Low- and High-Energy Excitations in the Unitary Fermi Gas

Introduction / Motivation
Homogeneous Gas
  Momentum Distribution
  Quasi-Particle Spectrum

Low Energy Excitations and
  Static Structure Function
Inhomogeneous Gas
  and Density Functional
Small Trapped Systems

High Energy Excitations
  and Response
  Spin Response
  Density Response

Conclusions / Future

Fermi Condensates

C. Regal et al. PRL 2004

Kevin Schmidt
Shiwei Zhang
Stefano Gandolfi
J. Carlson
Homogeneous Unitary Fermi Gas

\[ H = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} V_0 \delta(r_{ij}) \]

\( V_0 \) can be tuned across BCS (\( |V_0| \sim 0 \)) to BEC (\(-V_0 \gg E_F\))

Concentrate on unitarity: zero energy bound state
    infinite scattering length

\[ E = \xi \quad E_{FG} = \xi \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} \]

\[ \Delta = \delta \frac{\hbar^2 k_F^2}{2m} \]

\[ T_c = t \frac{\hbar^2 k_F^2}{2m} \]

Values of \( \xi, \delta, t \) are independent of \( \rho \)
Equations of State for uniform gas: $\xi$: Experiments, Analytic, and Computational

PRA (2013)
T=0 Algorithm: Branching random walk (diffusion) using AFMC

\[ H = \frac{1}{N^3} \sum_{k,j,m,s} \psi_{j,s}^\dagger \psi_{m,s} \epsilon_k e^{i k \cdot (r_j - r_m)} + U \sum_i n_{i\uparrow} n_{i\downarrow}. \]

Lattice actions

\[ \epsilon_k^{(2)} = \frac{\hbar^2 k^2}{2m}, \quad \epsilon_k^{(4)} = \frac{\hbar^2 k^2}{2m} \left[ 1 - \beta^2 k^2 \alpha^2 \right] \]

\[ \epsilon_k^{(h)} = \frac{\hbar^2}{ma^2} \left[ 3 - \cos(k_x \alpha) - \cos(k_y \alpha) - \cos(k_z \alpha) \right]. \]

Each step: multiply by exp [ -T dt / 2 ] momentum space

Auxiliary field for exp [ -V dt ] coordinate space

multiply by exp [ -T dt /2 ] momentum space

Use importance sampling with BCS wave function

<table>
<thead>
<tr>
<th>Energy</th>
<th>( \frac{U 2m \alpha^2}{\hbar^2} )</th>
<th>( \beta )</th>
<th>( r_e \alpha^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_k^{(h)} )</td>
<td>-7.91355</td>
<td>-</td>
<td>-0.30572</td>
</tr>
<tr>
<td>( \epsilon_k^{(2)} )</td>
<td>-10.28871</td>
<td>-</td>
<td>0.33687</td>
</tr>
<tr>
<td>( \epsilon_k^{(4)} )</td>
<td>-8.66605</td>
<td>0.16137</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

\[ |BCS\rangle = \left[ \sum_k f_k c_{k\uparrow} c_{-k\downarrow}^\dagger \right]^{N/2} |0\rangle \]

\[ \langle W|BCS\rangle = \langle 0| \prod_{n=1}^{N/2} w_{n\downarrow} w_{n\uparrow} \left[ \sum_k f_k c_{k\uparrow} c_{-k\downarrow}^\dagger \right]^{N/2} |0\rangle \]

\[ \langle W|BCS\rangle = \det A, \]

\[ A_{nm} = w_{n\downarrow} w_{m\uparrow} \sum_k f_k c_{k\uparrow} c_{-k\downarrow}^\dagger \]

\[ = \sum_k \phi_{n,-k} f_k \phi_{m,k} = \sum_k \phi_{n,k}^* f_k \phi_{m,k}. \]
Homogeneous Gas: $\xi$ and effective range $S$

MIT exp.: $\xi = 0.376(0.005)$


$E(k_F r_e)/E_{FG} = \xi + S k_F r_e + \ldots$

$\xi = 0.372 \pm 0.005$
$S = 0.12 \pm 0.01$

Carlson, Gandolfi, Schmidt, and Zhang, PRA 2011
Carlson, Gandolfi, and Gezerlis, PTEP 2012
Important for spin excitations and high-energy behavior
Nuclear Momentum Distributions
(see Wiringa’s talk)

Wiringa, et al, 2013 (PRC)
Spin Excitations: requires high energy

Spin up, down densities in a trap

\[ \delta = 0.45(05) \]

JC and Reddy, PRL 2007
analyzing MIT data

\[ \Delta = \delta \frac{\hbar^2 k_F^2}{2m} \]

\[ \delta = 0.50(03) \]

\[ \left( \frac{k_{min}}{k_f} \right)^2 = 0.80(10) \]

JC and Reddy, PRL 2005
Simple BCS wave function does not describe low-q behavior. Added long-range Jastrow to better describe phonons.

Carlson and Gandolfi, PRA, 2014
S(q) for Unitary Fermi Gas

Hoinka, Lingham, Fenech, Hu, Vale, Drut, Gandolfi, PRL 2013
Inhomogeneous Matter: Neutron Drops

Harmonic Potential
Requires gradient terms, pairing, L.S splitting, …

Gandolfi, et al, PRL 2011
Low Energy Excitations in the UFG

Low-lying excitations (phonons)
Transitions from 3 dimensions to 2 dimensions
How do finite systems behave - bulk vs. finite
Density functional
Necessary to understand `exotic' (LOFF,...) phases

Add a spin-independent background potential
Examine the system for:

weak periodic potentials
Harmonic Oscillator potentials (1d, 2d, 3d,...)
EFT treatment

Son and Wingate, Annal. of Physics 2006

Static Density Response

\[ \chi(q) = -\frac{k_F}{\pi^2 \xi} \left[ 1 + 2\pi^2 \sqrt{2\xi} \left( c_1 - \frac{9}{2}c_2 \right) \frac{q^2}{k_F^2} \right]. \]

Phonon Dispersion relation

\[ \omega = \sqrt{\frac{\xi}{3}} v_F q \left[ 1 - \pi^2 \sqrt{2\xi} \left( c_1 + \frac{3}{2}c_2 \right) \frac{q^2}{k_F^2} \right]. \]

Lowest order determined by \( \xi \)
next order involves \( c_1 \) and \( c_2 \)
Density Functional

\[ \mathcal{E}(x) = n(x)V(x) + \frac{3 \cdot 2^{2/3}}{5^{5/3}mc_0^{2/3}}n(x)^{5/3} - \frac{4}{45} \frac{2c_1 - 9c_2}{mc_0} \frac{(\nabla n(x))^2}{n(x)} - \frac{12}{5} \frac{c_2}{mc_0} \nabla^2 n(x). \]

**Epsilon expansion at unitarity**

\[ \mathcal{E}(x) = n(x)V(x) + 1.364 \frac{n(x)^{5/3}}{m} + 0.022 \frac{(\nabla n(x))^2}{mn(x)} + O(\nabla^4 n). \]

**compare to free fermions**

\[ \mathcal{E}_{ETF}(x) = n(x)V(x) + 2.871 \frac{n(x)^{5/3}}{m} + 0.014 \frac{(\nabla n(x))^2}{mn(x)} + 0.167 \frac{\nabla^2 n(x)}{m} + O(\nabla^4 n). \]

**Note increase in coefficient of gradient term at unitarity compared to free Fermi gas**
Change notation:

\[ E = V(r)\rho(r) + \xi \frac{4}{5}(3\pi^2)^{2/3}\rho^{5/3} + c_g \nabla \rho^{1/2} \cdot \nabla \rho^{1/2} + ... \]

\[ \hbar^2/(2m) \to 1 \]

Free fermions (BCS limit) \quad c_g = 0.111

Free bosons (BEC limit \( M = 2m \)) \quad c_g = 0.50

The gradient term is exactly like the kinetic term in the Gross–Pitaevskii equation (BEC).

see also M. Forbes [arXiv:1211.3779](https://arxiv.org/abs/1211.3779) for treatment with Superfluid Local Density Approximation

We use only bosonic degrees of freedom

no single-particle orbital summation for the density.
Static response from weak external potentials

\[ V(r) = V_0 \, E_F \, \cos(k \cdot r) \]

\[ E(V_0) = E_0 - \frac{\sum_f \langle 0 | V(r) | f \rangle \langle f | V(r) | 0 \rangle}{E_f - E_0} \]

\[ E(V_0) = E_0 - \int d\omega \, S(k, \omega) / \omega \]

At low \( q \) \( E(V0) \) determined by compressibility (\( \xi \))

Next order in \( q \) determined by \( c_g \)
Static Response for BCS and BEC limits

Periodic Potential with momentum $k$ and strength $V_0$

Graph showing the static response for Fermions and Bosons with different values of $k$. The y-axis represents $E/E_{RG}$ and the x-axis represents $V_0$. The graph includes lines for $k = 0.0256$, $k = 0.51$, $k = 1.02$, and $k = 2.04$.
Why does this work so well far beyond weak coupling?

Assume that density functional is local (not true for Fermions) and scale invariant places strong constraints on $\epsilon(\rho)$.

$$\epsilon(\rho) \propto d^{-5/3}$$

gradient term alone works if

$$\left| \frac{\nabla \rho}{\rho} \right| << k_F$$
Calculation of $c_g$ from weak external potential

$N=66$, $k/k_F = 0.5$, $V_0 = 0.25$

fit gives $c_g = 0.30 \pm 0.05$
Beyond the lowest order gradient term

Assume density functional is local
Must respect scale invariance
Negele-Vautherin-like form (density matrix expansion)

\[ \mathcal{E} = V(r)\rho(r) + \xi (3\pi^2)^{2/3} \rho^{5/3} + c_g \nabla \rho^{1/2} \cdot \nabla \rho^{1/2} + \\
+ c_4 \frac{\nabla^2 \rho^{1/2} \nabla^2 \rho^{1/2}}{\rho^{2/3}} \]

Requires gradient term \((c_g)\) be treated non-perturbatively; \(c_4\) in perturbation theory
Eventually larger $q$, $V_0$ require higher gradient terms
What about small systems of trapped Fermions?

Need ~ 10x greater accuracy than for bulk
Improved DMC results for trapped fermions

More sophisticated trial wavefunction which includes additional single-particle orbitals & terms which go to SLDA-like pairing.

Approaches FN bulk limit of 0.39

No obvious shell closures
AFMC compared to DMC

Approaching bulk limit of 0.37, no shell closures
Trapped fermions: LDA + gradient + $q^4$

Bands correspond to errors from inhomogeneous bulk
Conclusions (Low-Energy Excitations)

Low-Energy degrees of freedom are phonons in UFG

Scale invariance ties linear response to complete functional

\[ c_g = 0.30(5) \]

compared to 0.111 for BCS (free fermions)

0.50 for BEC (free bosons of mass 2m)

Quadratic corrections important for trapped fermions

No evidence for shell structure (large pairing gap)
in the unitary Fermi Gas, even for small systems

Can treat large systems and dynamics with GP-like equation
High-Energy Excitations and Quasi-Elastic Response

Spin Response:

\[ S_\sigma (q, \omega) = \int d\omega \sum_f \langle 0|\hat{\rho}_\sigma^\dagger (q)\rho_\sigma(q)|0\rangle \delta(\omega - (E_f - E_0)) \]

\[ \rho_\sigma = \sum_i \exp[iq \cdot r] \sigma_x(i) \]

Density Response:

\[ S_\rho (q, \omega) = \int d\omega \sum_f \langle 0|\hat{\rho}_\rho^\dagger (q)\rho_\rho(q)|0\rangle \delta(\omega - (E_f - E_0)) \]

\[ \rho_\rho = \sum_i \exp[iq \cdot r] \]

Flips spin little FSI

Important FSI
Analogies to Electron / Neutrino Scattering

$(e,e')$ Inclusive Response: Scaling Analysis

Donnelly and Sick (1999)

$^3$He

$^4$He

A=3,4

$^{12}$C

Transverse

Longitudinal

See Lovato talk
Traditional Ingredients to the Response

Momentum Distribution

Spectral Function
From quasiparticle spectrum

Can calculate PWIA or Spectral Function Models
General picture correct, low energy tail overestimated
Spin Response : Spectral Function Approach

Good description of data with spectral function and (adjusted) BCS
Spin versus Density response (Experiment)

Spin, density responses are identical in PWIA, SF
BCS calculations give 2\textsuperscript{nd} peak (pair propagation)
But underestimates width
Inversion of Euclidean Response w/ GIFT algorithm

Qualitatively correct, but more physics input needed for exact reproduction
Conclusions

• Low Energy excitations of UFG understood
• Can be used to predict even very small clusters
• Can be used to calculate dynamics with GP-like equations
• High-energy response understood from QMC+BCS, FSI can be very important …
• Many analogies to nuclear physics

Future

• Dynamics with fermionic `GP’ equations
• Better treatment of density response
• Multiple species
• …
• Improvements to nuclear DFT
• Applications to neutrino (nuclear) physics
• …
Inhomogeneous Matter: Cold atoms and Neutron Drops

Comparison of different Hamiltonians

Constrain Isovector Gradient Terms in the density functional