Hadron Spectroscopy

Mathematical Techniques

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Amplitude Analysis for Two-pseudoscalar Systems

Primary References:

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General Angular Distributions

Consider the following reaction

(1) $$\pi^- + p \to \pi^-\eta + p$$

In the Jackson frame, the amplitudes may be expanded in terms of the partial waves for the $\pi\eta$ system:

(2) $$U_k(\Omega) = \sum_{\ell m} V_{\ell m k} A_{\ell m}(\Omega), \qquad \Omega = (\theta, \phi)$$

where $V_{\ell m k}$ stands for the production amplitude for a state $|\ell m\rangle$ and k represents the spin degrees of freedom for the initial and final nucleons ($k = 1, 2$ for spin-flip and spin-nonflip amplitudes). $A_{\ell m}(\Omega)$ is the decay amplitude given by

(3) $$A_{\ell m}(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}}\, D^{\ell\,*}_{m0}(\phi, \theta, 0) = Y^m_\ell(\Omega)$$

where the angles $\Omega = (\theta, \phi)$ describe the direction of the η in the Jackson frame. The angular distribution is given by

(4) $$I(\Omega) = \sum_k |U_k(\Omega)|^2$$

The eigenstates of this reflection operator are

$$|\epsilon\ell m\rangle = \theta(m)\Big\{|\ell m\rangle - \epsilon(-)^m|\ell - m\rangle\Big\} \tag{5}$$

where

$$\begin{aligned} \theta(m) &= \frac{1}{\sqrt{2}}, \quad m > 0 \\ &= \frac{1}{2}, \quad m = 0 \\ &= 0, \quad m < 0 \end{aligned} \tag{6}$$

For a positive reflectivity, the $m = 0$ states are not allowed, i.e.

$$|\epsilon\ell 0\rangle = 0, \quad \text{if} \quad \epsilon = + \tag{7}$$

The reflectivity quantum number ϵ has been defined so that it coincides with the naturality of the exchanged particle in Reaction (1).

$${}^{\epsilon}U_k(\Omega) = \sum_{\ell m} {}^{\epsilon}V_{\ell m k} \sqrt{\frac{2\ell+1}{4\pi}}\ {}^{\epsilon}D^{\ell\ *}_{m0}(\phi, \theta, 0) \tag{8}$$

and the resulting angular distribution is

(9) $$I(\Omega) = \sum_{\epsilon k} |{}^{\epsilon}U_k(\Omega)|^2$$

The angular distribution may be expanded in terms of the moments $H(LM)$ via

(10) $$I(\Omega) = \sum_{LM} \left(\frac{2L+1}{4\pi}\right) H(LM) D^{L\,*}_{M0}(\phi,\theta,0)$$

so that

(11) $$H(LM) = \int \mathrm{d}\Omega\; I(\Omega)\; D^{L}_{M0}(\phi,\theta,0) \rightarrow \boxed{\text{measurable experimentally}}$$

$$H(LM) = \sum_{\epsilon k} \sum_{\substack{\ell m \\ \ell' m'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} {}^{\epsilon}V_{\ell m k}\, {}^{\epsilon}V^{*}_{\ell' m' k}\, {}^{\epsilon}b(\ell' m' L M \ell m)\; (\ell' 0 L 0|\ell 0)$$

where a new function ${}^{\epsilon}b$ is a sum of Clebsch-Gordan coefficients:

(12) $$\begin{aligned} {}^{\epsilon}b(\ell' m' L M \ell m) = \theta(m')\theta(m)\Big[&(\ell' m' L M|\ell m) + (-)^{M}\,(\ell' m' L\, -M|\ell m) \\ &- \epsilon\,(-)^{m'}\,(\ell'\, -m' L M|\ell m) - \epsilon\,(-)^{m}\,(\ell' m' L M|\ell\, -m)\Big] \end{aligned}$$

Two assumptions: (1) ${}^{\epsilon}V_{\ell m k} = 0$ if $m \geq 2$ and (2) rank=1, i.e. k=1.
The angular distribution now is

(13) $$I(\Omega) = \frac{1}{4\pi}\left[f_0(\theta) + 2f_1(\theta)\cos\phi + 2f_2(\theta)\cos 2\phi\right]$$

The f-functions are experimentally measurable, as they are completely determined given a set of moments $\{H\}$. Indeed one finds

(14) $$f_M(\theta) = \sum_{L=0}^{2\ell_m}(2L+1)H(LM)d^L_{M0}(\theta)$$

where ℓ_m is the maximum ℓ in the problem. An alternative expression for $I(\Omega)$ as a function of the partial waves

(15) $$I(\Omega) = \frac{1}{4\pi}\left\{\left|h_0(\theta) + \sqrt{2}h_-(\theta)\cos\phi\right|^2 + \left|\sqrt{2}h_+(\theta)\sin\phi\right|^2\right\}$$

where

(16)
$$\begin{cases} h_0(\theta) = \sum_{\ell=0}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_0\, d^{\ell}_{00}(\theta), & [\,\ell\,]_0 = {}^{(-)}V_{\ell 0} \\ h_-(\theta) = \sum_{\ell=1}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_-\, d^{\ell}_{10}(\theta), & [\,\ell\,]_- = {}^{(-)}V_{\ell 1} \\ h_+(\theta) = \sum_{\ell=1}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_+\, d^{\ell}_{10}(\theta), & [\,\ell\,]_+ = {}^{(+)}V_{\ell 1} \end{cases}$$

Note that

(17)
$$h_0(-\theta) = +h_0(\theta) \quad \text{and} \quad h_\pm(-\theta) = -h_\pm(\theta)$$

Comparing the two expressions for $I(\Omega)$, one finds

(18)
$$\begin{cases} f_0(\theta) = |h_0(\theta)|^2 + |h_-(\theta)|^2 + |h_+(\theta)|^2 \\ f_1(\theta) = \sqrt{2}\,\mathrm{Re}\{h_0(\theta)h^*_-(\theta)\} \\ f_2(\theta) = \frac{1}{2}\Big\{|h_-(\theta)|^2 - |h_+(\theta)|^2\Big\} \end{cases}$$

Define

(19)
$$f_a(\theta) \equiv f_0(\theta) + 2f_2(\theta) = |h_0(\theta)|^2 + |\sqrt{2}\,h_-(\theta)|^2$$
$$f_b(\theta) \equiv 2f_1(\theta) = 2\mathrm{Re}\{h_0(\theta)\sqrt{2}\,h_-^*(\theta)\}$$

The form of f_a and f_b suggests that one can define and find

(20)
$$\begin{cases} f_a(\theta) = |g(\theta)|^2 + |g(-\theta)|^2 \\ f_b(\theta) = |g(\theta)|^2 - |g(-\theta)|^2 \end{cases}$$
where
$$\begin{cases} g(\theta) = \frac{1}{\sqrt{2}}\left[h_0(\theta) + \sqrt{2}\,h_-(\theta)\right] \\ g(-\theta) = \frac{1}{\sqrt{2}}\left[h_0(\theta) - \sqrt{2}\,h_-(\theta)\right] \end{cases}$$

With $u = \tan(\theta/2)$, we see that

(21)
$$(1+u^2)^{\ell_m}\,h_0(u) = \sum_{\ell=0}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_0\,(1+u^2)^{\ell_m-\ell}\,e_{00}^{\ell}(u)$$
$$(1+u^2)^{\ell_m}\,h_-(u) = \sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_-\,(1+u^2)^{\ell_m-\ell}\,e_{10}^{\ell}(u)$$
$$(1+u^2)^{\ell_m}\,h_+(u) = \sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_+\,(1+u^2)^{\ell_m-\ell}\,e_{10}^{\ell}(u)$$

Suppose now that a set of $[\,\ell\,]$ has been found. One can then find $2\ell_m$ roots of the function

$$(1+u^2)^{\ell_m}\, g(u) = c_0 \prod_{k=1}^{2\ell_m} (u-u_k) \tag{22}$$

where u_k's are complex roots—these are the so-called 'Barrelet' zeroes—and c_0 is a complex constant. Next solve for h's in terms of g's

$$\begin{cases} h_0(\theta) = \dfrac{1}{\sqrt{2}}\Big[g(\theta)+g(-\theta)\Big] \\ h_-(\theta) = \dfrac{1}{2}\Big[g(\theta)-g(-\theta)\Big] \end{cases} \tag{23}$$

and find $|h_+(\theta)|^2 = |h_-(\theta)|^2 - 2f_2(\theta)$

E. Barrelet, Nuovo Cimento **8A**, 331 (1972)

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The ambiguity problem for $[\,\ell\,]_+$ can be dealt with by setting

$$(1+u^2)^{\ell_m}\, h_+(u) = c_+\, u \prod_{k=1}^{\ell_m-1} (u^2 - r_k) \tag{24}$$

where r_k's are the complex roots in u^2 and c_+ is a complex constant. For $\ell_m > 1$, there must be in general 2^{ℓ_m-2} solutions for the partial waves with natural-parity exchange, i.e. $[\,\ell\,]_+$.

The total number of ambiguous solutions is, for $\ell_m \leq 4$,

ℓ_m	0	1	2	3	4
N_a	1	2	8	64	512

An Example with S-, P- and D-waves

Consider an example of the $\pi\eta$ system with $\ell_m = 2$, produced in Reaction (1).

(25)

$$H(00) = S_0^2 + P_0^2 + P_-^2 + D_0^2 + D_-^2 + P_+^2 + D_+^2$$

$$H(10) = \frac{1}{\sqrt{3}} S_0 P_0 + \frac{2}{\sqrt{15}} P_0 D_0 + \frac{1}{\sqrt{5}} (P_- D_- + P_+ D_+)$$

$$H(11) = \frac{1}{\sqrt{6}} S_0 P_- + \frac{1}{\sqrt{10}} P_0 D_- - \frac{1}{\sqrt{30}} P_- D_0$$

$$H(20) = \frac{1}{\sqrt{5}} S_0 D_0 + \frac{2}{5} P_0^2 - \frac{1}{5} (P_-^2 + P_+^2) + \frac{2}{7} D_0^2 + \frac{1}{7} (D_-^2 + D_+^2)$$

$$H(21) = \frac{1}{\sqrt{10}} S_0 D_- + \frac{1}{5}\sqrt{\frac{3}{2}} P_0 P_- + \frac{1}{7\sqrt{2}} D_0 D_-$$

$$H(22) = \frac{1}{5}\sqrt{\frac{3}{2}} (P_-^2 - P_+^2) + \frac{1}{7}\sqrt{\frac{3}{2}} (D_-^2 - D_+^2)$$

The $H(LM)$'s above are unnormalized moments.

and

(26)
$$\begin{aligned}
H(30) &= \frac{3}{7\sqrt{5}}(\sqrt{3}P_0D_0 - P_-D_- - P_+D_+)\\
H(31) &= \frac{1}{7}\sqrt{\frac{3}{5}}(2P_0D_- + \sqrt{3}P_-D_0)\\
H(32) &= \frac{1}{7}\sqrt{\frac{3}{2}}(P_-D_- - P_+D_+)\\
H(40) &= \frac{2}{7}D_0^2 - \frac{4}{21}(D_-^2 + D_+^2)\\
H(41) &= \frac{1}{7}\sqrt{\frac{5}{3}}D_0D_-\\
H(42) &= \frac{\sqrt{10}}{21}(D_-^2 - D_+^2)
\end{aligned}$$

One should note that the moments $H(4M)$ have contributions from the D-wave only, while the moments $H(3M)$ result from interference between P- and D-waves.

(27) $$d^{\ell}_{m'm}(\theta) = \left(\cos\frac{\theta}{2}\right)^{2\ell} e^{\ell}_{m'm}\left(\tan\frac{\theta}{2}\right) \quad \rightarrow \quad e^{\ell}_{m'm}(u) = (1+u^2)^{\ell}\, d^{\ell}_{m'm}(u)$$

where $u = \tan\theta/2$. The $e^{\ell}_{m'm}(u)$ functions are polynomials of order 2ℓ in u.

Suppose now that one has found a set of solutions $\{S_0, P_0, P_-, D_0, D_-\}$ for unnatural-parity exchange and $\{P_+, D_+\}$ for natural-parity exchange. It is helpful to write down the h's explicitly:

$$\left\{\begin{aligned} (1+u^2)^2\, h_0(u) &= S_0\,(1+u^2)^2 + \sqrt{3}P_0\,(1-u^4) + \sqrt{5}D_0\,(1-4u^2+u^4) \\ \sqrt{2}(1+u^2)^2\, h_-(u) &= -2u\left[\sqrt{3}P_-\,(1+u^2) + \sqrt{15}D_-\,(1-u^2)\right] \\ \sqrt{2}(1+u^2)^2\, h_+(u) &= -2u\left[\sqrt{3}P_+\,(1+u^2) + \sqrt{15}D_+\,(1-u^2)\right] \end{aligned}\right. \tag{28}$$

The last equation above shows that there are no ambiguities for the partial waves P_+ and D_+, since the expression inside the square bracket is linear in u^2. On the other hand, from the first two equations, one finds that the function $g(u)$ is given by

$$\begin{aligned} G(u) \equiv \sqrt{2}(1+u^2)^2\, g(u) &= S_0\,(1+u^2)^2 + \sqrt{3}P_0\,(1-u^4) + \sqrt{5}D_0\,(1-4u^2+u^4) \\ &\quad - 2\sqrt{3}P_-\,(u+u^3) - 2\sqrt{15}D_-\,(u-u^3) \end{aligned}$$

which is a polynomial of order 4 in u and thus gives rise to the ambiguities in the unnatural-parity partial-waves through the Barrelet zeroes.

One may write

$$G(u) = a_4\, u^4 - a_3\, u^3 + a_2\, u^2 - a_1\, u + a_0 \tag{29}$$

with

$$\begin{cases} a_4 = S_0 - \sqrt{3}P_0 + \sqrt{5}D_0 \\ a_3 = 2\sqrt{3}(P_- - \sqrt{5}D_-) \\ a_2 = 2S_0 - 4\sqrt{5}D_0 \\ a_1 = 2\sqrt{3}(P_- + \sqrt{5}D_-) \\ a_0 = S_0 + \sqrt{3}P_0 + \sqrt{5}D_0 \end{cases} \tag{30}$$

The inverse is

$$\begin{cases} 6S_0 = 2a_0 + a_2 + 2a_4 \\ 2\sqrt{3}P_0 = a_0 - a_4 \\ 6\sqrt{5}D_0 = a_0 - a_2 + a_4 \\ 4\sqrt{3}P_- = a_1 + a_3 \\ 4\sqrt{15}D_- = a_1 - a_3 \end{cases} \tag{31}$$

Since $G(u)$ is a 4th-order polynomial in u with 4 complex roots $\{u_1, u_2, u_3, u_4\}$, it is given by

(32)
$$G(u) = a_4\, u^4 - a_3\, u^3 + a_2\, u^2 - a_1\, u + a_0$$
$$= a_4(u - u_1)(u - u_2)(u - u_3)(u - u_4)$$

so that

(33)
$$a_3 = a_4(u_1 + u_2 + u_3 + u_4)$$
$$a_2 = a_4(u_1 u_2 + u_1 u_3 + u_1 u_4 + u_2 u_3 + u_2 u_4 + u_3 u_4)$$
$$a_1 = a_4(u_1 u_2 u_3 + u_2 u_3 u_4 + u_3 u_4 u_1 + u_4 u_1 u_2)$$
$$a_0 = a_4(u_1 u_2 u_3 u_4)$$

Finally, the partial waves can be expressed in terms of the roots or the Barrelet zeroes:

$$6S_0 = a_4(2u_1u_2u_3u_4 + u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4 + 2)$$
$$2\sqrt{3}P_0 = a_4(u_1u_2u_3u_4 - 1)$$
$$6\sqrt{5}D_0 = a_4(u_1u_2u_3u_4 - u_1u_2 - u_1u_3 - u_1u_4 - u_2u_3 - u_2u_4 - u_3u_4 + 1)$$
$$4\sqrt{3}P_- = a_4(u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2 + u_1 + u_2 + u_3 + u_4)$$
$$4\sqrt{15}D_- = a_4(u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2 - u_1 - u_2 - u_3 - u_4)$$

There should be in general 8 ambiguous solutions involving the partial waves S_0, P_0, P_-, D_0 and D_-. The 8 solutions are enumerated below in two columns:

$$\begin{array}{cc}
\{u_1, u_2, u_3^*, u_4^*\} & \{u_1, u_2, u_3^*, u_4\} \\
\{u_1, u_2, u_3, u_4\} & \{u_1, u_2, u_3, u_4\} \\
\{u_1, u_2, u_3, u_4^*\} & \{u_1^*, u_2, u_3, u_4\} \\
\{u_1, u_2, u_3^*, u_4\} & \{u_1, u_2^*, u_3, u_4\} \\
\{u_1, u_2, u_3^*, u_4^*\} & \{u_1, u_2, u_3^*, u_4\} \\
\{u_1, u_2^*, u_3, u_4\} & \{u_1, u_2, u_3, u_4^*\} \\
\{u_1, u_2^*, u_3, u_4^*\} & \{u_1^*, u_2^*, u_3, u_4\} \\
\{u_1, u_2^*, u_3^*, u_4\} & \{u_1^*, u_2, u_3^*, u_4\} \\
\{u_1, u_2^*, u_3^*, u_4^*\} & \{u_1^*, u_2, u_3, u_4^*\}
\end{array} \tag{34}$$

The first column results from a procedure in which u_1 is left invariant and the remaining three roots u_2, u_3 and u_4 are allowed to undergo complex conjugation—one sees that there are 2^3=8 ways of doing this.

BNL-E852

Reaction: $\pi^- p \to \eta\pi^- p$ at 18 GeV/c, $\eta \to \gamma\gamma$, $\sigma(\eta \to \gamma\gamma) \sim 30$ MeV
$\sim 47\,200$ events

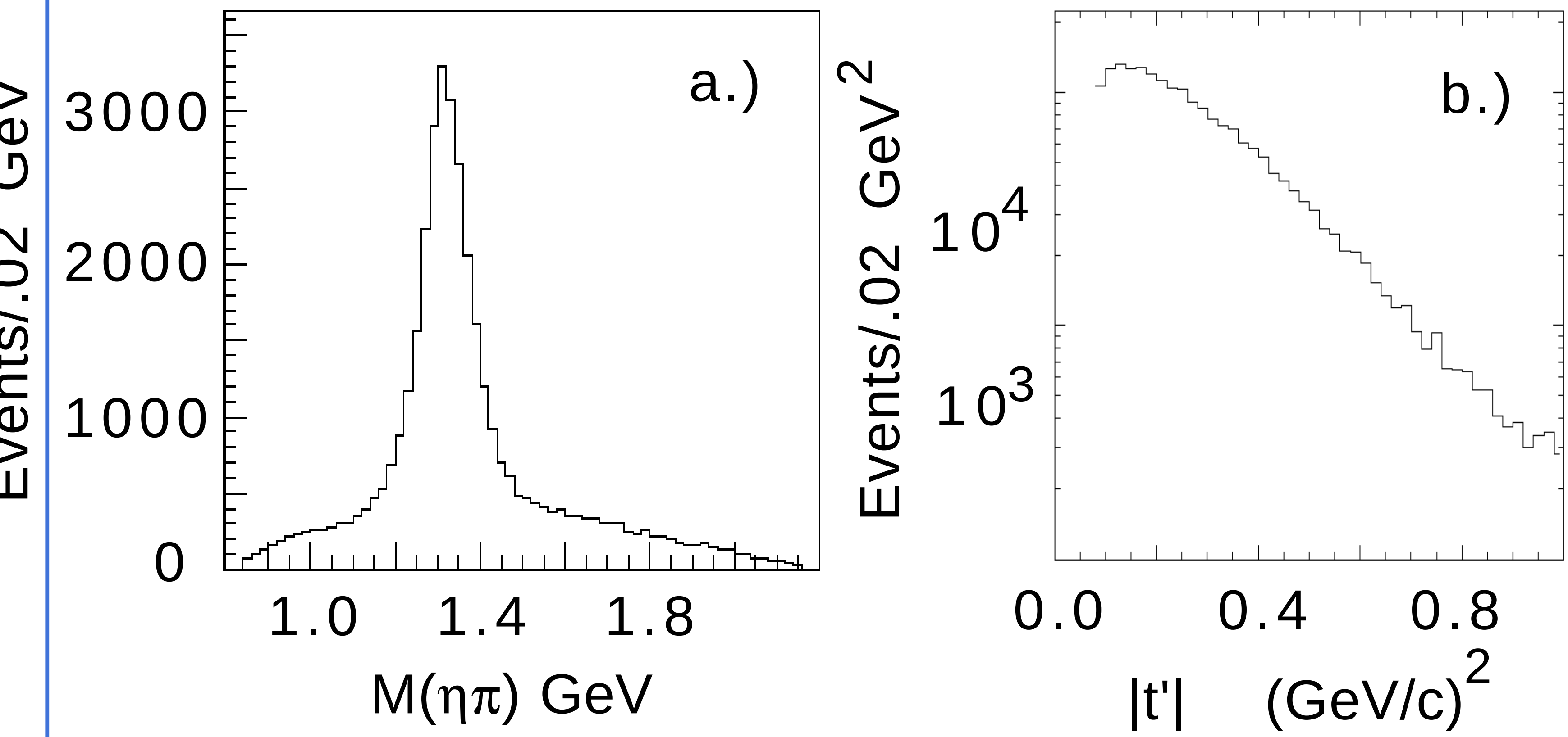
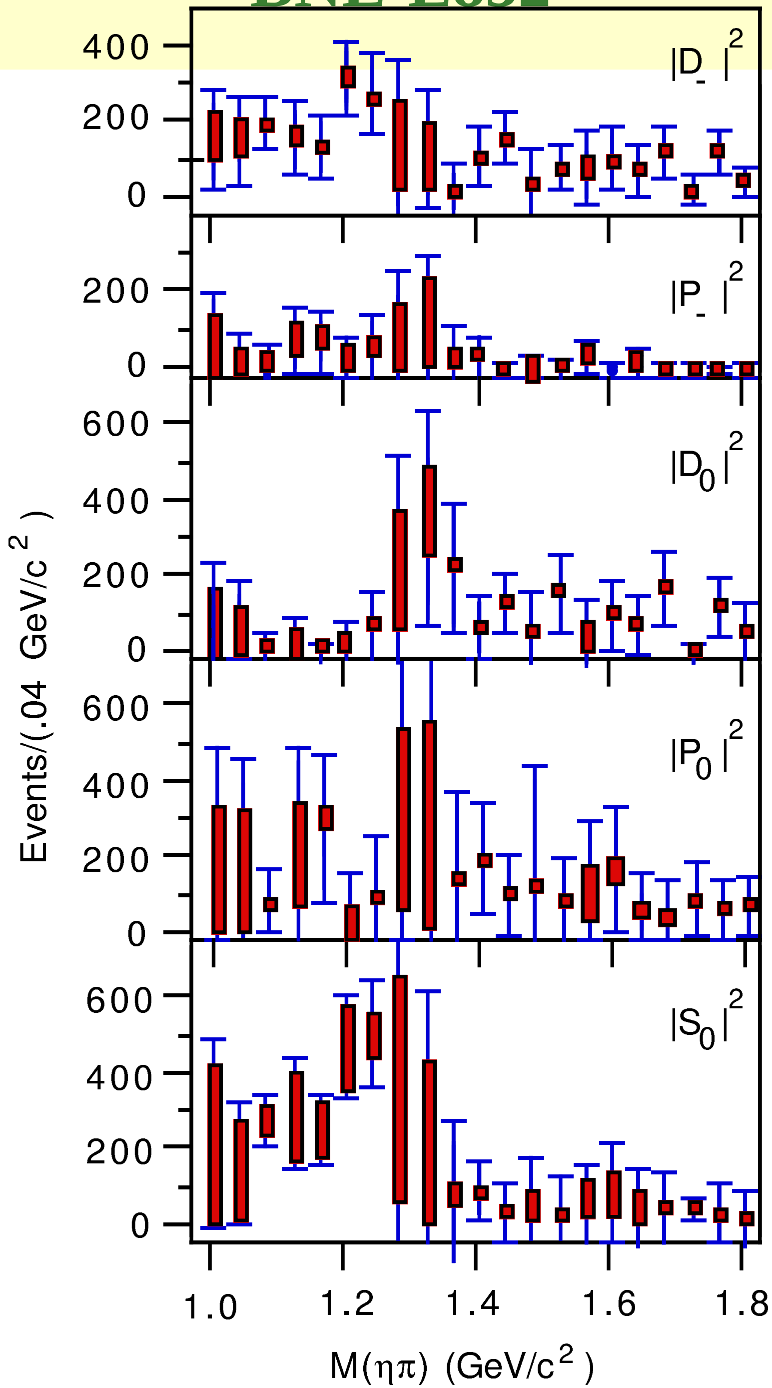
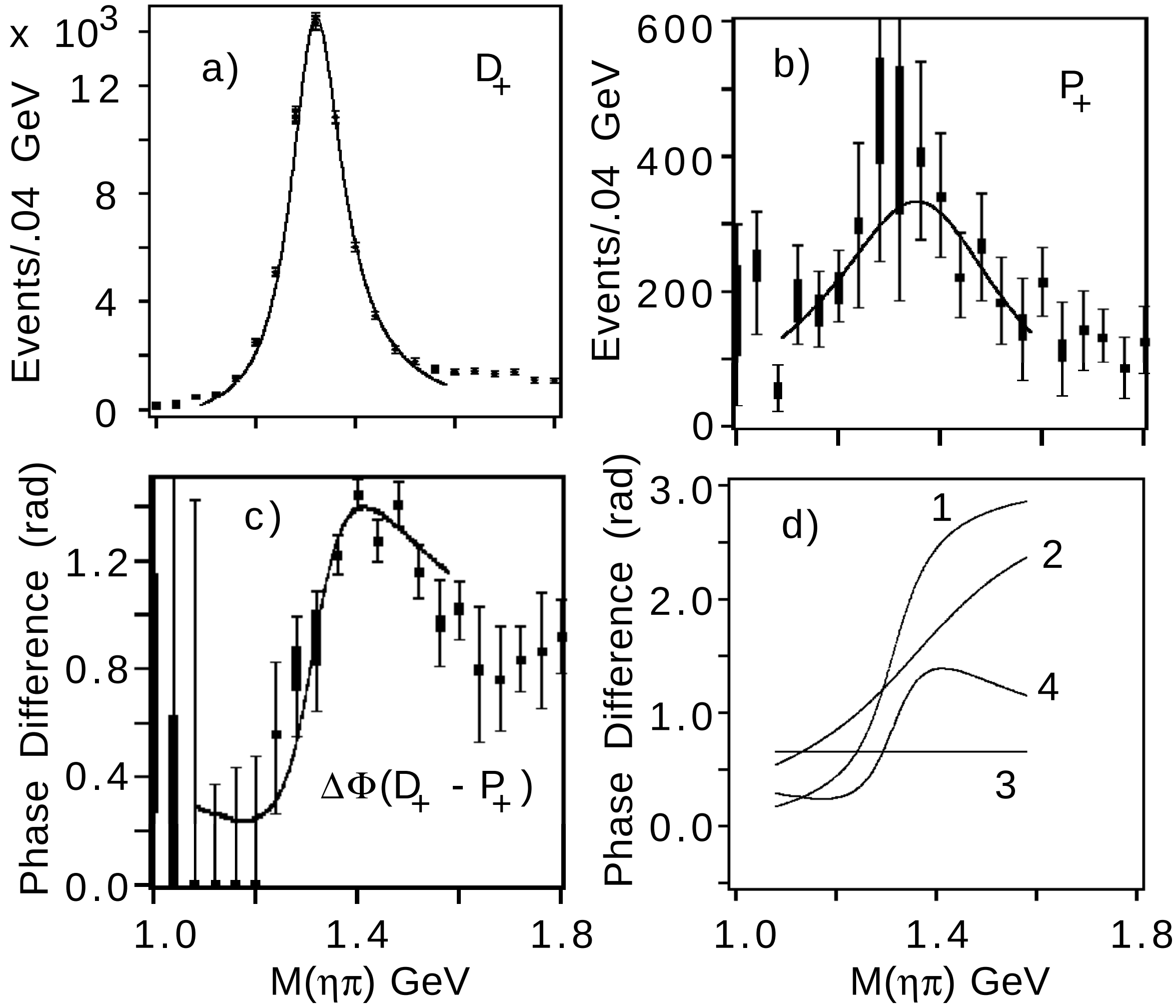


Figure 1

BNL-E852





$$\begin{cases} M(P_+) = 1370 \pm 16 {}^{+50}_{-30} \\ \Gamma(P_+) = 385 \pm 40 {}^{+\ 65}_{-105} \end{cases}$$

PRL 79, 1630 (1997)
PRD 60, 092001 (1999)

Two Identical Paticles

Reaction:

(35) $$\pi^- + p \to (\eta\eta) + (\pi^- p)$$

Let ℓ_m be the maximum spin present in a given $\eta\eta$ mass bin. It is easy to show that the number of independent non-zero H's are

(36) $$N^{\mathrm{e}} = 3\ell_m + 1$$

N^{e} as a function of ℓ_m is given below as a table:

ℓ_m	0	2	4	6
N^{e}	1	7	13	19

Consider a system which consists of two identical spinless particles with $\ell_m = 4$. This problem requires a total of 12 parameters to be fitted, consisting of S_0, D_0, D_-, G_0, G_-, D_+ and G_+ which are complex in general. As in previous examples, one may set one wave to be real in each group of a given naturality, e.g. S_0 and D_+.

The relevant moments are $H(00)$, $H(20)$, $H(21)$, $H(22)$, $H(40)$, $H(41)$, $H(42)$, $H(60)$, $H(61)$, $H(62)$, $H(80)$, $H(81)$, $H(82)$.

(37) $$16\sqrt{70}\,H(22) - 24\sqrt{42}\,H(42) + 182\,H(62) - 119\sqrt{3}\,H(82) = 0$$

Introduce a new function $\varepsilon^{\ell}_{m}(u)$

(38) $$\varepsilon^{\ell}_{m}(u) = \frac{1}{u^{\ell}}\, e^{\ell}_{m\,0}(u)$$

and a new variable $v = 2\cot\theta$ so that

(39) $$v = \frac{1}{u} - u$$

The $\varepsilon^{\ell}_{m}(v)$'s, as functions of v, are polynomials of order ℓ=even in v.

They are the functions needed to search for the ambiguities

when the two particles are identical.

$$
\begin{aligned}
\varepsilon_0^2(u) &= v^2 - 2 \\
\varepsilon_1^2(u) &= -\sqrt{6}\, v \\
\varepsilon_0^4(u) &= v^4 - 12v^2 + 6 \\
\varepsilon_1^4(u) &= -2\sqrt{5}\, v(v^2 - 3) \\
\varepsilon_0^6(u) &= v^6 - 30v^4 + 90v^2 - 20 \\
\varepsilon_1^6(u) &= -\sqrt{42}\, v(v^4 - 10v^2 + 10)
\end{aligned} \tag{40}
$$

The functions $\varepsilon_m^\ell(u)$ are new to physics—to the best of my knowledge.

The amiguous solutions are obtained by examining

$$\begin{aligned}\mathcal{G}_-(v) &= S_0\left(\frac{1}{u}+u\right)^4+\sqrt{5}D_0\left(\frac{1}{u}+u\right)^2\varepsilon_0^2(u)+3G_0\,\varepsilon_0^4(u)\\ &\qquad +\sqrt{10}D_-\left(\frac{1}{u}+u\right)^2\varepsilon_1^2(u)+3\sqrt{2}G_-\,\varepsilon_1^4(u)\\ &= a_4\,v^4-a_3\,v^3+a_2\,v^2-a_1\,v+a_0\\ &= a_4(v-v_1)(v-v_2)(v-v_3)(v-v_4)\end{aligned} \tag{41}$$

Let v_1, v_2, v_3, v_4 be the roots of this polynomial. Then

$$\begin{aligned}a_3 &= a_4(v_1+v_2+v_3+v_4)\\ a_2 &= a_4(v_1v_2+v_1v_3+v_1v_4+v_2v_3+v_2v_4+v_3v_4)\\ a_1 &= a_4(v_1v_2v_3+v_2v_3v_4+v_3v_4v_1+v_4v_1v_2)\\ a_0 &= a_4(v_1v_2v_3v_4)\end{aligned} \tag{42}$$

The partial waves for unnatural-parity exchange are obtained by

(43)

$$\begin{aligned} 30S_0 &= a_0 + a_2 + 6a_4 \\ 42\sqrt{5}D_0 &= -2a_0 + a_2 + 24a_4 \\ 210G_0 &= a_0 - 4a_2 + 16a_4 \\ 14\sqrt{15}D_- &= a_1 + 3a_3 \\ 42\sqrt{10}G_- &= -a_1 + 4a_3 \end{aligned}$$

There are 8 ambiguous solutions. The partial waves with natural-parity exchanges are

(44)

$$\begin{aligned} |G_+|^2 &= |G_-|^2 - \frac{2431}{42\sqrt{35}} H(82) \\ |D_+|^2 &= |D_-|^2 - \frac{7\sqrt{2}}{\sqrt{3}} H(22) + \frac{221}{4\sqrt{35}} \left[\frac{17}{14} H(82) - \frac{11}{17\sqrt{3}} H(62) \right] \\ 2\,\mathrm{Re}\{D_+\, G_+^*\} &= 2\,\mathrm{Re}\{D_-\, G_-^*\} + \frac{143}{2\sqrt{70}} \left[\frac{17}{14\sqrt{3}} H(82) - H(62) \right] \end{aligned}$$

Thank you for your patience and hospitality. . .