Isospin-Given Charge Formalism for Three-Body Nuclear Systems

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Outline

Formalism: two isospin bases for three-body system
\( I_z = \pm 1/2 \)

Three-nucleon systems, \textit{pnn} and \textit{npp}: given charge basis and “isospinless” model

Kaonic clusters \textit{KNN}: particle representation/given charge basis, averaged potential/isospinless model
Motivation

Based on the given charge formalism we propose an interpretation for

“Isospinless model”


The isospin-averaged model [3] as example of an isospinless approach for kaonic clusters

“Particle representation”

Isospin basis for NNN system ($I_z=\pm 1/2$)

The three-body isospin basis for the configuration $1+(2+3)$

$$\eta_1 = \frac{1}{\sqrt{2}} (\eta_{++} - \eta_{-+}), \text{ singlet,} \quad I=1/2$$

$$\eta_2 = \sqrt{\frac{2}{3}} (\eta_{++} - \frac{1}{2} \eta_{++} - \frac{1}{2} \eta_{-+}), \text{ triplet,} \quad I=1/2$$

$$\eta_3 = \frac{1}{\sqrt{3}} (\eta_{++} + \eta_{-+} + \eta_{++}), \text{ triplet,} \quad I=3/2$$

the isospin functions: $\eta_{++} = \eta_{+}(1)\eta_{+}(2)\eta_{+}(3)$,

$\eta_{-+} = \eta_{-}(1)\eta_{+}(2)\eta_{+}(3)$,

$\eta_{++} = \eta_{+}(1)\eta_{+}(2)\eta_{-}(3)$.

Where, for example, $\eta_{-}(k)$ is eigenfunction of the isospin of $k$-th particle with projection of $-\frac{1}{2}$. 
**Isospin given charge basis**

The isospin functions $\eta_{+-+}$, $\eta_{-++}$ and $\eta_{+++}$ represent new isospin basis $\tau$ with the elements $\tau_1$, $\tau_2$, $\tau_3$, respectively.

The matrix of transformation of the $\eta$ and $\tau$ bases is given by flowing relation:

$$\eta = S \tau,$$

where

$$\tau = (\tau_1, \tau_2, \tau_3)^T,$$

$$\tau_1 = \eta_{+-+}, \quad \tau_2 = \eta_{-++}, \quad \tau_3 = \eta_{+++},$$

and

$$S = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}.$$
Two particles are identical

(Differential Faddeev equations)

(18)

\[
(H_0 + V_{23} - E)U = -V_{23}(W - P \ W)
\]

\[
(H_0 + V_{12} - E)W = -V_{12}(U - P \ W)
\]

\[
\Psi = U + (1 + P)W
\]

$P$ is the permutation operator for the identical particles

Jacobi coordinates
Representation operators within the given charge basis

The unitary transformation given by the matrix $S$ in (18) leads to the matrices:

There are non-diagonal elements

$$S^T V I^{(1,2)} S = \begin{pmatrix} 0 & V^+ & V^- \\ 0 & V^- & V^+ \\ v^t & 0 & 0 \end{pmatrix}, \quad S^T V P^{(1,2)} S = \begin{pmatrix} 0 & V^- & V^+ \\ 0 & V^+ & V^- \\ v^t & 0 & 0 \end{pmatrix},$$

$$S^T V I^{(2,1)} S = \begin{pmatrix} V^- & 0 & V^+ \\ V^+ & 0 & V^- \\ 0 & v^t & 0 \end{pmatrix}, \quad S^T V P^{(2,2)} S = \begin{pmatrix} V^+ & 0 & V^- \\ V^- & 0 & V^+ \\ 0 & v^t & 0 \end{pmatrix},$$

where $V^+ = \frac{1}{2}(v^t + v^s)$ and $V^- = \frac{1}{2}(v^t - v^s)$. 

$v^t, v^s$ - triplet and singlet component of potentials
**Application for three-nucleon systems**

We can assume that 
\[ v_{23}^s = v_{23}^t = v_{23} \]
\[ v_{12}^s = v_{12}^t = v_{12} \]

The corresponding set of the Faddeev equations can be reduced using the matrix transformation:
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & -1 \\
1 & 0 & 0
\end{pmatrix}
\]
and we obtain the "isospin-less" model

\[
(H_0 + V_{23} - E)U = -V_{23}(W - P_{23}W)
\]
\[
(H_0 + V_{12} - E)W = -V_{12}(U - P_{23}W)
\]

after separation of isospin variables.

The final equations were obtained by using a spin basis and taking into account the spin-splitting nucleon-nucleon potential.
Isospin models for Faddeev equations for pnn system

**pnn system total spin S=1/2 within S-wave approach**

**within the isospin formalism (all particle are identical):**

\[
(H_0 + V^{s}_{np} - E)W_1 = -V^{s}_{np} \left( \frac{1}{4} (p^+ + p^-) W_1 - \frac{3}{4} (p^+ + p^-) W_2 \right),
\]

\[
(H_0 + V^{t}_{np} - E)W_2 = -V^{t}_{np} \left( -\frac{3}{4} (p^+ + p^-) W_1 + \frac{1}{4} (p^+ + p^-) W_2 \right),
\]

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**Within given charge formalism (two particle are identical):**

\[
(H_0 + V^{s}_{nn} - E)U = -V^{s}_{nn} \left( -\frac{1}{2} (1 + p_{23}) W_1 + \frac{\sqrt{3}}{2} (1 + p_{23}) W_2 \right),
\]

\[
(H_0 + V^{s}_{np} - E)W_1 = -V^{s}_{np} \left( -\frac{1}{2} U - \frac{1}{2} p_{23} W_1 - \frac{\sqrt{3}}{2} p_{23} W_2 \right),
\]

\[
(H_0 + V^{t}_{np} - E)W_2 = -V^{t}_{np} \left( \frac{\sqrt{3}}{2} U - \frac{\sqrt{3}}{2} p_{23} W_1 + \frac{1}{2} p_{23} W_2 \right),
\]
NN-forces Isospin structure within AAA model:

**Class I** (isospin invariant forces): $[V_{II}^{2N}, T] = 0$ \[\Rightarrow V_{II}^{2N} = \alpha + \beta(\tau_1 \cdot \tau_2)\]

**Class II** (charge independence breaking):
$[V_{II}^{2N}, T] \neq 0, \quad [V_{II}^{2N}, P_{cr}] = [V_{II}^{2N}, (T)^2] = 0$ \[\Rightarrow V_{II}^{2N} = \alpha \tau_1^3 \tau_2^3\]

**Evidence:** $1/2(\delta_{nn}^\alpha + \delta_{pp, str}^\alpha) \neq \delta_{np}^\alpha$

In particular: $a_{nn}^{1S0} \approx -18.9$ fm, $a_{pp, str}^{1S0} \approx -17.5$ fm, $a_{np}^{1S0} = -23.74(2)$ fm

**Class III** (charge symmetry breaking, no isospin mixing):
$[V_{III}^{2N}, T] \neq 0, \quad [V_{III}^{2N}, P_{cr}] \neq 0, \quad [V_{III}^{2N}, (T)^2] = 0$ \[\Rightarrow V_{III}^{2N} = \alpha (\tau_1^3 + \tau_2^3)\]

**Evidence:** $\delta_{nn}^\alpha \neq \delta_{pp, str}^\alpha$, BE difference of mirror nuclei, ...

**Class IV** (charge symmetry breaking and isospin mixing):
$[V_{IV}^{2N}, T] \neq 0, \quad [V_{IV}^{2N}, P_{cr}] \neq 0, \quad [V_{IV}^{2N}, (T)^2] \neq 0$ \[\Rightarrow V_{IV}^{2N} = \alpha (\tau_1^3 - \tau_2^3) + \beta(\tau_1 \times \tau_2)^3\]

**Evidence:** different neutron/proton analyzing powers in np scattering, ...

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from E. Epelbaum presentation, Lacanau, 28.09.2009
Modification of the s-wave MT-I-III potential

Structure of the Faddeev equations (S=1/2)

\[
(H_0 + V^s_{nn} - E)U = -V^s_{nn} \left( -\frac{1}{2} \left( 1 + p_{23} \right) W_1 + \frac{\sqrt{3}}{2} \left( 1 + p_{23} \right) W_2 \right),
\]

\[
(H_0 + V^s_{np} - E)W_1 = -V^s_{np} \left( -\frac{1}{2} U - \frac{1}{2} p_{23} W_1 - \frac{\sqrt{3}}{2} p_{23} W_2 \right),
\]

\[
(H_0 + V^t_{np} - E)W_2 = -V^t_{np} \left( \frac{\sqrt{3}}{2} U - \frac{\sqrt{3}}{2} p_{23} W_1 + \frac{1}{2} p_{23} W_2 \right),
\]

Deuteron in subsystem

MT-I-III

is defined for spin singlet and spin triplet component of \( np \) pair

\[V^s_{nn} = \gamma_1 V^s_{np}; \quad V^s_{pp} = \gamma_2 V^s_{np}\]
Singlet $NN$ scattering lengths and effective radii (in fm, given in brackets) for modified MT I-III and ATS3 potentials.

<table>
<thead>
<tr>
<th>NN</th>
<th>Exp.</th>
<th>MT I-III</th>
<th>ATS3(M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>np</td>
<td>-23.748±0.009 (2.75±0.05)</td>
<td>-23.55 (2.84)</td>
<td>-23.75 (2.69)</td>
</tr>
<tr>
<td>nn</td>
<td>-18.8±0.3 (2.75±0.11)</td>
<td>-18.80 (2.95)</td>
<td>-18.87 (2.72)</td>
</tr>
<tr>
<td></td>
<td>-18.95±0.4 (2.75±0.11)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>pp</td>
<td>-17.3±0.4 (2.85±0.04)</td>
<td>-17.35 (2.96)</td>
<td>-17.30 (2.73)</td>
</tr>
</tbody>
</table>


Singlet $NN$ scattering lengths $a_{NN}$ for different $NN$ potentials are used for the calculation of $\xi$-parameter: $\xi = (a_{pp} - a_{nn})/(a_{nn} - a_{np})$. The value $\xi=1/3$ is presented by the horizontal dashed line. The data are from R. Machleidt, D.R. Entem / Physics Reports 503 (2011) 1-75.
Charge symmetry breaking effect for $^3$He and $^3$H

$B(^3\text{H})$ and $B(^3\text{He})$ binding energies (in MeV), the Coulomb energy $\Delta B_c$ (in keV), CSB effect for energy $\Delta B_c$ (in keV). The results of Ref. [46] (Ref. [34]) are given in brackets (square brackets). $m_n$ ($m_p$) is the mass of neutron (proton).

<table>
<thead>
<tr>
<th>Potential</th>
<th>Masses</th>
<th>$B(^3\text{H})$</th>
<th>$B(^3\text{He})$</th>
<th>$\Delta B$</th>
<th>$\Delta B_c$</th>
<th>$\Delta B_c$ (CSB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MT I-III</td>
<td>np</td>
<td>8.534</td>
<td>7.878</td>
<td>656</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m_N$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>–</td>
<td>(8.535)</td>
<td>(7.868)</td>
<td>667</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>–</td>
<td>[8.54]</td>
<td>[7.88]</td>
<td>660</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m_n \neq m_p$</td>
<td>8.536</td>
<td>7.871</td>
<td>665</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modified</td>
<td>nn</td>
<td>$m_n \neq m_p$</td>
<td>8.376</td>
<td>7.715</td>
<td>661</td>
<td></td>
</tr>
<tr>
<td>MT I-III</td>
<td>pp</td>
<td>–</td>
<td>7.666</td>
<td>710</td>
<td>49</td>
<td></td>
</tr>
</tbody>
</table>

Our result is about 50 keV. The similar value have been previously obtained* without the magnetic and other Breit interaction effects.

pd break-up, 14.1 MeV

Isospin formalism

MT I-III

CD effect within given charge formalism

Modified MT I-III vs MT I-III
Kaonic clusters $K$-NN

The given charge formalism and “particle representation”

As an example of the particle representation:

Constraining the $\vec{K}N$ interaction from the 1S level shift of kaonic deuterium

Tsubasa Hoshino, Shota Ohnishi, Wataru Horiuchi, Tetsuo Hyodo, Wolfram Weise
arXiv:1706.06857 [nucl-th]
(or arXiv:1705.06857v3 [nucl-th] for this version)

nucleon with index $i = 2$ ($i = 3$). The isospin part, $\psi^{(\text{isospin})}$, of the wave function written in the particle basis includes the following two channels:

$$|K^-pn\rangle = |\uparrow\uparrow\downarrow\rangle, \quad |\bar{K}^0nn\rangle = |\uparrow\downarrow\downarrow\rangle. \quad (9)$$

- elements of given charge basis are differ by single cyclic permutation

Due to particle transition does not incorporate into isospin formalism

$\tau_1 = (+-)$ \quad $\tau_2 = (-++)$

Conclusion: there is a problem with “channel interpretation”; the results has to be corrected by the mass differences
The given charge formalism and “particle representation”

The unitary transformation given by the matrix $S$ in Eq. (18) leads to the matrices:

$$S^T V I^{(1,2)} S = \begin{pmatrix}
0 & V^+ & V^- \\
V^- & 0 & V^+ \\
0 & v^t & 0
\end{pmatrix}, \quad S^T V P^{(1,2)} S = \begin{pmatrix}
0 & V_- & V^+ \\
V^+ & 0 & V^- \\
0 & v^t & 0
\end{pmatrix}$$

$$S^T V I^{(2,1)} S = \begin{pmatrix}
V^- & 0 & V^+ \\
V^+ & 0 & V^- \\
0 & v^t & 0
\end{pmatrix}, \quad S^T V P^{(2,2)} S = \begin{pmatrix}
V^+ & 0 & V^- \\
V^- & 0 & V^+ \\
0 & v^t & 0
\end{pmatrix}.$$

**Conclusion:** there is a problem with “channel interpretation”; the results has to be corrected by the mass differences.
Averaged potential model as reduction to the isospinless approach

The isospinless approach is working for $pnn$ and $nnp$ system due to weak dependence of nucleon-nucleon potential on isospin variables.

$K\cdot N$ interaction essentially depends on pair isospin

**Kaonic cluster $K\cdot pp$**

The isospin averaged potential $V_{KN}^{av}$ is defined as:

$$V_{KN}^{av} = \frac{3}{4} u_{KN}^s + \frac{1}{4} u_{KN}^t.$$  

The model based on the averaged potential = the isospinless approach.

After separation of isospin variables the Faddeev equations are written as

$$(H_0^U + V_{AA} - E)U = -V_{AA}(1+p)W, $$

$$(H_0^W + V_{AB}^{av} - E)W = -V_{AB}^{av}(U + pW).$$

Here, $V_{AA}$ is pp potential, $V_{AB}$ is averaged K·p pot.
Averaged potential model as a reduction to the isospinless approach

Kaonic cluster K-pp

Ground state energy $E_3$ of the $NN\bar{K}$ system calculated within the averaged potential and isospin models. The energies are given in MeV.

<table>
<thead>
<tr>
<th>Model</th>
<th>[27]</th>
<th>[17]</th>
<th>Our</th>
<th>[3]</th>
<th>[33]</th>
</tr>
</thead>
<tbody>
<tr>
<td>KWW</td>
<td>AY+T</td>
<td>AY+MT</td>
<td>AY+T</td>
<td>AY+AV14</td>
<td></td>
</tr>
<tr>
<td>Averaged potential</td>
<td>-35.5</td>
<td>-39.1</td>
<td>-33.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Isospin</td>
<td>-46.0</td>
<td>-48</td>
<td>-47.34</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Conclusion: the averaged potential model understates binding energy of K-pp cluster.
Conclusions

• The given charge formalism was proposed to describe three-body nuclear systems.

• For three-nucleon systems the formalism allows us to reduce the isospin model to effective “isospinless” approach.

• Within this approach, the charge symmetry breaking effect was calculated for $^3$H and $^3$He nuclei with $nn$, $pp$ and $np$ interactions modeled by s-wave phenomenological potential MT-I-III. Results of the calculation close to the existing evaluations.

• For kaonic cluster, such reduction is possible for averaged potential model. Due to essential dependence of the antikaon-nucleon interaction on pair isospin, the obtained model is failure to describe the kaonic clusters.

• The particle representation for kaonic clusters proposed in many works has problem with channel interpretation. These works have used the given charge formalism in fact. The formalism does not include particle transitions.

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