Universality in few-body systems:
from few atoms to few nucleons

A. Kievsky

INFN, Sezione di Pisa (Italy)

Dedicated to Prof. S. Rosati
Electron-Nucleus Scattering XIII
Marciana Marina, June 2014

Collaborators

- M. Viviani - INFN & Pisa University, Pisa (Italy)
- M. Gattobigio - INLN & Nice University, Nice (France)
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Correlations in few-nucleon systems

In the 80’s and 90’s strong efforts have been done to:

- Construction of realistic interactions:
  - AV18, CDBonn, NimI and II
  - Chiral potentials appear few years later

- Solution of the $A = 3, 4$-nucleon problem:
  - VMC and GFMC methods
  - Faddeev and Faddeev-Yakuboski methods
  - Hyperspherical Harmonics
  - Expansion in Gaussians
  - NCSM

- all these methods can treat the strong repulsion of the NN interaction

- three-body forces can be included as well
### $^3$H and $^4$He Bound States and $n – d$ scattering length

<table>
<thead>
<tr>
<th>Potential(NN)</th>
<th>Method</th>
<th>$^3$H[MeV]</th>
<th>$^4$He[MeV]</th>
<th>$^2a_{nd}$[fm]</th>
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<td>NCSM</td>
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<tr>
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<td>NCSM</td>
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<td>25.39(1)</td>
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### Potential(NN+NNN)

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<th>Potential(NN+NNN)</th>
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<th>$^3$H[MeV]</th>
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<td>N3LO-EM/3N-N2LO</td>
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<td>0.645±0.010</td>
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</table>
Universality in few-body systems
Marciana, June 2014
A = 3, 4 Systems

The A = 3, 4 systems with NN+NNN interactions

- 2N system: $\chi^2$ per datum $\approx 1$
- 3N and 4N systems using a NN+NNN interaction: $\chi^2$ per datum $>> 1$

The NNN interaction

- Urbana IX → 2 parameters
- Tucson-Melbourne → 1 parameter
- 3N-N2LO → 2 parameters
- Illinois → 4 parameters

- further work in determining the NNN interaction is probably needed
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Some unexplained questions

In Chen et al., PRC39, 1261 (1989) n-d scattering has been calculated. The experimental values are $^2 a_{nd} = 0.65 \pm 0.04$ fm and $^4 a_{nd} = 6.35 \pm 0.02$ fm

Effective range formula:

$$k \cot \delta = -\frac{1}{a_{nd}} + \frac{1}{2} r_s k^2$$

$$E_p = \frac{3}{4}(\hbar^2/m)k_0^2 \approx 160 \text{ keV}$$

$$r_s \approx -127 \text{ fm}$$

$$r_s \approx 3 \text{ fm}$$
Some questions

- Why the doublet scattering length is small?
- Why there is a pole in effective range function?
- There is a virtual state in the thre-nucleon system with $J = 1/2^+$?
- If yes, what is the origin of this virtual state?

and more general

- a correlation between the doublet scattering length $a_{nd}$ and the triton binding energy $\rightarrow$ Phillips line
- a correlation between the $A = 3$ and $A = 4$ binding energies $\rightarrow$ Tjon line
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A different point of view

Universality: General Aspects

- Physical systems that are different at short distances exhibit identical behavior at long distances
- For example:
  - A liquid-gas transition near the critical point
  - A ferromagnetic material near the critical point
- Few-body systems with large scattering length

Universality in Few-Body Systems

- When the two-body scattering length is much bigger than the range of the interaction: \( r_0/a \to 0 \)
- S-wave phase shift: \( k \cot \delta \to -1/a \)
- Shallow dimer: \( E_D \to \hbar^2/ma^2 \)
- Cross section: \( \frac{d\sigma}{d\Omega} \to \frac{4a^2}{1+a^2k^2} \)
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The two-body scattering length appears as a control parameter

**Two-Body System: Continuous Scale Invariance**

- \( \Psi_D \rightarrow e^{-r/a} \frac{r}{r} \) \( \langle r^2 \rangle = a^2 / 2 \)
- \( a \rightarrow \lambda a \)
  - \( E \rightarrow \lambda^{-2} E \)
  - \( \sigma \rightarrow \lambda^2 \sigma \)
  - \( \langle r^2 \rangle \rightarrow \lambda^2 \langle r^2 \rangle \)

**Three-Body System: Discrete Scale Invariance**

- Efimov Effect: a series of bound states having the ratio:
  - \( E_3^{(n+1)} / E_3^{(n)} \rightarrow e^{-2\pi / s_0} \)
- The series of states is infinite at the unitary limit, \( a \rightarrow \infty \)
- \( s_0 \) is an **universal** number
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**Two-Body System: Continuous Scale Invariance**

- \[ \psi_D \rightarrow \frac{e^{-r/a}}{r} \quad < r^2 > = \frac{a^2}{2} \quad \sigma = 2\pi \frac{4a^2}{1+a^2k^2} \]
- \[ a \rightarrow \lambda a \]
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Three-boson spectrum for $r_0/a << 1$

Zero-Range Theory

$$E_3^n/(\hbar^2/ma^2) = \tan^2 \xi$$

$$\kappa_\ast a = e^{\pi(n-n_\ast)/s_0} e^{-\Delta(\xi)/2s_0} / \cos \xi$$

- $\kappa_\ast$ is the three-body parameter
- $\Delta(\xi)$ is an universal function

$$E_3^n + \frac{\hbar^2}{ma^2} = \left[ e^{-2\pi(n-n_\ast)/s_0} \right] \left[ e^{\Delta(\xi)/s_0} \right] \frac{\hbar^2 \kappa_\ast^2}{m}$$

$$E_3^{n_\ast} = \frac{\hbar^2 \kappa_\ast^2}{m}$$

- the ratios at $\xi = const.$ are $E_3^{(n+1)}/E_3^{(n)} = e^{-2\pi/s_0} \approx 1/22.7^2$
Discrete Scale Invariance

Polar Coordinates: \( 1/a = H \cos \xi \), \( K = H \sin \xi \)

\[
E_n^3 + \frac{\hbar^2}{ma^2} = e^{-2n\pi/s_0} e^{\Delta(\xi)/s_0} \frac{\hbar^2 \kappa_0^2}{m} \rightarrow H = \kappa_0 \ e^{-n\pi/s_0} \ e^{\Delta(\xi)/2s_0}
\]
Universality: $N$-boson bound states

- There is a tree of two attached states: one shallow and one deep
Universality: $N$-boson bound states

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\[
\text{sgn}(E) \sqrt{|E|/E} = \frac{\text{sgn}(a) \sqrt{\ell/|a|}}{E}
\]
Characteristics of few-boson systems having a large value of $a$

- Two-Body System: Continuous Scale Invariance
- Three-Body System: Discrete Scale Invariance
- Geometrical series of bound states: Efimov Effect
- Scale parameter: $\kappa_*$
- The universal function: $\Delta(\xi)$
- $N$-boson system: a tree of two attached states
- The $N$-boson spectrum is determined by two parameters: the control parameter $a$ and the scale parameter $\kappa_*$

Searching universal behavior in few-nucleon systems

- The $n-p$ scattering lengths are large:
  $a_{n-p} \approx 5.5$ fm ($S = 1$) and $a_{n-p} \approx -23$ fm ($S = 0$)
- The deuteron is shallow: $E_d \approx -\frac{h}{ma^2} = 1.4$ MeV
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Universality in atom-dimer scattering

- Atom-dimer scattering length:
  $$a_{AD} = a(d_1 + d_2 \tan[s_0 \ln(\kappa_* a) + d_3])$$ (Efimov 1979)
  with $d_1$, $d_2$, $d_3$ universal constants (Braaten and Hammer, 2006)

- Atom-dimer effective range function:
  $$k \cot \delta = a^{-1}(c_1(ka) + c_2(ka) \cot[s_0 \ln(\kappa_* a) + \phi(ka)])$$
  with $c_1$, $c_2$, $\phi$ universal functions
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  with $c_1$, $c_2$, $\phi$ universal functions
\[ ka \cot \delta = c_1(ka) + c_2(ka) \cot[s_0 \ln(\kappa_*a) + \phi(ka)] \]
The scattering lengths as control parameters

The $2N$ sector

Low Energy data:

\[ E_d = -2.2245 \text{ MeV} \]

\[ a_1 = 5.424 \pm 0.003 \text{ fm} \quad r_1^{\text{eff}} = 1.760 \pm 0.005 \text{ fm} \]

\[ a_0 = -23.740 \pm 0.020 \text{ fm} \quad r_0^{\text{eff}} = 2.77 \pm 0.05 \text{ fm} \]

Constructing the LO $2N$ potential

Two parameters corresponding to the $l = 0$ partial waves with $S = 0, 1$:

\[ V_0(r) = -V_0 e^{-r^2/r_0^2}, \quad V_1(r) = -V_1 e^{-r^2/r_1^2} \]

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<thead>
<tr>
<th>$V_0$[MeV]</th>
<th>$r_0$[fm]</th>
<th>$a_0$[fm]</th>
<th>$r_0^{\text{eff}}$[fm]</th>
<th>$V_1$[MeV]</th>
<th>$r_1$[fm]</th>
<th>$a_1$[fm]</th>
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### The 3N sector

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<td>79.600</td>
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#### Introducing a Three-Body Force

We choose a simple (two-parameter) form:

$$W(\rho) = W_0 e^{-\rho^2/\rho_0^2}$$

with $\rho^2 = \frac{2}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2)$

$W_0$ and $\rho_0$ fixed to describe $E(^3\text{H})$ and $^2a_{nd}$.
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with $\rho^2 = \frac{2}{3} (r_{12}^2 + r_{23}^2 + r_{31}^2)$

$W_0$ and $\rho_0$ fixed to describe $E(\text{^3H})$ and $^2a_{nd}$
$$V(r) = [V(S=1) + V(S=0)] \times \exp\left(-\frac{r^2}{r_1^2}\right) + W_0 \times \exp\left(-\frac{\rho^2}{\rho_0^2}\right)$$
The N=4 ground and excited state

\[ V(r) = [V(S=1) + V(S=0)] \exp(-r^2/r_1^2) + W_{SR}(\rho) \]

\[ V(r) = [V(S=1) + V(S=0)] \exp(-r^2/r_1^2) + W_{LR}(\rho) \]

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Summary of the LO potential

<table>
<thead>
<tr>
<th>LO</th>
<th>$E_d$</th>
<th>$B(^3\text{H})$</th>
<th>$B(^3\text{He})$</th>
<th>$B(^3\text{He}^*)$</th>
<th>$2a_{nd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>-2.225</td>
<td>-8.482</td>
<td>-28.296</td>
<td>-8.10</td>
<td>0.645</td>
</tr>
</tbody>
</table>

No bad for a 4-parameter $2N$ potential + 2-parameter $3N$ potential!

next step (in progress) → $^6\text{He}$ and $^6\text{Li}$ ground states
Conclusions

Bosons systems with large scattering length

- natural \((E \approx \hbar^2 / m r_0^2)\) vs unnatural \((E \approx \hbar^2 / m a^2)\) energy scale
- The particles are far to each other
- Insensitivity to the short-range part of the interaction
- The spectrum is governed by two parameters: \(a, \kappa_*\)
- The systems present universal behavior

Shallow states with Potential models

- Use of soft two- and three-body potentials
- Intensity of the force fixed to reproduce some experimental data
- Treatment of finite range corrections
- Halo nuclei ↔ Efimov physics

Evolution with \(N\)

- General solution of the \(N\)-boson problem with contact interactions
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Evolution with $N$

The DSI imposes that

$$E_N^0 + \frac{\hbar^2}{ma^2} = \exp \left[ \frac{\Delta(\xi)}{s_0} \right] \frac{\hbar^2 \kappa_N^2}{m}$$

Linear dependence of $\kappa_N$

$$\kappa_N/\kappa_3 = 1 + \gamma (N - 3)$$

$\gamma = 1.147$ is an universal quantity (A. Deltuva, FBS 54, 569 (2013)).

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that can be transformed to an energy ratio:

$$\frac{E_N}{E_3} = [1 + 1.147(N - 3)]^2 \approx (1.15N - 2.44)^2$$

Preliminary result for the $N$-boson energy at unitary limit
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Preliminary result for the $N$-boson energy at unitary limit
Testing the $E_N/E_3$ formula at $\xi = -\pi/2$
Testing the linear dependence on $\kappa_N$ at constant $a$

$$E_N / E_2 = \tan^2 \xi$$

$$\kappa_N a_B + \Gamma_N = \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}$$


<table>
<thead>
<tr>
<th></th>
<th>TTY</th>
<th>TBG</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$ (a.u.)</td>
<td>189.0</td>
<td>189.4</td>
</tr>
<tr>
<td>$r_0$ (a.u.)</td>
<td>13.94</td>
<td>13.84</td>
</tr>
<tr>
<td>$E_2$ (mK)</td>
<td>-1.310</td>
<td>-1.303</td>
</tr>
<tr>
<td>$E_3$ (mK)</td>
<td>-126</td>
<td>-151</td>
</tr>
<tr>
<td>$E_4$ (mK)</td>
<td>-558</td>
<td>-751</td>
</tr>
<tr>
<td>$E_5$ (mK)</td>
<td>-1302</td>
<td>-1945</td>
</tr>
</tbody>
</table>


\[ \text{T} \]

\[ M. \text{ Lewerenz, J. Chem. Phys. 106, 4596 (1997) for the TTY potential:} \]