Breakdown of large-$N$ reduction in the quenched Eguchi-Kawai model

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arXiv:0807.1275 for an adaptation of Wang-Landau algorithm to LGT
Large-N volume reduction

- Starting point: pure SU(N) on a single point.

\[ S_{EK} = Nb \sum_{\mu < \nu} 2\text{Re} \text{Tr} (U_{\mu} U_{\nu} U_{\mu}^\dagger U_{\nu}^\dagger) \quad \text{with} \quad b = (g^2 N)^{-1} \quad \text{symmetric under} \quad U_{\mu} \rightarrow U_{\mu} z_{\mu} ; \quad z_{\mu} \in \mathbb{Z}_N \]

- Observables:

\[ W_C = \frac{1}{N} \text{tr} U_{x,\mu} U_{x+\hat{\nu},\nu} \cdots U_{x-\hat{\rho},\rho} U_{x-\hat{\nu},\nu} , \quad W_C^{\text{reduced}} = \frac{1}{N} \text{tr} U_{\mu} U_{\nu} \cdots U_{\rho} U_{\nu} . \]

\[ \langle W_C \rangle_{\text{gauge theory}} = \langle W_C^{\text{reduced}} \rangle_{\text{reduced}} + O(1/N^2). \]

- Dyson-Schwinger Eqs.:

\[ \langle \text{tr} (U_{\mu} U_{\nu}^\dagger) \text{tr} (U_{\mu}^\dagger U_{\nu}) \rangle_{\text{reduced}} = 0 \]

1. \[ \langle W_{C_1} W_{C_2} \rangle_{\text{reduced}} = \langle W_{C_1} \rangle_{\text{reduced}} \langle W_{C_2} \rangle_{\text{reduced}} + O(1/N^2) , \]

2. \[ \langle W_{\text{open}} \rangle_{\text{reduced}} = 0. \quad \text{if } \mathbb{Z}_N \text{ intact} \]

- However, weak-coupling analysis: Bhanot, Heller & Neuberger '82 (also later Kazakov & Migdal '82)

\[ \text{Eig} (U_{\mu}) = \left( e^{ip_{\mu}}^1 , e^{ip_{\mu}}^2 , \ldots , e^{ip_{\mu}}^N \right) \quad \text{attract and break of } \mathbb{Z}_N \]
## Alternatives to EK

**Name of the game:** cause $p$ to repel each other

<table>
<thead>
<tr>
<th>Method</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quench the $p$’s to be uniform - the QEK</td>
<td>BHN <code>82, Migdal </code>82, Gross and Kitazawa <code>82, Parisi </code>82, Bars <code>83, Okawa </code>82, Parsons <code>84, Carlson </code>83, Lewis '84, Greensite and co. '83-'86</td>
</tr>
<tr>
<td>Twisted BC’s - the TEK</td>
<td>Gonzalez-Arroyo &amp; Okawa `82</td>
</tr>
<tr>
<td>Partial reduction - $L^4$ instead of $1^4$</td>
<td>but Teper and Vairinhos <code>06, Ishikawa et al.</code>07</td>
</tr>
<tr>
<td>Adjoint fermions - the AEK</td>
<td>Kovtun, Unsal and Yaffe `07</td>
</tr>
<tr>
<td>Deform the action - the DEK</td>
<td>Unsal and Yaffe `08</td>
</tr>
</tbody>
</table>
The QEK model

- **Definition of the model:**

(I) \[ \langle \mathcal{O}(U) \rangle_p = \frac{\int \prod_{\mu} DV_\mu \ e^{S_{\text{QEK}}(p)} \mathcal{O}(U)}{\int \prod_{\mu} DV_\mu \exp(S_{\text{QEK}}(p))} \]

(II) \[ S_{\text{QEK}}(p) = Nb \sum_{\mu < \nu} 2\text{Re} \ \text{Tr} \ (U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) \]

(III) \[ \langle \mathcal{O}(U) \rangle_{\text{QEK}} = \int dp \ \langle \mathcal{O}(U) \rangle_p \]

*invariant to* \( p_\mu \rightarrow p_\mu + 2\pi k_\mu / N \); \( k_\mu = 1, 2, \ldots, N \)

- **with**

\[ U_\mu = V_\mu^\dagger \Lambda_\mu V_\mu , \]

and

\[ \Lambda_\mu(p) = \begin{pmatrix} e^{ip_\mu^1} & 0 & \cdots & 0 \\ 0 & e^{ip_\mu^1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^{ip_\mu^N} \end{pmatrix} \]

\[ p_\mu^{\alpha} \in [0, 2\pi) \]
Formal proofs

✓

Planar perturbation theory.

• Perturbation theory: **integrands** of all planar diagrams in gauge theory.

• $\int dp \Rightarrow$ all planar diagrams in gauge theory

✓

W-loop's Dyson-Schwinger equations.

• $\int dp$ is $\mathbb{Z}_N$ invariant $p_\mu \rightarrow p_\mu + 2\pi k_\mu/N ; \quad k_\mu = 1, 2, \ldots, N$

• But $W_{\text{open}}$ is not and so

$$\langle W_{\text{open}} \rangle_{QEK} = \int dp \langle W_{\text{open}} \rangle_p = 0$$
Is that enough? No! due to non-perturbative effects.

\[ \exists V_{\mu} \in SU(N) \] such that:

\[ V_{\mu} \left( \begin{array}{cccccc}
    e^{i p^1_{\mu}} & 0 & \cdots & 0 \\
    0 & e^{i p^2_{\mu}} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & e^{i p^N_{\mu}}
  \end{array} \right) V_{\mu}^\dagger = \left( \begin{array}{cccccc}
    e^{i p^{(1)}_{\mu}} & 0 & \cdots & 0 \\
    0 & e^{i p^{(2)}_{\mu}} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & e^{i p^{(N)}_{\mu}}
  \end{array} \right) \]

- Focus on one point in \( \int \prod_{\mu} \prod_i dp^i_{\mu} \) (imagine N=6)

\[ \begin{array}{cccccc}
    p^x_{i=1,2,3,4,5,6} & 6 & 2 & 5 & 3 & 4 & 1 \\
    \times & \times & \times & \times & \times & \times & \\
    p^y_{i=1,2,3,4,5,6} & 3 & 6 & 1 & 5 & 4 & 2 \\
    \times & \times & \times & \times & \times & \times & \\
    0 & 2\pi & 0 & 2\pi & 0 & 2\pi & 0 \\
  \end{array} \] non-perturbatively

- “Locking” occurs in weak-coupling: minima defined by \( p^i_{\mu} - p^i_{\nu} \approx \alpha_{\mu\nu} \)

\[ M_{\mu,\nu} \equiv \text{tr} \left( U_{\mu} U_{\nu}^\dagger \right) / N \quad \text{and} \quad M_{\mu,\nu} \equiv \text{tr} \left( U_{\mu} U_{\nu}^\dagger \right) / N \quad (\mu > \nu) \]

\[ (M_{\mu,\nu})_{\text{locked}} \approx e^{i \alpha_{\mu\nu}} \]

\[ |M_{\mu,\nu}|_{\text{locked}} \approx 1 \]
Implications on Gross-Kitazawa-Parisi analysis

Planar perturbation theory

- Perturbatively fix $p$ and integrate uniformly over 4D BZ of $p$.
- Non-perturbatively can get locking and $p$ integral is not uniform.

DS equations: consider

$$W_{\text{open}} = M_{\mu,\nu} = \text{tr} \left( U_\mu U_\nu \right) / N$$

usually:

$$\langle M_{\mu\nu} M^*_{\mu\nu} \rangle_{\text{QEK}} = 0, \text{ because } \langle M_{\mu\nu} \rangle_{\text{QEK}} = \langle M^*_{\mu\nu} \rangle_{\text{QEK}} = 0$$

but if locked:

$$\int dp \langle M_{\mu\nu} M^*_{\mu\nu} \rangle_p \neq \int dp \langle M_{\mu\nu} \rangle_p \int dp \langle M^*_{\mu\nu} \rangle_p + O(1/N)$$

\[ O(1) \]
Does locking occurs? Non-perturbative studies

1. Fix \( p \), do MC to evaluate

\[
\langle \mathcal{O}(U) \rangle_p = \frac{\int \prod_{\mu} D V_{\mu} \; e^{S_{\text{QEK}}(p)} \; \mathcal{O}(U)}{\int \prod_{\mu} D V_{\mu} \; \exp(S_{\text{QEK}}(p))}
\]

2. Integrate over \( p \)

\[
\langle \mathcal{O}(U) \rangle_{\text{QEK}} = \int dp \; \langle \mathcal{O}(U) \rangle_p
\]

1. \( N=20,30,40,50,80,100,125,150,200, @ 100K \) measurements.


3. Various choices for \( p \) distributions (uniform, “clock” momenta, “BARS”)
Results: MC lattice studies of QEK

SU(40) (obtained by self-averaging)
Results: MC lattice studies of QEK

SU(40), b=0.5

Real($M_{\mu\nu}$)  

$M_{\mu\nu}$ in complex plane

Similar results $\forall N, b, dp$
Evidence for breakdown of quenched large-N reduction

- Weak-coupling: breakdown of Gross-Kitazawa-Parisi:
  - p’s chosen by non-perturbatively are locked, and not what your put in.

- Monte-Carlo studies of $N \leq 200$:
  1. Locking.
  2. Large discrepancies in plaquette of QEK vs. gauge.
  3. Large discrepancies in strong-to-weak transition coupling:
     - $b_{\text{transition}} = 0.3148(2)$ in QEK
     - $b_{\text{bulk}} \approx 0.36$
Other Large-N reductions on the lattice

- **DEK**: deform Yang-Mills action
  
  \[
  S_{DEK} = S_{YM} + \sum_{n_1, n_2} a_{n_1, n_2, n_3, n_4} \left| \text{tr} \left( U^{n_\mu}_{\mu} \cdot U^{n_\nu}_{\nu} \cdots U^{n_\rho}_{\rho} \right) \right|^2
  \]

  Unsal and Yaffe '08

- Numerically hard: naively scales like \( N^7 \) !!!!!

- **partial DEK**: for example 2+1 dimensions
  
  \[
  S_{DEK} = S_{YM} + \int_0^{1/T} d\tau \sum_{n_1, n_2} a_{n_1, n_2} \left| \text{tr} \left( U^{n_1}_{1}(\tau) \cdot U^{n_2}_{2}(\tau) \right) \right|^2
  \]

- **AEK**: dynamical adjoint fermions.
  
  Flips sign and p’s repel

Kovtun, Unsal and Yaffe '07
Minimizing \( F_{QEK}(p^{\sigma(a)}_{\mu}) \sim \sum_{a<b} \log \left[ \sum_{\mu} \sin^2 \left( \frac{p^{\sigma(a)}_{\mu} - p^{\sigma(b)}_{\mu}}{2} \right) \right] \)
Results: MC lattice studies of QEK

self averaging
Results: MC lattice studies of QEK
Results: MC lattice studies of QEK self averaging
Results: MC lattice studies of QEK

SU(80), b=0.4

self-averaging, p = locked start
Results: MC lattice studies of QEK

SU(80), b=0.4

Each 5000 have randomize p

self-averaging, p = locked start

QEK slowly tunnels to the locked state (20K updates)
The QEK model

| (I)  | \[ \langle \mathcal{O}(\mathcal{U}) \rangle_p = \frac{\int \prod_{\mu} DV_{\mu} e^{S_{\text{QEK}}(p)} \mathcal{O}(\mathcal{U})}{\int \prod_{\mu} DV_{\mu} \exp(S_{\text{QEK}}(p))} \] |
| (II) | \[ S_{\text{QEK}}(p) = Nb \sum_{\mu < \nu} 2\text{Re } \text{Tr} \left( U_{\mu} U_{\nu} U_{\mu}^\dagger U_{\nu}^\dagger \right) . \] |
| (III) | \[ \langle \mathcal{O}(\mathcal{U}) \rangle_{\text{QEK}} = \int dp \langle \mathcal{O}(\mathcal{U}) \rangle_p \] |

Uniform:
\[ \int dp \]

The original EK model

| (I)  | \[ \langle \mathcal{O}(\mathcal{U}) \rangle_p = \frac{\int \prod_{\mu} DV_{\mu} e^{S_{\text{EK}}(p)} \mathcal{O}(\mathcal{U})}{\int \prod_{\mu} DV_{\mu} \exp(S_{\text{EK}}(p))} \] |
| (II) | \[ S_{\text{EK}}(p) = Nb \sum_{\mu < \nu} 2\text{Re } \text{Tr} \left( U_{\mu} U_{\nu} U_{\mu}^\dagger U_{\nu}^\dagger \right) . \] |
| (III) | \[ \langle \mathcal{O}(\mathcal{U}) \rangle_{\text{EK}} = \int dp \langle \mathcal{O}(\mathcal{U}) \rangle_p Z(p) \] |

Non-uniform:
\[ \int dp \ Z(p) \sim \int \prod_{\mu,a} \frac{dp_{\mu}^a}{2\pi} e^{-F_{\text{EK}}(p)} \]
W-loop’s Dyson-Schwinger Equations

• Do a change of variables \( U_\mu \rightarrow U_\mu + i\mathcal{O}(\epsilon U_\mu) \)

• Again get source terms, which with the p-integral are

\[
\langle W_{\text{open}} W'_{\text{open}} \rangle_{\text{QEK}} = \int dp \left\langle W_{\text{open}} W'_{\text{open}} \right\rangle_p \quad \text{with}
\]

• These are zero if quenched large-N factorization holds

\[
\int dp \left\langle W_{\text{open}} W'_{\text{open}} \right\rangle_p = \int dp \left\langle W_{\text{open}} \right\rangle_p \int dp' \left\langle W'_{\text{open}} \right\rangle_{p'} + O(1/N)
\]

Because \( \int dp \left\langle W_{\text{open}} \right\rangle_p \) vanishes.
Quenched factorization - why?

\[ \int dp \left\langle W_{\text{open}} \, W'_{\text{open}} \right\rangle_p = \int dp \left\langle W_{\text{open}} \right\rangle_p \int dp' \left\langle W'_{\text{open}} \right\rangle_{p'} + O(1/N) \]

- Perturbation theory to (L+M)-loop order

\[ \int dp \sum_{a_1, a_2, \ldots, a_L} \sum_{b_1, b_2, \ldots, b_M} f(p_{a_1}, p_{a_2}, \ldots, p_{a_L}) g(p_{b_1}, p_{b_2}, \ldots, p_{b_M}). \]

- For most terms in the sum, with the exception of \( O(1/N) \) have

\[ (a_1, a_2, a_3, \ldots, a_L) \neq (b_1, b_2, b_3, \ldots, b_M) \]

\[ \int dp \, f(p_{a_1}, p_{a_2}, \ldots, p_{a_L}) \, g(p_{b_1}, p_{b_2}, \ldots, p_{b_M}) = \int dp \, f(p_{a_1}, p_{a_2}, \ldots, p_{a_L}) \int dq \, g(q_{b_1}, q_{b_2}, \ldots, q_{b_M}). \]
Free energy along path

Path 1

\( F_{1/N} \)

\( x \)

Path 2

\( F_{1/N^2} \)

\( x \)
Plaquette vs MC time

SU(40) at $b=0.5$

SU(80) at $b=0.4$

fixed $p$  randomize $p$ every 5000
Tunneling event unlocked to locked

plaquette:

SU(40) at b=0.5
Non-perturbative locking

$SU(40,80)$ at $b=0.5$, uniform dist.

Real($M_{\mu\nu}$)  M$_{\mu\nu}$ in complex plane
Non-perturbative locking

SU(16,81) at $b=0.7$, dist. a la Bars

Real($M_{\mu\nu}$), SU(16)  \hspace{2cm} M_{\mu\nu} \text{ in complex plane, SU}(81)
Transition is very strongly 1st order

- First implementation of Wang-Landau algorithm for gauge theory

Large-N transition at

\[ b = 0.3148(2) \text{ QEK} \]

\[ b \approx 0.36 \text{ gauge theory} \]