) \right. \]

\[ \left. - \epsilon (-)^m (\ell' -m' LM |\ell m) - \epsilon (-)^m (\ell' m' LM |\ell -m) \right] \] (12)
Two assumptions: (1) $\epsilon V_{\ell m k} = 0$ if $m \geq 2$ and (2) rank=1, i.e. $k=1$.

The angular distribution now is

$$I(\Omega) = \frac{1}{4\pi} \left[ f_0(\theta) + 2 f_1(\theta) \cos \phi + 2 f_2(\theta) \cos 2\phi \right]$$  \hspace{1cm} (13)

The $f$-functions are experimentally measurable, as they are completely determined given a set of moments $\{H\}$. Indeed one finds

$$f_M(\theta) = \sum_{L=0}^{2\ell_m} (2L + 1) H(LM) d_{LM0}(\theta)$$  \hspace{1cm} (14)

where $\ell_m$ is the maximum $\ell$ in the problem. An alternative expression for $I(\Omega)$ as a function of the partial waves

$$I(\Omega) = \frac{1}{4\pi} \left\{ \left| h_0(\theta) + \sqrt{2} h_-(\theta) \cos \phi \right|^2 + \left| \sqrt{2} h_+(\theta) \sin \phi \right|^2 \right\}$$  \hspace{1cm} (15)
where

\[
\begin{align*}
  h_0(\theta) &= \sum_{\ell=0}^{\ell_m} \sqrt{2\ell + 1} \, [\ell]_0 \, d_{00}^\ell(\theta), \quad [\ell]_0 = (-) V_{\ell 0} \\
  h_-(\theta) &= \sum_{\ell=1}^{\ell_m} \sqrt{2\ell + 1} \, [\ell]_- \, d_{10}^\ell(\theta), \quad [\ell]_- = (-) V_{\ell 1} \\
  h_+(\theta) &= \sum_{\ell=1}^{\ell_m} \sqrt{2\ell + 1} \, [\ell]_+ \, d_{10}^\ell(\theta), \quad [\ell]_+ = (+) V_{\ell 1}
\end{align*}
\]  

(16)

Note that

\[
(17)
  h_0(-\theta) = +h_0(\theta) \quad \text{and} \quad h_\pm(-\theta) = -h_\pm(\theta)
\]

Comparing the two expressions for \( I(\Omega) \), one finds

\[
\begin{align*}
  f_0(\theta) &= |h_0(\theta)|^2 + |h_-(\theta)|^2 + |h_+(\theta)|^2 \\
  f_1(\theta) &= \sqrt{2} \, \text{Re}\{h_0(\theta)h^*_-(\theta)\} \\
  f_2(\theta) &= \frac{1}{2} \left\{ |h_-(\theta)|^2 - |h_+(\theta)|^2 \right\}
\end{align*}
\]  

(18)
Define

\[ f_a(\theta) \equiv f_0(\theta) + 2f_2(\theta) = |h_0(\theta)|^2 + |\sqrt{2} h_-(\theta)|^2 \]

\[ f_b(\theta) \equiv 2f_1(\theta) = 2\text{Re}\{h_0(\theta)\sqrt{2} h^*_-(\theta)\} \]

The form of \( f_a \) and \( f_b \) suggests that one can define and find

\[ \begin{align*}
  f_a(\theta) &= |g(\theta)|^2 + |g(-\theta)|^2 \\
  f_b(\theta) &= |g(\theta)|^2 - |g(-\theta)|^2
\end{align*} \]  

where

\[ \begin{align*}
  g(\theta) &= \frac{1}{\sqrt{2}} \left[ h_0(\theta) + \sqrt{2} h_-(\theta) \right] \\
  g(-\theta) &= \frac{1}{\sqrt{2}} \left[ h_0(\theta) - \sqrt{2} h_-(\theta) \right]
\end{align*} \]

With \( u = \tan(\theta/2) \), we see that

\[ (1 + u^2)^\ell_m h_0(u) = \sum_{\ell=0}^{\ell_m} \sqrt{2\ell + 1} [\ell]_0 (1 + u^2)^{\ell_m-\ell} e_{00}^\ell(u) \]

\[ (1 + u^2)^\ell_m h_-(u) = \sum_{\ell=1}^{\ell_m} \sqrt{2\ell + 1} [\ell]_- (1 + u^2)^{\ell_m-\ell} e_{10}^\ell(u) \]

\[ (1 + u^2)^\ell_m h_+(u) = \sum_{\ell=1}^{\ell_m} \sqrt{2\ell + 1} [\ell]_+ (1 + u^2)^{\ell_m-\ell} e_{10}^\ell(u) \]
Suppose now that a set of $[\ell]$ has been found. One can then find $2\ell_m$ roots of the function

$$(1 + u^2)^\ell_m g(u) = c_0 \prod_{k=1}^{2\ell_m} (u - u_k)$$

(22)

where $u_k$'s are complex roots—these are the so-called ‘Barrelet’ zeroes—and $c_0$ is a complex constant. Next solve for $h$'s in terms of $g$’s

$$
\begin{align*}
&h_0(\theta) = \frac{1}{\sqrt{2}} \left[ g(\theta) + g(-\theta) \right] \\
&h_-(\theta) = \frac{1}{2} \left[ g(\theta) - g(-\theta) \right]
\end{align*}
$$

(23)

and find $|h_+(\theta)|^2 = |h_-(\theta)|^2 - 2f_2(\theta)$

E. Barrelet, Nuovo Cimento 8A, 331 (1972)
The ambiguity problem for $[\ell]_+$ can be dealt with by setting

$$
(1 + u^2)^\ell_m h_+(u) = c_+ u \prod_{k=1}^{\ell_m-1} (u^2 - r_k)
$$

(24)

where $r_k$’s are the complex roots in $u^2$ and $c_+$ is a complex constant. For $\ell_m > 1$, there must be in general $2^{\ell_m-2}$ solutions for the partial waves with natural-parity exchange, i.e. $[\ell]_+$.

The total number of ambiguous solutions is, for $\ell_m \leq 4$,

<table>
<thead>
<tr>
<th>$\ell_m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_a$</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>64</td>
<td>512</td>
</tr>
</tbody>
</table>
An Example with $S$-, $P$- and $D$-waves

Consider an example of the $\pi\eta$ system with $\ell_m = 2$, produced in Reaction (1).

\[
H(00) = S_0^2 + P_0^2 + P_-^2 + D_0^2 + D_-^2 + P_+^2 + D_+^2
\]
\[
H(10) = \frac{1}{\sqrt{3}} S_0 P_0 + \frac{2}{\sqrt{15}} P_0 D_0 + \frac{1}{\sqrt{5}} (P_- D_- + P_+ D_+)
\]
\[
H(11) = \frac{1}{\sqrt{6}} S_0 P_- + \frac{1}{\sqrt{10}} P_0 D_- - \frac{1}{\sqrt{30}} P_- D_0
\]
\[
H(20) = \frac{1}{\sqrt{5}} S_0 D_0 + \frac{2}{5} P_0^2 - \frac{1}{5} (P_-^2 + P_+^2) + \frac{1}{7} D_0^2 + \frac{1}{7} (D_-^2 + D_+^2)
\]
\[
H(21) = \frac{1}{\sqrt{10}} S_0 D_- + \frac{1}{5} \sqrt{\frac{3}{2}} P_0 P_- + \frac{1}{7\sqrt{2}} D_0 D_-
\]
\[
H(22) = \frac{1}{5} \sqrt{\frac{3}{2}} (P_-^2 - P_+^2) + \frac{1}{7} \sqrt{\frac{3}{2}} (D_-^2 - D_+^2)
\]

The $H(LM)$’s above are unnormalized moments.
and

\[ H(30) = \frac{3}{7\sqrt{5}} (\sqrt{3}P_0D_0 - P_-D_- - P_+D_+) \]
\[ H(31) = \frac{1}{7} \sqrt{\frac{3}{5}} (2P_0D_- + \sqrt{3}P_-D_0) \]
\[ H(32) = \frac{1}{7} \sqrt{\frac{3}{2}} (P_-D_- - P_+D_+) \]
\[ H(40) = \frac{2}{7} D_0^2 - \frac{4}{21} (D_-^2 + D_+^2) \]
\[ H(41) = \frac{1}{7} \sqrt{\frac{5}{3}} D_0D_- \]
\[ H(42) = \frac{\sqrt{10}}{21} (D_-^2 - D_+^2) \]

One should note that the moments \( H(4M) \) have contributions from the \( D \)-wave only, while the moments \( H(3M) \) result from interference between \( P \)- and \( D \)-waves.

\[ d_{\ell m', m}^\ell (\theta) = \left( \cos \frac{\theta}{2} \right)^{2\ell} e_{\ell m', m}^\ell \left( \tan \frac{\theta}{2} \right) \rightarrow e_{\ell m', m}^\ell (u) = (1 + u^2)^\ell d_{\ell m', m}^\ell (u) \]

where \( u = \tan \theta/2 \). The \( e_{\ell m', m}^\ell (u) \) functions are polynomials of order \( 2\ell \) in \( u \).
Suppose now that one has found a set of solutions \( \{ S_0, P_0, P_-, D_0, D_- \} \) for unnatural-parity exchange and \( \{ P_+, D_+ \} \) for natural-parity exchange. It is helpful to write down the \( h \)'s explicitly:

\[
\begin{align*}
(1 + u^2)^2 h_0(u) &= S_0 (1 + u^2)^2 + \sqrt{3} P_0 (1 - u^4) + \sqrt{5} D_0 (1 - 4u^2 + u^4) \\
\sqrt{2} (1 + u^2)^2 h_-(u) &= -2u \left[ \sqrt{3} P_- (1 + u^2) + \sqrt{15} D_- (1 - u^2) \right] \\
\sqrt{2} (1 + u^2)^2 h_+(u) &= -2u \left[ \sqrt{3} P_+ (1 + u^2) + \sqrt{15} D_+ (1 - u^2) \right]
\end{align*}
\]

The last equation above shows that there are no ambiguities for the partial waves \( P_+ \) and \( D_+ \), since the expression inside the square bracket is linear in \( u^2 \). On the other hand, from the first two equations, one finds that the function \( g(u) \) is given by

\[
G(u) \equiv \sqrt{2} (1 + u^2)^2 g(u) = S_0 (1 + u^2)^2 + \sqrt{3} P_0 (1 - u^4) + \sqrt{5} D_0 (1 - 4u^2 + u^4) \\
- 2\sqrt{3} P_- (u + u^3) - 2\sqrt{15} D_- (u - u^3)
\]

which is a polynomial of order 4 in \( u \) and thus gives rise to the ambiguities in the unnatural-parity partial-waves through the Barrelet zeroes.
One may write

\[ G(u) = a_4 u^4 - a_3 u^3 + a_2 u^2 - a_1 u + a_0 \]  

with

\[
\begin{align*}
  a_4 &= S_0 - \sqrt{3}P_0 + \sqrt{5}D_0 \\
  a_3 &= 2\sqrt{3}(P_- - \sqrt{5}D_-) \\
  a_2 &= 2S_0 - 4\sqrt{5}D_0 \\
  a_1 &= 2\sqrt{3}(P_- + \sqrt{5}D_-) \\
  a_0 &= S_0 + \sqrt{3}P_0 + \sqrt{5}D_0
\end{align*}
\]

The inverse is

\[
\begin{align*}
  6S_0 &= 2a_0 + a_2 + 2a_4 \\
  2\sqrt{3}P_0 &= a_0 - a_4 \\
  6\sqrt{5}D_0 &= a_0 - a_2 + a_4 \\
  4\sqrt{3}P_- &= a_1 + a_3 \\
  4\sqrt{15}D_- &= a_1 - a_3
\end{align*}
\]
Since $G(u)$ is a 4th-order polynomial in $u$ with 4 complex roots \( \{u_1, u_2, u_3, u_4\} \), it is given by

\[
G(u) = a_4 u^4 - a_3 u^3 + a_2 u^2 - a_1 u + a_0
\]

\[= a_4 (u - u_1)(u - u_2)(u - u_3)(u - u_4)\]  

(32)

so that

\[
a_3 = a_4 (u_1 + u_2 + u_3 + u_4)
\]

\[
a_2 = a_4 (u_1 u_2 + u_1 u_3 + u_1 u_4 + u_2 u_3 + u_2 u_4 + u_3 u_4)
\]

(33)

\[
a_1 = a_4 (u_1 u_2 u_3 + u_2 u_3 u_4 + u_3 u_4 u_1 + u_4 u_1 u_2)
\]

\[
a_0 = a_4 (u_1 u_2 u_3 u_4)
\]

Finally, the partial waves can be expressed in terms of the roots or the Barrelet zeroes:

\[
6S_0 = a_4 (2u_1 u_2 u_3 u_4 + u_1 u_2 + u_1 u_3 + u_1 u_4 + u_2 u_3 + u_2 u_4 + u_3 u_4 + 2)
\]

\[2\sqrt{3}P_0 = a_4 (u_1 u_2 u_3 u_4 - 1)\]

\[
6\sqrt{5}D_0 = a_4 (u_1 u_2 u_3 u_4 - u_1 u_2 - u_1 u_3 - u_1 u_4 - u_2 u_3 - u_2 u_4 - u_3 u_4 + 1)
\]

\[4\sqrt{3}P_- = a_4 (u_1 u_2 u_3 + u_2 u_3 u_4 + u_3 u_4 u_1 + u_4 u_1 u_2 + u_1 + u_2 + u_3 + u_4)\]

\[4\sqrt{15}D_- = a_4 (u_1 u_2 u_3 + u_2 u_3 u_4 + u_3 u_4 u_1 + u_4 u_1 u_2 - u_1 - u_2 - u_3 - u_4)\]
There should be in general 8 ambiguous solutions involving the partial waves $S_0$, $P_0$, $P_-$, $D_0$ and $D_-$. The 8 solutions are enumerated below in two columns:

\begin{align*}
\{u_1, u_2, u_3^*, u_4^*\} & \quad \{u_1, u_2, u_3^*, u_4^*\} \\
\{u_1, u_2, u_3, u_4\} & \quad \{u_1, u_2, u_3, u_4\} \\
\{u_1, u_2, u_3, u_4^*\} & \quad \{u_1, u_2, u_3, u_4^*\} \\
\{u_1, u_2, u_3^*, u_4\} & \quad \{u_1, u_2, u_3^*, u_4\} \\
\{u_1, u_2, u_3^*, u_4^*\} & \quad \{u_1, u_2, u_3^*, u_4^*\} \\
\{u_1, u_2^*, u_3, u_4\} & \quad \{u_1, u_2^*, u_3, u_4\} \\
\{u_1, u_2^*, u_3, u_4^*\} & \quad \{u_1, u_2^*, u_3, u_4^*\} \\
\{u_1, u_2^*, u_3^*, u_4\} & \quad \{u_1, u_2^*, u_3^*, u_4\} \\
\{u_1, u_2^*, u_3^*, u_4^*\} & \quad \{u_1, u_2^*, u_3^*, u_4^*\} \\
\end{align*}

(34)

The first column results from a procedure in which $u_1$ is left invariant and the remaining three roots $u_2$, $u_3$ and $u_4$ are allowed to undergo complex conjugation—one sees that there are $2^3 = 8$ ways of doing this.
Reaction: $\pi^- p \rightarrow \eta \pi^- p$ at 18 GeV/c, $\eta \rightarrow \gamma\gamma$, $\sigma(\eta \rightarrow \gamma\gamma) \sim 30$ MeV

$\sim 47,200$ events
\begin{align*}
\{ & M(P_+) = 1370 \pm 16^{+50}_{-30} \\
& \Gamma(P_+) = 385 \pm 40^{+65}_{-105} \}
\end{align*}

PRL \textbf{79}, 1630 (1997)
PRD \textbf{60}, 092001 (1999)
Two Identical Particles

Reaction:

\[ \pi^- + p \rightarrow (\eta\eta) + (\pi^- p) \]  

Let \( \ell_m \) be the maximum spin present in a given \( \eta\eta \) mass bin. It is easy to show that the number of independent non-zero \( H \)'s are

\[ N^e = 3\ell_m + 1 \]

\( N^e \) as a function of \( \ell_m \) is given below as a table:

<table>
<thead>
<tr>
<th>( \ell_m )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N^e )</td>
<td>1</td>
<td>7</td>
<td>13</td>
<td>19</td>
</tr>
</tbody>
</table>
Consider a system which consists of two identical spinless particles with $\ell_m = 4$. This problem requires a total of 12 parameters to be fitted, consisting of $S_0$, $D_0$, $D_-$, $G_0$, $G_-$, $D_+$ and $G_+$ which are complex in general. As in previous examples, one may set one wave to be real in each group of a given naturality, e.g. $S_0$ and $D_+$.

The relevant moments are $H(00)$, $H(20)$, $H(21)$, $H(22)$, $H(40)$, $H(41)$, $H(42)$, $H(60)$, $H(61)$, $H(62)$, $H(80)$, $H(81)$, $H(82)$.

\begin{equation}
16\sqrt{70} H(22) - 24\sqrt{42} H(42) + 182 H(62) - 119\sqrt{3} H(82) = 0
\end{equation}

Introduce a new function $\epsilon^\ell_m(u)$

\begin{equation}
\epsilon^\ell_m(u) = \frac{1}{u^\ell} \epsilon^\ell_m 0(u)
\end{equation}

and a new variable $v = 2 \cot \theta$ so that

\begin{equation}
v = \frac{1}{u} - u
\end{equation}

The $\epsilon^\ell_m(v)$’s, as functions of $v$, are polynomials of order $\ell$=even in $v$.

They are the functions needed to search for the ambiguities when the two particles are identical.
\begin{align*}
\varepsilon_0^2(u) &= v^2 - 2 \\
\varepsilon_1^2(u) &= -\sqrt{6} v \\
\varepsilon_0^4(u) &= v^4 - 12v^2 + 6 \\
\varepsilon_1^4(u) &= -2\sqrt{5} v(v^2 - 3) \\
\varepsilon_0^6(u) &= v^6 - 30v^4 + 90v^2 - 20 \\
\varepsilon_1^6(u) &= -\sqrt{42} v(v^4 - 10v^2 + 10)
\end{align*}

The functions \( \varepsilon_m^\ell(u) \) are new to physics—to the best of my knowledge.
The ambiguous solutions are obtained by examining

\[ G_-(v) = S_0 \left( \frac{1}{u} + u \right)^4 + \sqrt{5}D_0 \left( \frac{1}{u} + u \right)^2 \varepsilon_0^2(u) + 3G_0 \varepsilon_0^4(u) \]

\[ + \sqrt{10}D_- \left( \frac{1}{u} + u \right)^2 \varepsilon_1^2(u) + 3\sqrt{2}G_- \varepsilon_1^4(u) \]

(41)

\[ = a_4 v^4 - a_3 v^3 + a_2 v^2 - a_1 v + a_0 \]

\[ = a_4 (v - v_1)(v - v_2)(v - v_3)(v - v_4) \]

Let \( v_1, v_2, v_3, v_4 \) be the roots of this polynomial. Then

\[ a_3 = a_4 (v_1 + v_2 + v_3 + v_4) \]

\[ a_2 = a_4 (v_1v_2 + v_1v_3 + v_1v_4 + v_2v_3 + v_2v_4 + v_3v_4) \]

(42)

\[ a_1 = a_4 (v_1v_2v_3 + v_2v_3v_4 + v_3v_4v_1 + v_4v_1v_2) \]

\[ a_0 = a_4 (v_1v_2v_3v_4) \]
The partial waves for unnatural-parity exchange are obtained by

\[
30S_0 = a_0 + a_2 + 6a_4 \\
42\sqrt{5}D_0 = -2a_0 + a_2 + 24a_4 \\
210G_0 = a_0 - 4a_2 + 16a_4 \\
14\sqrt{15}D_- = a_1 + 3a_3 \\
42\sqrt{10}G_- = -a_1 + 4a_3
\]

(43)

There are 8 ambiguous solutions. The partial waves with natural-parity exchanges are

\[
|G_+|^2 = |G_-|^2 - \frac{2431}{42\sqrt{35}}H(82) \\
|D_+|^2 = |D_-|^2 - \frac{7\sqrt{2}}{\sqrt{3}}H(22) + \frac{221}{4\sqrt{35}} \left[ \frac{17}{14}H(82) - \frac{11}{17\sqrt{3}}H(62) \right] \\
2\text{Re}\{D_+ G_+^*\} = 2\text{Re}\{D_- G_-^*\} + \frac{143}{2\sqrt{70}} \left[ \frac{17}{14\sqrt{3}}H(82) - H(62) \right]
\]

(44)
Thank you for your patience and hospitality...