Orientifold Planar Equivalence: the chiral condensate

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The antisymmetric and the antifundamental representations coincide for $SU(3)$ (but not in general for $SU(N)$) ⇒ different $SU(N)$ generalizations of QCD.

In the planar limit, the (anti)symmetric representation is equivalent to another gauge theory with the same number of Majorana fermions in the adjoint representation (in a common sector). In particular, QCD with one massless fermion in the antisymmetric representation is equivalent to $\mathcal{N} = 1$ SYM in the planar limit ⇒ copy analytical predictions from SUSY to QCD.

The orientifold planar equivalence holds if and only if the $C$-symmetry is not spontaneously broken in both theories ⇒ a calculation from first principles is mandatory.

Assuming that planar equivalence works, how large are the $1/N$ corrections?

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Dynamical fermions difficult to simulate $\Rightarrow$ start with the quenched theory.
Outline

1. Condensates on the lattice
2. Proof of the "quenched" equivalence
3. Lattice setup
4. Results
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Aim

To measure the bare quark condensate with staggered fermions in the two-index representations of the gauge group, in the quenched lattice theory.

- Wilson action.
- Staggered Dirac operator $D = m - K$.
- The two-index representations.
- The bare condensate.
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\[
S_{YM} = -\frac{2N}{\lambda} \sum_p \Re \text{tr } U(p)
\]
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\[
D_{xy} = m\delta_{xy} - K_{xy} =
\]

\[
= m\delta_{xy} + \frac{1}{2} \sum_{\mu} \eta_{\mu}(x) \left\{ R[U_{\mu}(x)]\delta_{x+\hat{\mu},y} - R[U_{\mu}(x - \hat{\mu})]^\dagger \delta_{x-\hat{\mu},y} \right\}
\]
Condensates on the lattice

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\[
\text{tr Adj}[U] = |\text{tr } U|^2 - 1
\]

\[
\text{tr S/AS}[U] = \frac{(\text{tr } U)^2 \pm \text{tr}(U^2)}{2}
\]
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For S/AS representations:

$$\langle \bar{\psi}\psi \rangle_q = \frac{1}{V} \langle \text{Tr}(m - K)^{-1} \rangle_{YM}$$

For the adjoint representation:

$$\langle \lambda\lambda \rangle_q = \frac{1}{2V} \langle \text{Tr}(m - K)^{-1} \rangle_{YM}$$
Proof of the “quenched” equivalence

\[
\lim_{N \to \infty} \frac{1}{V N^2} \langle \text{Tr} (m - K_{S/AS})^{-1} \rangle = \lim_{N \to \infty} \frac{1}{2V N^2} \langle \text{Tr} (m - K_{Adj})^{-1} \rangle
\]

- Expand in \( m^{-1} \).
- Replace the two-index representations.
- Take the large-\( N \) limit.
- Mathematical details. The condensate is an analytical function of each real mass. The large-\( N \) limit can be exchanged with the series.
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\[
\frac{1}{VN^2} \langle \text{Tr}(m - K)^{-1} \rangle = \frac{1}{VN^2} \sum_{n=0}^{\infty} \frac{1}{m^{n+1}} \langle \text{Tr} K^n \rangle = \\
= \frac{1}{VN^2} \sum_{\omega \in \mathcal{C}} \frac{c(\omega)}{m^{L(\omega)+1}} \langle \text{tr} R[U(\omega)] \rangle
\]
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\[
\frac{1}{VN^2} \langle \text{Tr}(m - K_{S/AS})^{-1} \rangle = \frac{1}{2V} \sum_{\omega \in C} \frac{c(\omega)}{m^{L(\omega)}+1} \frac{\langle [\text{tr} U(\omega)]^2 \rangle \pm \langle [\text{tr} U(\omega)]^2 \rangle}{N^2}
\]

\[
\frac{1}{2VN^2} \langle \text{Tr}(m - K_{\text{Adj}})^{-1} \rangle = \frac{1}{2V} \sum_{\omega \in C} \frac{c(\omega)}{m^{L(\omega)}+1} \frac{\langle |\text{tr} U(\omega)|^2 \rangle - 1}{N^2}
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\]

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4. Results
A convenient parameterization

\[
\frac{1}{N^2} \langle \bar{\psi} \psi \rangle_{S/\text{AS}} = \frac{1}{2V} \sum_{\omega \in \mathcal{C}} \frac{c(\omega)}{m^L(\omega) + 1} \frac{\langle [\text{tr} U(\omega)]^2 \rangle \pm \langle \text{tr}[U(\omega)^2] \rangle}{N^2}
\]

\[
\frac{1}{N^2} \langle \lambda \lambda \rangle_{\text{Adj}} = \frac{1}{2V} \sum_{\omega \in \mathcal{C}} \frac{c(\omega)}{m^L(\omega) + 1} \frac{\langle |\text{tr} U(\omega)|^2 \rangle - 1}{N^2}
\]
A convenient parameterization

\[ \frac{1}{N^2} \langle \bar{\psi} \psi \rangle_{S/AS} = f \left( m, \frac{1}{N^2} \right) \pm \frac{1}{N} g \left( m, \frac{1}{N^2} \right) \]

\[ \frac{1}{N^2} \langle \lambda \lambda \rangle_{\text{Adj}} = \tilde{f} \left( m, \frac{1}{N^2} \right) - \frac{1}{2N^2} \langle \bar{\psi} \psi \rangle_{\text{free}} \]

Planar equivalence: \( f(m, 0) = \tilde{f}(m, 0) \).

**Strategy**

1. Simulate the condensates at various values of the mass.
2. Extract the functions \( f, g, \tilde{f} \).
3. Fit at fixed mass:

\[ \tilde{f} = a_0 + \frac{b_0}{N^2} \quad g = a_1 + \frac{b_1}{N^2} \quad f - \tilde{f} = \frac{a_2}{N^2} + \frac{b_2}{N^4} \]
$N = 2, 3, 4, 6, 8$

$\beta(N)$ chosen in such a way that $(aT_c)^{-1} = 5 (a \simeq 0.145 \text{ fm})$

$14^4$ lattice, which corresponds to $L \simeq 2.0 \text{ fm}$

22 values of the bare mass in the range $0.012 \cdots 8.0$
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For $m \leq 0.2$ we get $\chi^2 / \text{dof} \leq 0.53$ (we use $N = 4, 6, 8$).
For $m \leq 0.2$ we get $\chi^2 / \text{dof} \leq 0.37$ (we are fitting here $f - \tilde{f}$; we use $N = 4, 6, 8$).
For $m \leq 0.2$ we get $\chi^2$/dof $\leq 0.17$ (we use $N = 4, 6, 8$).
Condensate in the adjoint representation

\[ \frac{\langle \lambda \lambda \rangle_\text{Adj}(m = 0.012)}{N^2} = 0.23050(22) - \frac{0.3134(72)}{N^2} \]

At \( N = 3 \), relative error \( \simeq 0.8\% \).
Condensate in the antisymmetric representation

\[
\langle \bar{\psi}\psi \rangle_{\text{AS}}(m = 0.012) = \frac{0.23050(22)}{N^2} - \frac{0.4242(11)}{N} - \frac{0.612(43)}{N^2} - \frac{0.811(25)}{N^3}
\]

At \( N = 3 \), condensate < 0!
Condensate in the symmetric representation

\[ \frac{\langle \bar{\psi} \psi \rangle_{S}(m = 0.012)}{N^2} = \frac{0.23050(22)}{N} + \frac{0.4242(11)}{N^2} - \frac{0.612(43)}{N^3} + \frac{0.811(25)}{N^4} \]

At \( N = 3 \), relative error \( \simeq 4\% \).
Conclusions and perspectives

- First lattice calculation involving fermions in the two-index representations at $N \geq 4$.
- Check of the orientifold planar equivalence in a simple case.
- Computation of the quark condensate
  - For fermions in the adjoint and symmetric representations, the leading $1/N^2$ correction describes the data at $N \geq 3$ with an accuracy of a few percents;
  - For fermions in the antisymmetric representation higher order corrections play a major role.

- Current and future developments
  - Dynamical fermions;
  - Renormalization of the condensate and continuum limit.