Solving the DSE for the Minkowski Fermion Propagator in Quenched QED

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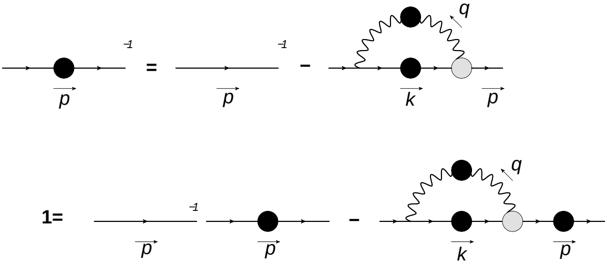

Figure : Diagrammatic representations of fermion propagator DSE in propagator form (above) and spectral form (below).

Spectral Representation for Propagator Functions

spectral function $\rho(W)$

$$S_F(p) = \int_{|W|\geq m}^{+\infty} dW \frac{\rho(W)}{\not{p} - W + i\epsilon}, \quad \rho(W) = \mathrm{sign}(W)[W\rho_1(W^2) + \rho_2(W^2)]$$

$$D(p^2 + i\epsilon) \Longleftrightarrow \rho(s) \quad (1)$$

$$D(p^2) = \int_{s\geq m^2}^{+\infty} ds \frac{\rho(s)}{p^2 - s + i\epsilon}$$

$$\rho(s) = -\frac{1}{\pi}\mathrm{Im}\,\{D(s + i\epsilon)\}$$

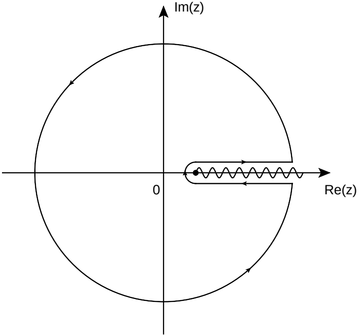

Figure : the analytic structure of propagator function for massive particles

Spectral Representation for $S_F(k)\Gamma^\mu(k,p)S_F(p)$

The original Gauge Technique

$$S_F(k)\Gamma^\mu(k,p)S_F(p) = \int dW \rho(W) \frac{1}{\not{k} - W}\gamma^\mu \frac{1}{\not{p} - W}$$

satisfies longitudinal Ward-Takahashi identity.

Delbourgo and West [1977 J. Phys. A: Math. Gen. 10 1049(1977)]

$$1 = Z_2 \int dW \rho(W), \quad mZ_m = Z_2 \int dW\, W \rho(W).$$

$$\rho(W) = \delta(W - m) + r(W), \quad a = 3\alpha/(4\pi)$$

$$r(W) = -\mathrm{sign}(W)\theta(W^2 - m^2)\frac{2a}{W}\left(\frac{W^2 - m^2}{\mu^2}\right)^{-2a} \frac{m^2}{W^2 - m^2}\times$$
$$\left\{ {}_2F_1\left(-a, -a; -2a; 1 - \frac{W^2}{m^2}\right) + \frac{W}{m}\, {}_2F_1\left(-a, 1 - a, -2a, 1 - \frac{W^2}{m^2}\right) \right\} \quad (2)$$

Fermion Propagator DSE

Unrenormalized DSE for fermion propagator

$$1 = (\not p - m)S_F(p) + ie^2 \int d\underline{k} \int dW \gamma^\nu \frac{1}{\not k - W} \mathcal{K}^\mu(k,p) \frac{1}{\not p - W} D_{\mu\nu}(q)\rho(W). \quad (3)$$

$\mathcal{K}^\mu = \gamma^\mu$ recovers the original Gauge Technique.
Define $\sigma_{1,2}(p^2)$ linear in ρ,

$$\sigma_1(p^2) + \not p \sigma_2(p^2) = ie^2 \int d\underline{k} \int dW \gamma^\nu \frac{1}{\not k - W} \mathcal{K}^\mu(k,p) \frac{1}{\not p - W} D_{\mu\nu}(q)\rho(W).$$

Eq (3) rewritten as two coupled equations

$$1 + mS_2(p^2) = p^2 S_1(p^2) + \sigma_1(p^2) \quad (4)$$

$$mS_1(p^2) = S_2(p^2) + \sigma_2(p^2). \quad (5)$$

Renormalizability Requirements

Loop-Renormalizability

Divergences from integral over loop momentum k should be removable after renormalization conditions.

$$\sigma_1(p^2) = p^2\Sigma_1(p^2)S_1(p^2) + \Sigma_2(p^2)S_2(p^2) = \frac{\lambda\alpha}{4\pi\epsilon}Z_2^{-1} + \overline{\sigma}_1(p^2), \quad (6)$$

$$\sigma_2(p^2) = \Sigma_1(p^2)S_2(p^2) + \Sigma_2(p^2)S_1(p^2) = \overline{\sigma}_2(p^2), \quad (7)$$

where $Z_2^{-1} = \int ds\rho_1(s)$, $\Sigma_1(p^2)\not{p} + \Sigma_2(p^2) = (\sigma_1 + \not{p}\sigma_2)S_F^{-1}$.

Renormalized equations become

$$\left(1 - \frac{\lambda\alpha}{4\pi\epsilon}\right)Z_2^{-1} + mZ_mS_2 = p^2S_1 + \overline{\sigma}_1 \quad (8)$$

$$mZ_mS_1 = S_2 + \overline{\sigma}_2. \quad (9)$$

At quenched approximation,

$$\sigma_1(p^2) = -\frac{3\alpha}{4\pi}\int ds\frac{sK(p^2,s)}{p^2-s}\rho_1(s) + \frac{\alpha\xi}{4\pi}\int ds\left(C_{div} + 1 + \ln\frac{\nu^2}{s-p^2}\right)\rho_1(s) \tag{10}$$

$$\sigma_2(p^2) = -\frac{3\alpha}{4\pi}\int ds\frac{K(p^2,s)}{p^2-s}\rho_2(s) + \frac{\alpha\xi}{4\pi}\int ds\frac{1}{p^2}\left(-1 + \frac{s}{p^2}\ln\frac{s}{s-p^2}\right)\rho_2(s), \tag{11}$$

where

$$K(p^2,s) = C_{div} + \frac{4}{3} + \ln\frac{\nu^2}{s-p^2} - \frac{s}{p^2}\ln\frac{s}{s-p^2}, \quad C_{div} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi. \tag{12}$$

Modification to the original gauge technique

$$K(p^2,s) = \frac{4}{3} + \ln\frac{s}{s-p^2} - \frac{s}{p^2}\ln\frac{s}{s-p^2} \tag{13}$$

such that loop divergence for fermion propagator DSE can be removed.

On-Shell Renormalization

On-shell renormalization conditions

$$\lim_{\mu^2 \to m^2} (\mu^2 - m^2) S_1(\mu^2) = 1, \quad \lim_{\mu^2 \to m^2} (\mu^2 - m^2) S_1(\mu^2) = m. \tag{14}$$

Equivalently,

$$S_1(p^2) = \frac{1}{p^2 - m^2} + P_1(p^2), \quad S_2(p^2) = \frac{m}{p^2 - m^2} + P_2(p^2), \tag{15}$$

with

$$\lim_{\mu^2 \to m^2} (\mu^2 - m^2) P_1(\mu^2) = 0, \quad \lim_{\mu^2 \to m^2} (\mu^2 - m^2) P_2(\mu^2) = 0.$$

On-Shell renormalization conditions indicate

$$\rho_1(s) = \delta(s - m^2) + r_1(s), \quad \rho_2(s) = m\delta(s - m^2) + r_2(s), \tag{16}$$

with $r_{1,2}(s)$ regular at $s = m^2$.

Reparameterization of Kernel Functions

Fermion DSE is an elegant system because spectral representation can be used consistently on both propagator functions and kernel functions.

$$\frac{s'K(s,s')}{s-s'} = \frac{s'}{s-s'}\left(\frac{4}{3} + \ln\frac{\nu^2}{s'}\right) + \frac{s'}{s}\ln\frac{s'}{s'-s}$$

To find out the imaginary part of $\sigma(p^2)$,

$$\frac{-1}{\pi}\mathrm{Im}\left\{\frac{1}{s-s'+i\epsilon}\right\} = \delta(s-s') \quad (17)$$

$$\frac{-1}{\pi}\mathrm{Im}\left\{\int ds'\left(C_{div} + 1 + \ln\frac{\nu^2}{s'}\right)\rho_1(s')\right\} = 0. \quad (18)$$

$$\frac{s'}{s}\ln\frac{s'}{s'-s} = \frac{1}{z}\ln\frac{1}{1-z} = \int d\zeta - \frac{1}{\zeta}\frac{\theta(\zeta-1)}{z-\zeta+i\epsilon} \quad (19)$$

$$\ln\frac{s'}{s'-s} = \ln\frac{1}{1-z} = \int d\zeta\frac{-\theta(\zeta-1)}{z-\zeta+i\epsilon}. \quad (20)$$

After on-shell renormalization, DSE for $r_1(s)$ and $r_2(s)$ becomes

$$\left(1-\frac{\alpha}{\pi}\right)[s^2 r_1(s)-msr_2(s)]+\frac{3\alpha}{4\pi}\left[m^2\theta(s-m^2)+\int_{m^2}^{s} ds' s' r_1(s')\right]$$
$$=\frac{(3-\xi)\alpha}{2\pi}s+\frac{\alpha\xi}{4\pi}\left[s\theta(s-m^2)+s\int_{m^2}^{s} ds' r_1(s')\right]$$

$$\left(1-\frac{\alpha}{\pi}\right)[-msr_1(s)+sr_2(s)]+\frac{3\alpha}{4\pi}\left[m\theta(s-m^2)+\int_{m^2}^{s} ds' r_2(s')\right]$$
$$=\frac{\alpha\xi}{4\pi}\left[\frac{m^3}{s}\theta(s-m^2)+\frac{1}{s}\int_{m^2}^{s} ds' s' r_2(s')\right]. \tag{21}$$

DSE for $r_{1,2}$ in the Landau Gauge

Define coupling parameter

$$a = \frac{3\alpha/(4\pi)}{1 - \alpha/\pi}. \tag{22}$$

After taking the derivative on s, eq (21) becomes

$$\left[\begin{pmatrix} s & -ms \\ -m & s \end{pmatrix} \frac{d}{ds} + \begin{pmatrix} a+1 & -m \\ 0 & a+1 \end{pmatrix}\right] \begin{pmatrix} sr_1(s) \\ r_2(s) \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix}. \tag{23}$$

When decoupled

$$sr_1(s) = g_1(s)f_1(s), \quad r_2(s) = g_2(s)f_2(s).$$

$$g_1(s) = \left(\frac{m^2}{s - m^2}\right)^{a+1}, \quad g_2(s) = \left(\frac{m^2}{s - m^2}\right)^{a} \frac{m}{s}$$

$$\frac{d}{ds}f_1(s) + \frac{a}{s}f_2(s) = \frac{2a}{m^2}\left(\frac{s - m^2}{m^2}\right)^{a} \tag{24}$$

$$\frac{d}{ds}f_2(s) + \frac{(a+1)m^2}{(s - m^2)^2}f_1(s) = \frac{2a}{s - m^2}\left(\frac{s - m^2}{m^2}\right)^{a}. \tag{25}$$

Taking another derivative with respect to s yields

$$\frac{d}{ds} s \frac{d}{ds} f_1(s) - \frac{a(a+1)m^2}{(s-m^2)^2} f_1(s) = \frac{2a(a+1)}{m^2} \left(\frac{s-m^2}{m^2} \right)^a \tag{26}$$

Solutions depend on parameter a. Because $\alpha > 0$, $a > 0$ or $a < -3/4$.

For $a > 0$ or equivalently $\alpha < \pi$,

$$\begin{cases} r_1(s) = \frac{2a}{(a+1)s} \left[1 + \frac{a^2}{(2a+1)} \, {}_2F_1 \left(a+1, a+1; 2a+2; -\frac{s-m^2}{m^2} \right) \right] \\ r_2(s) = -\frac{2a^2}{(2a+1)m} \, {}_2F_1 \left(a+1, a+2; 2a+2; -\frac{s-m^2}{m^2} \right) \end{cases} \tag{27}$$

Another linearly independent solution for $r_1(s)$

$$r_1(s) = \frac{2a}{(a+1)s} + \frac{c}{s} (s/m^2 - 1)^{-2a-1} \, {}_2F_1(-a, -a; -2a; 1 - s/m^2)$$

happens to agree with Delbourgo and West [1977 J. Phys. A: Math. Gen. 10 1049(1977)]

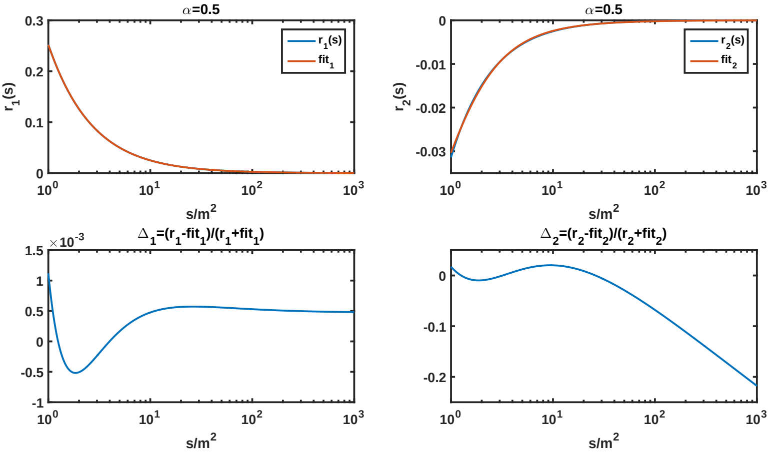

Figure : Complex conjugate pole fits to $r_{1,2}(s)$ with $\alpha = 0.5$

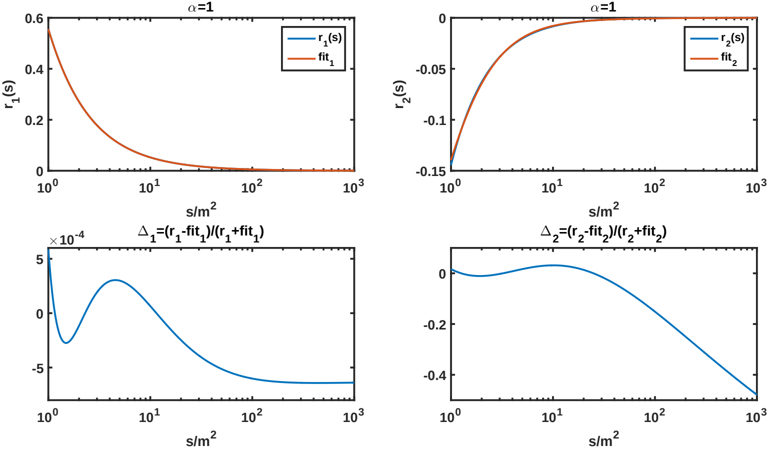

Figure : Complex conjugate pole fits to $r_{1,2}(s)$ with $\alpha = 1.0$

$$r_{fit}(s) = \frac{x + iy}{s - (\xi + i\zeta)} + (c.c.)$$

Fitting Parameters for $r_1(s)$

α	x_1	y_1	ξ_1	ζ_1
0.5	0.1242	-0.0280	0.0028	0.1418
1	0.2597	-0.1282	0.0142	0.1746
2	0.5555	-6.7860	0.1380	0.0218
3	0.5134	0.1682	0.9700	0.0237

Fitting Parameters for $r_2(s)$

α	x_2	y_2	ξ_2	ζ_2
0.5	-0.0106	0.0321	-0.3908	0.6673
1	-0.0260	1.0371	-0.7898	0.1726
2	0.0036	4.7054	-0.2218	0.1537
3	0.0831	0.9964	0.9617	0.1185

Summary and Outlook

Summary

Spectral Representation for $S_F(p)$ and $S_F\Gamma^\mu S_F$
DSE for Fermion Propagator Spectral Functions
Loop-renormalizability
Exact Solutions in the Landau Gauge

Outlook

Improvement to the original Gauge Technique
Consistency with LKFT