Wilson twisted mass in the epsilon regime

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Introduction

- Why $\epsilon$ regime? Alternative and complementary to $p$ regime

Wilson (twisted mass) fermions in the $\epsilon$ regime

\[ S[\chi, \bar{\chi}, U] = S_G[U] + S_F[\chi, \bar{\chi}, U] \]

\[ S_G[U] = \frac{\beta}{3} \sum_x \left\{ b_0 \sum_{\mu < \nu} \Re \text{Tr} \left[ 1 - P^{(1 \times 1)}(x; \mu, \nu) \right] + b_1 \sum_{\mu \neq \nu} \Re \text{Tr} \left[ 1 - P^{(2 \times 1)}(x; \mu, \nu) \right] \right\} , \]

\[ S_F[\chi, \bar{\chi}, U] = a^4 \sum_x \bar{\chi}(x) \left[ D_W + i\mu_q \gamma_5 \tau^3 \right] \chi(x), \]

\[ D_W = \frac{1}{2} \left\{ \gamma_\mu (\nabla_\mu + \nabla_\mu^*) - a\nabla_\mu^* \nabla_\mu \right\} + m_0, \]

- Sample all topological sectors
- PHMC with exact reweighting
- Decrease quark mass without encountering instabilities and/or metastabilities
Outline

1. Well known facts
   - Wilson fermions phase diagram
   - $\epsilon$ expansion

2. $\epsilon$ expansion of $W\chi$PT
   - Phase diagram
   - Cutoff effects

3. Simulations with $N_f = 2$ mass degenerate light Wtm quarks in the $\epsilon$ regime
   - Algorithm
   - Low energy constants

4. Conclusions and outlooks
**Phase diagram**

Wilson twisted mass fermions → metastabilities

\[ \mu_q > \frac{a^2 |w'|}{B_0} \]

What happens in the \( \epsilon \) regime?

Wilson fermions → instabilities

\[ m \geq \frac{n}{Z} \frac{a}{\sqrt{V}} \]

Reweighting can be useful

\[ \frac{2\mu B}{a^2 w'} \quad \frac{2m'B}{a^2 w'} \]

(Aoki:1984;Sharpe,Singleton:1998)

\( \epsilon \) expansion in the continuum

- Integrate exactly over the constant zero modes, and treat the non-zero modes as standard perturbations
- Modify the power counting of the \( p \) regime, in a power counting where the pion mass \( M_\pi \) is small compared to the linear sizes of the box

\[
\frac{1}{T} = O(\epsilon), \quad \frac{1}{L} = O(\epsilon), \quad M_\pi = O(\epsilon^2).
\]

The order parameter, vanishes in the chiral limit at fixed finite volume

\[
R = \frac{\langle \bar{q}q \rangle}{B_0 F^2}
\]

(Gasser, Leutwyler: 1987)

\[ R \]

\[ m_\text{R} [\text{MeV}] \]

\[ m_\text{R} [\text{MeV}] \]

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\[ m_\text{R} [\text{MeV}] \]
**ε expansion with Wilson fermions**

- Continuum → chiral symmetry restoration
- Include the effects of the non vanishing lattice spacing in the ε expansion
- Study the mass dependence of the chiral condensate

New suitable power counting in Wilson chiral perturbation theory (WχPT)

\[ M = O(\epsilon^4), \quad \frac{1}{L} = O(\epsilon), \quad \frac{1}{T} = O(\epsilon) \quad a^2 = O(\epsilon^4) \]

Is this the appropriate power counting?

\[ M \approx 5\text{MeV}, \quad a \approx 0.1\text{fm}, \quad L \approx 1.5\text{fm} \]

\[ F \approx 90\text{MeV}, \quad B_0 \approx 5.5\text{GeV}, \quad |w'| \approx (570\text{MeV})^4 \]

\[ \Rightarrow \quad MF^2B_0V \approx 0.75, \quad a^2F^2|w'|V \approx 0.75, \quad \frac{MB_0}{a^2|w'|} \approx 1 \]

- We are inside the Aoki region but with a finite volume
- The dependence of the “order parameter” on the “external field” is smooth
The partition function at leading order

\[ \mathcal{Z} = \int \mathcal{D}[\Sigma_0] e^{\frac{c_1 V}{2} \text{Tr} [\Sigma_0 + \Sigma_0^\dagger] - \frac{c_2 V}{4} \text{Tr} [\Sigma_0 + \Sigma_0^\dagger]^2 + \frac{c_3 V}{2} \text{Tr} [i \tau^3 (\Sigma_0^\dagger - \Sigma_0)]} \]

New scaling variable \( z_2 \)

\[ z_1 = c_1 V = B_0 F^2 m' V, \quad z_2 = c_2 V = -\frac{F^2 w' V a^2}{4}, \quad z_3 = c_3 V = B_0 F^2 \mu_R V. \]

We can compute the chiral condensate

\[ R = \frac{\langle \bar{q} q \rangle}{B_0 F^2} \]

\[ R = \frac{1}{N_f} \frac{\partial}{\partial z_3} \log \mathcal{Z}, \quad z_1 = 0. \]
\( \epsilon \) expansion with Wilson fermions

**Smooth dependence on the quark mass.**

**No phase transition**

**Cutoff effects under control for the order parameter**
\( \epsilon \) expansion with Wilson fermions

- General power counting: valid also for Wilson fermions
- Extension to NLO in progress
- No phase transition/minimal twisted mass in the \( \epsilon \) regime
- It could be used to attack other problems like: interplay between \( a \) and \( V \) in the eigenvalues distribution
- Alternative way to extract the LEC \( |w'| \) which parametrized the \( O(a^2) \) effects
- Is this the correct power counting?
Simulate in the $\epsilon$ regime with Wtm using PHMC with exact reweighting

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[U] e^{-S_G[U]} \det\left( QQ^\dagger [U] \right) O [U] \quad Q = \gamma_5 \left[ D_W + i\mu_q \gamma_5 \right],$$

$Q$ single flavour operator

$$\det\left( QQ^\dagger [U] \right) = \frac{\det \left[ QQ^\dagger P_{n,\tilde{\epsilon}} \left( QQ^\dagger \right) \right]}{\det \left[ P_{n,\tilde{\epsilon}} \left( QQ^\dagger \right) \right]}, \quad P_{n,\tilde{\epsilon}} \left( QQ^\dagger \right) \simeq \left[ QQ^\dagger \right]^{-1},$$

$$P_{n,\tilde{\epsilon}} \left( QQ^\dagger \right) \simeq \left[ QQ^\dagger \right]^{-1} \quad \{ \lambda \} \in [\tilde{\epsilon}, 1]$$

**Observables**

$$\langle O \rangle = \frac{\langle OW \rangle_P}{\langle W \rangle_P}, \quad W = \det \left[ QQ^\dagger P_{n,\tilde{\epsilon}} \left( QQ^\dagger \right) \right] \simeq \prod_{\lambda_i < \tilde{\epsilon}} [\lambda_i P_{n,\tilde{\epsilon}} (\lambda_i)],$$

- Better sampling of configuration space
- With a twisted mass no instabilities issues
- In the $\epsilon$ regime no metastabilities issues
Simulation details

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<tr>
<th>( \beta )</th>
<th>( \kappa )</th>
<th>( L/\alpha )</th>
<th>( T/\alpha )</th>
<th>( \sigma_{\mu q} )</th>
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<th>( N_{\text{traj}} )</th>
<th>( N_{\text{ana}} )</th>
<th>( \tau_{\text{int}}(P) )</th>
<th>( \tau_{\text{int}}(m_{\text{PCAC}}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2500</td>
<td>421</td>
<td>( \sim 0.5 )</td>
<td>( \sim 0.5 )</td>
</tr>
</tbody>
</table>

- \( \beta \), \( \kappa \), \( L/\alpha \), \( T/\alpha \), \( \sigma_{\mu q} \), \( N_{\text{traj}} \), \( N_{\text{ana}} \), \( \tau_{\text{int}}(P) \), \( \tau_{\text{int}}(m_{\text{PCAC}}) \), \( r_0/\alpha \), \( a [\text{fm}] \), \( L [\text{fm}] \), \( a m_{\text{PCAC}} \)

\( \lambda_{\min} \)
PCAC mass

\[ M_R = \frac{1}{Z_P} M \]
\[ M = \sqrt{(Z_A m_{PCAC})^2 + \mu_q^2} \]

\[ a m_{PCAC} = 0.00045(12) \]
\[ \rightarrow a M_R = 0.0012(2) \]
\( S_{\text{eff}} = S_0 + \alpha S_1 + \ldots \quad S_0 = \int d^4 x \bar{\chi}(x) \left[ \gamma_{\mu} D_{\mu} + i \mu_R \gamma_5 \tau^3 \right] \chi(x) \)

\[ S_1 = \int d^4 y \mathcal{L}_1(y) \quad \mathcal{L}_1(y) = \sum_i c_i \mathcal{O}_i(y) \]

\[ \mathcal{O}_1 = i \bar{\chi} \sigma_{\mu \nu} F_{\mu \nu} \chi \quad \mathcal{O}_5 = \mu^2 \bar{\chi} \chi \]

\[ \langle \Phi \rangle = \langle \Phi \rangle_0 - \alpha \int d^4 y \langle \Phi \mathcal{L}_1(y) \rangle_0 + \alpha \langle \Phi_1 \rangle_0 + \ldots \]

**R**^1,2 in the \( \epsilon \) regime

LAT08

A. Shindler

Outline

Known facts

\( \epsilon W \chi \) PT

Numerical simulations

Runs

Analysis

LEC

Conclusions

Contact terms amount to a redifinition of \( \Phi_1 \)

\( \mathcal{R}^1,2 \) is not spontaneously broken

Symmetry restoration region (SRR) \( \rightarrow \) Automatic \( O(\alpha) \) improvement in the chiral limit for Wilson fermions

In the chiral limit of SRR the form of the Wilson term is actually irrelevant

In the SRR only \( O(\alpha M) \) expected \( \Rightarrow \) very small \( O(\alpha) \) even out full twist

If the mass is of \( O(\alpha^2) \) cutoff effects can become visible (observable dependent)
NLO $\varepsilon$ expansion $N_f = 2$

$$P^a(x) = \bar{\chi}(x)i\gamma_5 \frac{\tau^a}{2}\chi(x)$$

$$C_P(x_0) = \frac{1}{L^3} \int d^3x C_P(x, x_0) \delta^{ab} C_P(x, x_0) = \langle P^a(x, x_0) P^a(0, 0) \rangle$$

$$C_P(x_0) = a_P + \frac{T}{L^3} b_P \left[ \frac{y^2}{2} - \frac{1}{24} \right] + \ldots \quad y = \frac{x_0}{T} - \frac{1}{2}$$

$$a_P = \frac{B_0^2 F^4 \rho_2^2}{8} G_1(u), \quad b_P = F^2 B_0^2 \left[ 1 - \frac{1}{8} G_1(u) \right]$$

$$u = 2B_0 F^2 MV \rho, \quad \rho = 1 + \frac{3}{2} \frac{\beta_1}{F^2 \sqrt{V}}, \quad G_1(u) = \frac{8}{u} \frac{Y'(u)}{Y(u)}, \quad Y(u) = \frac{2l_1(u)}{u}$$

Fit formulae

$$C_P(x_0) = A_0 + A_2 y^2 \quad \Rightarrow \quad a_P = A_0 + \frac{A_2}{12} b_P = A_2 \frac{2L^3}{T}$$
Pseudoscalar correlation function

\[ a = 0.066 \text{ fm} \quad M_R = 3.5 \text{ MeV} \]

\[ \sigma^3 L^3 A_0 = (5.94(36)) \cdot 10^{-3}, \quad \sigma^3 L^3 A_2 = (4.81(30)) \cdot 10^{-2} \]

Random source locations

Nested Jackknife/bootstrap errors

\[ Z_2 \times Z_2 \text{ stochastic sources} \]
Effective couplings plots

\[ r_0 \Sigma^{1/3} = 0.620(8), \quad r_0 F = 0.220(8)(10) \]  

[PRELIMINARY]

\[ r_0 \Sigma^{1/3} = 0.617(15), \quad r_0 F = 0.224(10) \]  

(Hasenfratz,Hoffmann,Schaefer:2008)
Comparisons

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<th>Group</th>
<th>$N_f$</th>
<th>$\Sigma(2\text{GeV})$</th>
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<tr>
<td>This work</td>
<td>2</td>
<td>$-(282 \pm 4 \text{ MeV})^3$</td>
</tr>
<tr>
<td>ETMC (2007)</td>
<td>2</td>
<td>$-(272 \pm 4 \pm 7 \text{ MeV})^3$</td>
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<tr>
<td>JLQCD (2007)</td>
<td>2</td>
<td>$-(251 \pm 7 \pm 11 \text{ MeV})^3$</td>
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<tr>
<td>Lang et al (2007)</td>
<td>2</td>
<td>$-(276 \pm 11 \pm 16 \text{ MeV})^3$</td>
</tr>
<tr>
<td>McNeile + MILC (2005)</td>
<td>2+1</td>
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<tr>
<td>McNeile + JLQCD (2005)</td>
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<td>2</td>
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<tr>
<td>JLQCD (2007)</td>
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<td>78(3)(1)MeV</td>
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Conclusions and outlooks

Conclusions
- We are establishing the basic knowledge to simulate with Wilson-like fermions in the $\epsilon$ regime
- Introduced a power counting to study the $\epsilon$ expansion with Wilson-like fermions
- LO Chiral condensate $\rightarrow$ no phase transitions
- NLO and other observables ongoing
- Numerical simulations in the $\epsilon$ regime with Wtm
- PHMC with exact reweighting
- Sampling of all the topological sectors
- Extraction of LEC ($\Sigma$, $F$) without contamination from chiral logs

Outlooks
- Understand the usual systematic errors: discretization errors, quark mass and volume dependence
- Extend to more observables the current analysis
- Combine $p$ and $\epsilon$ regime fits
- Alternatively compute LEC from spectral quantities
Comments on the power countings

\[ \mathcal{L}_{W\chi}^{(2)} = \frac{F^2}{4} \left[ \langle \partial_\mu \Sigma(x)^\dagger \partial_\mu \Sigma(x) \rangle + \langle \sigma(x) \Sigma(x)^\dagger + \sigma(x)^\dagger \Sigma(x) \rangle + \langle A(x) \Sigma(x)^\dagger + A(x)^\dagger \Sigma(x) \rangle \right] . \]

\[ \sigma' \equiv \sigma + A \quad m_R \rightarrow m_R + aW_0/B_0 \equiv m'. \]

\( (\Sigma)^{1/3} = 250 \text{MeV} \quad \Lambda = 200 \text{MeV} \quad m \sim a\Lambda^2 \sim 2 \text{MeV} \quad V = (1.5 \text{fm})^4 \div (2.5 \text{fm})^4 \]

\[ \Rightarrow m\Sigma V = 1 \div 8 \]

Is this region \( \epsilon \) or \( p \) regime?

How do we define in which regime we are?

If we lower the quark mass \( \rightarrow \) certainly \( \epsilon \) regime

\[ m \sim a^2 \Lambda^3 \sim 2 \text{MeV} \]

Different power counting has to be adopted and cutoff effects can become visible

\[
\begin{array}{ccc}
m\Sigma V & 1 & 10 \\
\hline
m \sim O(a^2) & \epsilon/p & m \sim O(a)
\end{array}
\]

[NNLO]