A small tribute to Gary Goldstein on the occasion of his 70th Birthday

On the validity of the transverse angular momentum sum rule

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The power of Gary’s early work

**IF** you have a theory

Theory → Amplitudes → Observables
The power of Gary’s early work

IF you have a theory

Theory $\rightarrow$ Amplitudes $\rightarrow$ Observables

if trying to LEARN about the dynamics

Observables $\rightarrow$ Amplitudes $\rightarrow$ Dynamics
Reactions with spin

MANY different Observables; MANY different Amplitudes
Reactions with spin

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Crucial to simplify. Reduce to minimal number of independent amplitudes, with most transparent implications
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Great work with Mike Moravcsik on this subject
The transverse angular momentum sum rule

My aim

to counter the myth that one cannot have a sum rule
for a transversely polarized nucleon

and

to argue that the Bakker, Leader Trueman Transverse
Angular Momentum Sum Rule is correct
Two steps in the derivation of a sum rule

1) Derive an expression for expectation value of angular momentum operators in nucleon state specified by 4-momentum $P^\mu$ and covariant spin vector $S^\mu$

$$\langle P, S \mid J_i \mid P, S \rangle \quad i = 1, 2, 3$$

i.e. show dependence of matrix element on variables $P^\mu$ and $S^\mu$
2) As stressed by Jaffe and Manohar: the parton model is a Fock space model. Therefore substitute for the nucleon state its expansion in terms of quark and gluon Fock states and identify the expressions which result.

This is origin of incorrect claim that there cannot exist an angular momentum sum rule for a transversely polarized nucleon.
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Correct result for the forward matrix elements of the angular momentum is

$$
\langle P', s | J_i | P, s \rangle = 2P^0 (2\pi)^3 \left[ \frac{1}{2} s_i + i (P \times \nabla_P)_i \right] \delta^3 (P' - P).
$$

and there is then no problem with the transverse case—it is quite analogous to the longitudinal case.
Jaffe-Manohar result has

\[
\frac{1}{4Mp^0} \left\{ (3p_0^2 - M^2)s_i - \frac{3p_0 + M}{p_0 + M}(p \cdot s)p_i \right\}
\]

instead of \( \frac{1}{2} s_i \)
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For longitudinal case exactly same as BLT, but for transverse case, for \( p_0 \gg m \)

\[ J-M \to \frac{3p_0}{4m} \to \infty \]

so no transverse sum rule.

This is NOT controversial: J-M agree with us!
Thus BLT were able to derive a sum rule relating the transverse spin of the nucleon to the **transverse polarized quark densities** $\Delta_T q(x)$ and the transverse orbital angular momentum carried by quarks and gluons, namely

$$\frac{1}{2} = \frac{1}{2} \sum_{\text{flavours}} \left\{ \int dx \left[ \Delta_T q^f(x) + \Delta_T \bar{q}^f \right] + \sum_{a=q,\bar{q},G} \langle L_{sT} \rangle^a \right\}$$

where $L_{sT}$ is the component of $L$ along $s_T$. 
Thus BLT were able to derive a sum rule relating the transverse spin of the nucleon to the transverse polarized quark densities $\Delta_T q(x)$ and the transverse orbital angular momentum carried by quarks and gluons, namely

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where $L_{sT}$ is the component of $L$ along $s_T$.

**NB** sum of quark and antiquark transversity densities, in contrast to the case of the tensor charge of the nucleon, where the difference appears.
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We shall show that the identification of terms on the RHS is correct.
Instant-form Fock expansion

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For simplicity give the proof only for unpolarized quark density $q(x)$, but the argument holds also for the polarized densities $\Delta q(x)$ and $\Delta_T q(x)$. 
The sophisticated expression for the quark correlator
\( \Phi_{\alpha\beta}(k; P, S) \), for a given flavour, integrated over \( k \) with
the constraint \( x = k^+/P^+ \) yields

\[
\Phi_{\alpha\beta}(x) = P^+ \int \frac{d\xi^-}{2\pi} e^{ix P^+ \xi^-} \langle P, S | \bar{\psi}_\beta(0) \psi_\alpha(0, \xi^-, 0_\perp) | P, S \rangle
\]

where

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P^\pm = \frac{1}{\sqrt{2}} (P_0 \pm P_z)
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At leading twist \( \Phi_{\alpha\beta}(x) \) expressed in terms of three LT quark distribution functions,

\[
\Phi(x) = \frac{1}{2} P \{ q(x) - 2 \lambda \Delta q(x) \gamma_5 + \Delta_T q(x) \gamma_5 \gamma_\perp \}
\]

where \( \lambda = \pm 1/2 \) is the nucleon helicity.
However, nucleon, mass $m$, is moving fast along $OZ$ so

$$P^\mu \approx (P, 0, 0, P)$$

so that

$$Tr[\gamma^0 \Phi(x)] \approx Tr[\gamma^3 \Phi(x)]$$

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Thus we may take

$$q(x) = \frac{1}{2 P_0} Tr[\gamma^0 \Phi(x)],$$
Using translation invariance, final expression is

\[ q(x) = \frac{1}{\sqrt{2}} \sum_{X, \alpha} |\langle X | \psi_\alpha(0) | P, S \rangle|^2 \delta[P_X^+ - (1 - x)P^+] \]

This corresponds to intuitive definition of quark density!
Matrix element involves field at only one time, so evaluate in Interaction Picture i.e use free-field expansion

\[ \psi_\alpha(0) = \sum_\lambda \int \frac{d^3 p}{(2\pi)^3 2E_p} b(p, \lambda) u_\alpha(p, \lambda) \]

\[ + \sum_\lambda \int \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} d^{\dagger}(\bar{p}, \lambda) u_\alpha(\bar{p}, \lambda) \]
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where we use $\bar{p}$ to emphasize that $d^\dagger$ creates an antiquark. Note that

$$E_p = \sqrt{p^2 + m_q^2} \quad \text{and} \quad E_{\bar{p}} = \sqrt{\bar{p}^2 + m_q^2}$$
The nucleon, $\mathbf{P} = (0, 0, P)$, expanded as superposition of $n$-parton Fock states

$$ |\mathbf{P}, m\rangle = \left[(2\pi)^3 2P_0\right]^{1/2} \sum_n \sum_\mu \int \frac{d^3 k_1}{\sqrt{(2\pi)^3 2k_1^0}} \cdots \frac{d^3 k_n}{\sqrt{(2\pi)^3 2k_n^0}}$$

$$\times \psi_{\mathbf{P}, m}(k_1, \mu_1, ... k_n, \mu_n)$$

$$\times \delta^{(3)}(\mathbf{P} - k_1 - ... - k_n) |k_1, \mu_1, ... k_n, \mu_n\rangle.$$ 

where $\mu_i$ denotes helicity.
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\times \psi_{\mathbf{P}, m}(k_1, \mu_1, \ldots k_n, \mu_n) \\
\times \delta^{(3)}(\mathbf{P} - k_1 - \ldots - k_n)|k_1, \mu_1, \ldots k_n, \mu_n\rangle
\]

where \( \mu_i \) denotes helicity.
\( \psi_{\mathbf{P}, m} \) is the partonic wave function of nucleon normalized so that

\[
\sum_\{\sigma\} \int d^3k_1 \ldots d^3k_n |\psi_{\mathbf{P}, m}(k_1, \mu_1, \ldots k_n, \mu_n)|^2 \delta^{(3)}(\mathbf{p} - k_1 - \ldots - k_n) = \mathcal{P}_n
\]

with \( \mathcal{P}_n \) the probability of the \( n \)-parton state.
Consider contribution to the matrix element $\langle X | \psi_\alpha(0) | P \rangle$ when the nucleon is represented by Fock state with $n$ constituents

$$| k_1, \mu_1; \ldots; k_n, \mu_n \rangle$$

with $\sum_j k_{jT} = 0$ and $\sum_j k_{jz} = P$
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Show that this is not quite a miracle and that there is no problem also in the Instant form.
Focus on the $d^\dagger$ term in $\psi_\alpha(0)$

\[
\langle X | d^\dagger(\bar{p}, \lambda) | k_1, \mu_1; \cdots - - - - k_n, \mu_n \rangle = \\
\langle X | \bar{p}, \lambda : k_1, \mu_1; \cdots - - - - k_n, \mu_n \rangle
\]
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Aside from combinatoric factors this implies

$$| X \rangle = | \bar{p}, \lambda : k_1, \mu_1; \cdots \cdots \cdots \cdots k_n, \mu_n \rangle$$
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For this state

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P_{Xz} = \sum_j k_{jz} + \bar{p}_z = P + \bar{p}_z
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$$

Its energy is

$$
E_X = \sum_j \sqrt{k_j^2 + m_q^2 + E_{\vec{p}}} \\
\geq \sum_j k_{jz} + E_{\vec{p}} \\
\geq P + E_{\vec{p}}
$$
This implies

\[ P_X^+ > P^+ + \bar{p}^+ > P^+ \]

Therefore \( P_X^+ \) cannot equal \((1 - x)P^+\) since \( x > 0 \) so that the delta-function condition cannot be fulfilled.
This implies
\[ P^+_X > P^+ + \bar{p}^+ > P^+ \]

Therefore \( P^+_X \) cannot equal \( (1 - x)P^+ \) since \( x > 0 \) so that the delta-function condition cannot be fulfilled. Thus the \( d^{\dagger} \) term does not contribute.
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Therefore \( P_X^+ \) cannot equal \((1 - x)P^+\) since \( x > 0 \) so that the delta-function condition cannot be fulfilled. Thus the \( d^\dagger \) term does not contribute.

The rest of the derivation is standard giving \( q(x) \) as the sum of wave-functions squared for helicity \( +1/2 \) + helicity \( -1/2 \).
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so that the delta-function condition cannot be fulfilled since \( x > 0 \).
Thus the \( d^\dagger \) term does not contribute.

The rest of the derivation is standard giving \( q(x) \) as the sum of wave-functions squared for helicity \(+1/2\) and helicity \(-1/2\).

Thus the identification of the wave functions used by BLT with the standard parton densities is correct.
First ever use of the transverse sum rule

First moment of $u$ and $d$ transversity from Anselmino et al arXiv:0812.4366 ....assumes sea quark transversity zero

$$J_{Tr} = \frac{1}{2} = 0.16^{+0.07}_{-0.14} + L_{Tr}$$

Compare with

$$J_z = \frac{1}{2} = 0.42 \pm 0.19 + L_z \quad \text{for} \quad \Delta G > 0$$

or

$$J_z = -0.21 \pm 0.46 + L_z \quad \text{for changing sign} \Delta G$$
Summary of Transverse Angular Momentum Sum Rule

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The simplification of using instant from wave functions does not affect the partonic interpretation of the terms in the sum rule.
Of course Gary did not drift into retirement after the beautiful work on spin-dependent amplitudes!
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I wish him many, many more years of fruitful research!
EXTRA SLIDES
In this section we shall draw attention to a peculiarity regarding the quark and gluon orbital angular momentum inside a transversely polarized nucleon, which does \textit{not} depend on the precise definition of quark vs gluon angular momentum.

The shortest and most direct way to obtain the correct expression for the angular momentum matrix element \( \langle P', s | J_i | P, s \rangle \) is actually from consideration of the effect of rotations on a state vector.
But if one uses the traditional approach, via the energy momentum tensor $T^{\mu\nu}$, then to begin with the matrix element of $M^{jk}$, where

$$J_{i=x,y,z} = \epsilon_{ijk}M^{jk}$$  \hspace{1cm} (1)$$

depends on some of the scalar functions appearing in the matrix element of $T^{\mu\nu}$. Using Ji’s notation $A$ and $B$ for these, one has

$$\langle P, s | M^{ij} | P, s \rangle = \frac{A}{2M(P_0 + M)}[P^i(P \times s)^j - P^j(P \times s)^i] + \frac{1}{2}(A + B)\frac{\epsilon^{ij\alpha\beta}}{M}S_\alpha P_\beta$$
Using the fact that $T^{00}$ is the energy density one can show that the energy of the nucleon fixes the value of $A$ so that

$$A\text{(nucleon)} = 1. \quad (2)$$

Applying Eqn. (2) to the case of a longitudinally polarized nucleon with helicity $1/2$ implies that $1/2(A + B) = 1/2$. It follows that

$$B\text{(nucleon)} = 0 \quad (3)$$
Suppose now that we split $A$ and $B$ into quark (meaning a sum over flavours of quarks and antiquarks) and gluon pieces:

$$A(\text{nucleon}) = A_q + A_g = 1 \quad (4)$$

$$B(\text{nucleon}) = B_q + B_g = 0 \quad (5)$$

Then the quark and gluon components of $M_{ij}$ will be given by
\[ \langle P, s | M_{q,g}^{ij} | P, s \rangle = \frac{A_{q,g}}{2M(P_0 + M)} [P^i(P \times s)^j - P^j(P \times s)^i] \]
\[ + \frac{1/2(A + B)_{q,g}}{M} \epsilon^{ij\alpha\beta} S_{\alpha} P_{\beta} \]  

(6)

For a transversely polarized nucleon, say along \( OX \), with \( s_T = (1, 0, 0) \) and \( P = (0, 0, P) \) this becomes

\[ \langle P, s_T | J_{q,x} | P, s_T \rangle = \frac{1}{2M} [(M - P_0)A_q + 2P_0 1/2(A + B)_q] \]
\[ = \frac{1}{2M} [MA_q + P_0 B_q] \]  

(7)
Thus for a fast moving nucleon, as $P_0 \to \infty$, $J^q_{x}$, and similarly $J^g_{x}$, becomes infinite. Of course $J^\text{nucleon}_x$ is finite, since, as mentioned above, $B_q + B_g = 0$. 
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Examination of the Fock expansion shows that the $P_0$ term can only come from the orbital angular momentum. This is supported by a purely classical picture where the orbital angular momentum is generated by the quark rotating about the CM of the nucleon.
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Thus irrespective of how the angular momentum is split into quark and gluon components, the separate quark and gluon orbital angular momentum in a fast moving transversely polarized nucleon becomes infinite. Their sum, however, is finite.