The Pion Form Factor in the Covariant Spectator Theory

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New challenges in meson physics

Motivation

- Upcoming intense experimental activity to explore meson structure
  - GlueX (Jlab), PANDA (GSI)
- Search for exotic mesons (hybrids, glueballs)
- Need to understand “conventional” $q\bar{q}$-mesons in more detail
- Pion transition form factors
  - Hadronic contributions to light-by-light scattering
    - Source of uncertainty in prediction of anomalous magnetic moment of the muon
  - Important in search for physics beyond the Standard Model
A unified model for all $q\bar{q}$ mesons

Much important work was done on meson structure

- **Cornell-type static potential models** (Isgur and Godfrey, Spence and Vary, etc.)
  But: nonrelativistic (or “relativized”); structure of constituent quark and relation to existence of zero-mass pion in chiral limit not addressed
- **Dyson-Schwinger** approach (C. Roberts et al.)
  But: **Euclidean space**; only Lorentz vector confining interaction
- **Lattice QCD** (also **Euclidean space**), EFT, Bethe-Salpeter, Light-front, Point-form, ...

Our objectives

- Construct a model to describe all $q\bar{q}$-type mesons
- **Covariant framework** (CST) - light quarks require relativistic treatment
  Work in **Minkowski space** (physical momenta — but have to confront singularities)
  Improve on previous work by Gross, Milana, Savkli
- **Quark self-energy** from $q\bar{q}$ interaction kernel (consistent quark mass function)
- **Chiral symmetry**: massless pion in chiral limit of vanishing bare quark mass (NJL)
- Calculate meson spectrum, bound-state vertex functions and form factors
- Learn about **confining interaction** (scalar vs. vector, etc.)
**Covariant two-body bound-state equation**

Start from the Bethe-Salpeter (BS) equation for the bound-state vertex function $\Gamma$

$$\Gamma_{BS}(p, P) = i \int \frac{d^4k}{(2\pi)^4} \mathcal{V}(p, k; P) S_1(k_1) \Gamma_{BS}(k, P) S_2(k_2)$$

The dressed propagator satisfies the Dyson equation

$$S(p) = \frac{1}{m_0 - \not{p} + \Sigma(p) - i\epsilon} = \frac{Z(p^2)}{M(p^2) - \not{p} - i\epsilon}$$

Dressed mass satisfies

$$m = M(m^2)$$

$$M(p^2) = \frac{m_0 + A(p^2)}{1 - B(p^2)}$$
Kernel truncation

The kernel contains all two-body irreducible diagrams

\[ \nu = \frac{1}{\mu} \left( \frac{1}{\mu} \right)^{\frac{1}{2}} \left( \frac{1}{\mu} \right)^{\frac{1}{2}} + \frac{1}{\mu} \left( \frac{1}{\mu} \right)^{\frac{1}{2}} \left( \frac{1}{\mu} \right)^{\frac{1}{2}} + \frac{1}{\mu} \left( \frac{1}{\mu} \right)^{\frac{1}{2}} \left( \frac{1}{\mu} \right)^{\frac{1}{2}} + \ldots \]

In the BS equation the kernel is effectively iterated to all orders

But the complete kernel is a sum of an infinite number of irreducible diagrams

→ has to be truncated (most often: ladder approximation)

Problems with the ladder truncation

- No one-body limit
  (missing crossed ladders)
- Not best suited to describe bound states
  (crossed-ladder contributions are significant)
  see Nieuwenhuis and Tjon, PRL77, 814 (1996)
From Bethe-Salpeter to CST

Covariant Spectator Theory (CST)

Integration over relative energy $k_0$:
- Keep only pole contributions from propagators
- Move kernel poles to higher-order kernels (reorganization of the BS series)
- Cancellations between ladder and crossed ladder diagrams can occur
- Reduction to 3D loop integrations, but covariant
- Works very well in few-nucleon systems

If bound-state mass $\mu$ is small:
both poles are close together (both important)

Symmetrize pole contributions from both half planes:
resulting equation is symmetric under charge conjugation

BS vertex (approx.)

CST vertices

Four-channel CST equation

Closed set of equations when external legs are systematically placed onshell

\[ \begin{align*}
2\times \quad & = \quad + \\
2\times \quad & = \quad + \\
2\times \quad & = \quad + \\
2\times \quad & = \quad + \\
\end{align*} \]

Solutions: bound state masses \( \mu \) and corresponding vertices \( \Gamma \)

Approximations for special cases

- Heavy-light quark systems: 2 channels
- Large bound-state mass: 1 channel

Smooth one-body limit: Dirac equation
Smooth nonrelativistic limit: Schrödinger equation
Confining potential in momentum space

Phenomenological $q\bar{q}$ kernel

Inspired by Cornell potential:

$$V(r) = \sigma r - C - \frac{\alpha_s}{r}$$

**NR linear potential in momentum space:**

Fourier transform of screened potential

Usually:

$$\sigma r = \lim_{\epsilon \to 0} \sigma \frac{\partial^2}{\partial \epsilon^2} \frac{e^{-\epsilon r}}{r}$$

But simpler:

$$\sigma r = \lim_{\epsilon \to 0} -\frac{\sigma}{\epsilon} \left(e^{-\epsilon r} - 1\right) \equiv \tilde{V}_A(r) - \tilde{V}_A(0)$$

**FT:**

$$V_L(q) = V_A(q) - (2\pi)^3 \delta(q) \int \frac{d^3 q'}{(2\pi)^3} V_A(q')$$

with

$$V_A(q) = -\frac{8\pi\sigma}{q^4}$$

$$\langle V_L \phi \rangle(p) = \int \frac{d^3 k}{(2\pi)^3} V_L(p - k)\phi(k) = -8\pi\sigma \int \frac{d^3 k}{(2\pi)^3} \frac{\phi(k) - \phi(p)}{(p - k)^4}$$

highly singular

automatically subtraction

only a Cauchy principal value singularity remains

Static QCD potential from the lattice


Gross, Milana, PRD 43, 2401 (1991)

Savkli, Gross, PRC 63, 035208 (2001)
Covariant confining kernel in CST

**Covariant generalization:** \( q^2 \rightarrow -q^2 \)

This leads to a kernel that acts like

\[
\langle V_L \phi \rangle (p) = \int \frac{d^3 k}{(2\pi)^3} \frac{m}{E_k} V_L(p, \hat{k}) \phi(\hat{k}) = -8\pi \sigma \int \frac{d^3 k}{(2\pi)^3} \frac{m}{E_k} \frac{\phi(\hat{k}) - \phi(\hat{p}_R)}{(p - \hat{k})^4}
\]

any regular function

\[
\hat{k} = (E_k, \mathbf{k}) \quad \hat{p}_R = (E_{p_R}, \mathbf{p}_R)
\]

**Properties:**
- Subtraction regularizes kernel to Cauchy principal value
- Nonrelativistic limit \( \rightarrow \) linear potential
- Satisfies the condition

\[
\langle V_L \rangle = \int_{\hat{k}} V_L(p, \hat{k}) = 0
\]

But does it still confine?

**Yes:** the vertex function vanishes if both quarks are onshell!

Spontaneous chiral symmetry breaking - pion equation

Pion bound-state equation for finite mass $\mu$

$$= \frac{1}{2} \left\{ \gamma^5 G(p^2) + \gamma^5 G_0 + \gamma^5 G_0 + \gamma^5 G_0 \right\}$$

For $\mu \to 0$ this becomes (in the rest frame)

$$\lim_{\mu \to 0} \gamma^5 G(p^2) = \frac{1}{2} \gamma^5 G_0 + \frac{1}{2} \gamma^5 G_0$$

For a kernel with Lorentz scalar and vector structure

$$\mathcal{V}(p, k) = V_S(p, k) \mathbf{1}_1 \otimes \mathbf{1}_2 + \frac{1}{4} V_V(p, k) g_{\mu\nu} \gamma_1^\mu \otimes \gamma_2^\nu$$

one gets a condition for the existence of a massless pion solution

$$1 = \frac{Z_0}{4m(1 - B_0)} \int_k \left[ V_V(\hat{p}, \hat{k}) + V_V(\hat{p}, -\hat{k}) - V_S(\hat{p}, \hat{k}) - V_S(\hat{p}, -\hat{k}) \right]$$

$$B_0 = B(m^2)$$

$$Z_0 = Z(m^2)$$
Spontaneous chiral symmetry breaking

\* Dressed quark propagator

\[ S(p) = \frac{1}{m_0 - \phi + \Sigma(p) - i\epsilon} = Z(p^2) \frac{M(p^2) + \phi}{M^2(p^2) - p^2 - i\epsilon} \]

\[ \Sigma(p) = A(p^2) + \phi B(p^2) \]

\[ Z(p^2) = \frac{1}{1 - B(p^2)} \]

\[ M(p^2) = \frac{m_0 + A(p^2)}{1 - B(p^2)} \]

The \( q\bar{q} \) kernel determines the quark self-energy

\[ m_0 - \phi + \Sigma(p) \]

CST-Dyson equation:

\[ (\quad )^{-1} = (\quad )^{-1} + \frac{1}{2} \quad + \frac{1}{2} \quad \]

\* The dressed quark mass is the solution of the gap equation

In rest frame, \( p = (p_0, \mathbf{0}) \):

\[ A(p_0^2) = \frac{Z_0}{4} \int_k \left[ V_S(p, \hat{k}) + V_S(p, -\hat{k}) + V_V(p, \hat{k}) + V_V(p, -\hat{k}) \right] \]

\[ B(p_0^2) = \frac{Z_0}{4p_0} \int_k \frac{E_k}{m} \left[ V_S(p, \hat{k}) - V_S(p, -\hat{k}) - \frac{1}{2} V_V(p, \hat{k}) + \frac{1}{2} V_V(p, -\hat{k}) \right] \]

We can use \( M(m^2) = m = m_0 + A_0 + mB_0 \) in the equation for \( A \) at \( p_0 = m \)

this yields another constraint
Spontaneous chiral symmetry breaking

Compare:

The two equations become identical for $m_0 = 0$

provided that

$$\int_k \left[ V_S(p, \hat{k}) + V_S(p, -\hat{k}) \right] = 0$$

When

**the Lorentz scalar interaction is zero**
or it satisfies the constraint

in the chiral limit ($m_0 \to 0$),

if a finite quark mass $m$ can be generated

then a zero-mass pion solution exists

(Goldstone boson)

The linear confining kernel satisfies

$$\int_k V_L(p, \hat{k}) = 0$$

the confining kernel can have a scalar component!

The confining interaction does not contribute to

$$\begin{cases} A(p^2) \\ \text{pion equation} \end{cases}$$

(m_0 = 0)

Also: if $m_0 > 0$ the condition for $A(p^2)$ guarantees that no zero-mass pion solution exists

$\to$ the real pion must have a finite mass
NJL mechanism in diagrammatic form

In the chiral limit:

1-body CST-Dyson equation

\[ S^{-1}(p)_{s.p.} = \frac{-1}{2} \]

Multiply by \( \gamma^5 \)

and attach quark lines

and a pion line with \( P=0 \)

2-body pion equation

\[ \gamma^5 G = \frac{1}{2} \]

Topologically equivalent
Lorentz structure of the kernel

We construct a simple exploratory model:

★ Mixed scalar-vector linear confining kernel (becomes linear potential in nonrel. limit)

\[
\mathcal{V}_L(p, \hat{k}) = [\lambda \mathbf{1}_1 \otimes \mathbf{1}_2 - (1 - \lambda) \gamma_1^\mu \otimes \gamma_2^\mu] V_L(p, \hat{k})
\]

★ Pure vector “constant” (in \(r\)-space) interaction

\[
\mathcal{V}_C(p, \hat{k}) = \frac{1}{4} g_{\mu\nu} \gamma_1^\mu \otimes \gamma_2^\nu V_C(p, \hat{k})
\]

with

\[
V_C(p, \hat{k}; P) = 2C h(p_1^2) h(p_2^2) h(\hat{k}_1^2) h(\hat{k}_2^2) (2\pi)^3 \frac{E_k}{m} \delta^3(p - k)
\]

Quark form factors \(h\) provide regularization (one factor for each quark line)

Can be viewed as vertex corrections

★ In the chiral limit, \(V_L\) does not contribute to \(A\) (or the pion vertex)

For \(\lambda = 2\) one gets also

\[
B_L(p^2) = 0 \quad \Sigma_L(p^2) = 0 \quad A_L(p^2) = 0
\]

no contribution from the confining interaction to the self-energy
Quark mass function

* The constant vector interaction contributes only to \( A(p^2) \)

\[
M(p^2) = A(p^2) + m_0 = C \ h^2(m^2) h^2(p^2) + m_0 \quad \text{with} \quad C = m_\chi + c_1 m_0 + \mathcal{O}(m_0^2)
\]

\[
m_\chi(p^2) = m_\chi h^2(p^2)
\]

* Fit to (Euclidean) LQCD data

(to 50 points with \( p^2 > -1.94 \text{ GeV}^2 \))

\[
\Lambda = 2.042 \text{ GeV} \quad m_\chi = 0.308 \text{ GeV}
\]

Lattice QCD data from Bowman et al PRD 71, 2005 extrapolated to chiral limit

Lattice data don’t converge to \( m_0 \) for large \(-p^2\) (finite lattice spacing effect)
Quark mass function for finite $m_0$

\[ C = m_X + c_1 m_0 + \mathcal{O}(m_0^2) \]

$c_1$ is fit (other parameters fixed in the chiral limit)
Pion form factor

The pion current in CST: triangle diagrams with 6 propagator poles (each diagram)

RIA (Relativistic Impulse Approximation): keep only the spectator poles

Analysis of pole structure: RIA is a good approximation for

- large $Q^2$ (any $\mu$)
- large $\mu$ (any $Q^2$)

For small $Q^2$ and small $\mu$, the remaining poles contribute significantly

→ need to calculate the CIA (Complete Impulse Approximation)

Needed ingredients:

- pion vertex function
- dressed quark current
A simple pion vertex function

In the chiral limit, the self-energy function $A(p^2)$ and the vertex $\Gamma(p)$ satisfy the same equation $\rightarrow$ we already have a pion vertex!

General BS vertex structure for pseudoscalar bound states

$$\Gamma_{BS}(p_1, p_2) = G_1(p_1^2, p_2^2)\gamma^5 + G_+(p_1^2, p_2^2)(\not{p}_1\gamma^5 + \gamma^5\not{p}_2) + G_-(p_1^2, p_2^2)(\not{p}_1\gamma^5 - \gamma^5\not{p}_2) + G_3(p_1^2, p_2^2)\not{p}_1\gamma^5\not{p}_2$$

For real pion, assume the chiral limit structure dominates

For real pion, assume the chiral limit structure dominates

$$\Gamma(p, p) = G(p^2)\gamma^5$$
in chiral limit (and rest frame)

near the chiral limit

$$\Gamma(p, p) = G_0 h(p^2)\gamma^5$$
Dressed quark current

Use the prescription by Gross & Riska to derive a conserved quark current

\[ S(p) \quad j^\mu \quad h(p^2) \]

\[ h^2 S(p) \quad h^2 S(p') \]

\[ h^{-1} j^\mu h'^{-1} \]

* Vertex form factors are absorbed into modified propagators

\[ \tilde{S}(p) = h^2 (p^2) S(p) \]

* Impose Ward-Takahashi identity on reduced current

\[
q^\mu j^\mu_R(p', p) = \tilde{S}^{-1}(p) - \tilde{S}^{-1}(p')
\]

\[
\tilde{j}_R^\mu(p', p) = h^{-1}(p'^2) j^\mu(p', p) h^{-1}(p^2)
\]

Structure of the current:

\[
j_R^\mu = f \left[ \gamma^\mu + \kappa \frac{i\sigma^{\mu\nu} q^\nu}{2m} \right] + \delta' \Lambda' \gamma^\mu + \delta \gamma^\mu \Lambda + g \Lambda' \gamma^\mu \Lambda
\]

\[ \Lambda = \frac{M(p^2) - \phi}{2M(p^2)} \]

off-shell form factors determined by WTI in terms of \( h(p^2) \)

* Differs from Ball-Chiu current (used by Tandy and Maris) by a transverse piece

* WTI cannot determine the current uniquely \( \rightarrow \) need a dynamical calculation
Results for the pion form factor

- The pion form factor is calculated with different pion masses
- Best fit with $\mu=0.42$ GeV (somewhat large, but allows us to use RIA at small $Q^2$!)

$\mu = 0.42$ GeV, $\mu = 0.28$ GeV, $\mu = 0.14$ GeV

- $F_\pi$ shows correct monopole fall-off
  
  $F_\pi(Q^2) \approx \frac{1}{Q^2 + \nu^2}$ with $\nu \approx 0.63$ GeV

- this is independent of $h(p^2)$

- Remarkable scaling relations hold at large $Q^2$
  (used in the figure for $\mu=0.14$ and 0.28 GeV)

  $F_\pi(Q^2, \lambda \mu) \approx \frac{Q^2}{\lambda \mu^2} \chi^2 F_\pi(Q^2, \mu)$

- Q: Why does the model work well without a $\rho$-pole (VMD)?

- Should calculate the dressed quark current dynamically (to be done soon!)
Summary

Formalism
- Developed the CST formalism for a dynamical $q\bar{q}$ model for all mesons
- CST equations are solved in Minkowski space
- Consistency between bound-state and mass gap equations
- Describes confinement and spontaneous chiral symmetry breaking
- Can be applied in the time-like region

Model calculations
- A very simple model can yield good agreement of the quark mass function with lattice calculations, and of the pion form factor with experimental data

Work in progress or planned for the near future
- Solution of the full $q\bar{q}$ CST equation and fit to the meson spectrum
- Study interaction kernels with different Lorentz structure; add gluon exchange
- Calculate quark-photon vertex dynamically
- Study $\pi\pi$ scattering (constraints from axial-vector WTI)
But what does the pion look like?

Most likely…