Scaling and Chiral Extrapolation

C. Urbach
for the ETM Collaboration

Humboldt-Universität zu Berlin

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Continuum, Chiral and Thermodynamic Limits

we need a good understanding of those for extrapolating
  • data at finite $a$ to the continuum
  • data from unphysical $m_q$ to the physical point ($\chi$PT)
  • data in a finite box to infinite volume ($\chi$PT)
in order to control systematic uncertainties

however, we also have very interest in $\chi$PT itself
  • e.g. to extract low energy constants
European Twisted Mass Collaboration

Members from all over Europe:
Cyprus, France, Germany, Great Britain, Italy, Netherlands, Spain, Switzerland

C. Alexandrou, R. Baron, B. Blossier,
Ph. Boucaud, M. Brinet, J. Carbonell,
P. Dimopoulos, V. Drach, A. Deuzeman,
F. Farchioni, R. Frezzotti, V. Gimenez, I. Hailperin,
G. Herdoiza, K. Jansen, X. Feng, J. Gonzalez Lopez, T. Korzec, G. Koutsou, Z. Liu, V. Lubicz,
G. Martinelli, C. McNeile, C. Michael, I. Montvay,
G. Münster, A. Nube, D. Palao, E. Pallante,
O. Pène, S. Reker, D. Renner, C. Richards,
G.C. Rossi, S. Schäfer, L. Scorzato, A. Shindler,
S. Simula, T. Sudmann, C. Tarantino, C. Urbach,
A. Vladikas, M. Wagner, U. Wenger
Wilson Twisted Mass Fermions

- Wilson Twisted Mass Dirac operator

\[
D_{tm} = \frac{1}{2} \sum_{\mu} \left[ \gamma_\mu (\nabla_\mu + \nabla^*_\mu) - a \nabla^*_\mu \nabla_\mu \right] + m_0 + i \mu_q \gamma_5 \tau_3
\]

[Frezzotti, Grassi, Sint, Weisz, '99]

- when \( m_0 = m_{\text{crit}} \) (maximal twist)
  physical observables are \( \mathcal{O}(a) \) improved

[Frezzotti, Rossi, 2003]

- bare twisted mass parameter \( \mu_q \)
  directly relates to physical quark mass
  only multiplicative renormalisation

Drawback:

- flavour symmetry explicitly broken at finite \( a \)-values
  appears at \( \mathcal{O}(a^2) \) in physical observables
## Overview

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a$ [fm]</th>
<th>$L^3 \cdot T$</th>
<th>$L$ [fm]</th>
<th>$a\mu$</th>
<th>$N_{\text{traj}} (\tau = 0.5)$</th>
<th>$m_{\text{PS}}$ [MeV]</th>
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<tbody>
<tr>
<td>4.05</td>
<td>$\sim 0.066$</td>
<td>$32^3 \cdot 64$</td>
<td>2.2</td>
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<td>5200</td>
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<td>0.0060</td>
<td>$\sim 420$</td>
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<td>0.0080</td>
<td>$\sim 480$</td>
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<td>0.0120</td>
<td>$\sim 600$</td>
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<td>$24^3 \cdot 48$</td>
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<td>1.6</td>
<td>0.0060</td>
<td>3000 $\times 2$</td>
<td>$\sim 420$</td>
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<td>$20^3 \cdot 48$</td>
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<td>1.3</td>
<td>0.0060</td>
<td>5300 $\times 2$</td>
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<td>$24^3 \cdot 48$</td>
<td>2.1</td>
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<td>10500</td>
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<td>$32^3 \cdot 64$</td>
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<td>$20^3 \cdot 48$</td>
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<td>2.0</td>
<td>0.0060</td>
<td>4000 $\times 2$</td>
<td>$\sim 360$</td>
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The Data

For each value of $\beta$ and $\mu_q$ we’ll analyse

- data for $a_{f_{PS}}$

\[ a_{f_{PS}} = \frac{2\mu}{m_{PS}^2} |\langle 0 | P^1(0) | \pi \rangle| \]

(no renormalisation needed)

- data for $a_{m_{PS}}$

- data for $a_{m_N}$

- data for $r_0/a$, extrapolate to $\mu_q = 0$

- data for $Z_P$, extrapolate to $\mu_q = 0$

obtained non-pertubatively using RI-MOM

The renormalised quark mass at some renormalisation scale is obtained from

\[ \mu_R = \frac{1}{Z_P} \mu_q \]
Flavour Symmetry Breaking

Flavour symmetry is broken at $\mathcal{O}(a^2)$  

\[ \Rightarrow am^0_{PS} \neq am^\pm_{PS} \]

- not easy to measure: disconnected contributions!
- $m^\pm_{PS}, m^0_{PS}$ mass splitting vanishes like $a^2$
- $am^0_{PS} < am^\pm_{PS}$ consistent with prediction from $\chi$PT for observed phase structure

at $\beta = 4.05$ splitting still a large effect
Flavour Symmetry Breaking

- splitting observed so far only in $m_{\pi^0}$
- for other observables $O$:

$$R_O = \frac{o^{\pm}}{o^{\pm}} - \frac{\sigma^{\pm}}{\sigma^{\pm}}$$

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$a\mu_q$</th>
<th>$R_O$</th>
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<tbody>
<tr>
<td>$a_f^{PS}$</td>
<td>3.90</td>
<td>0.004</td>
<td>0.04(06)</td>
</tr>
<tr>
<td></td>
<td>4.05</td>
<td>0.003</td>
<td>-0.03(06)</td>
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<tr>
<td>$a_m^{V}$</td>
<td>3.90</td>
<td>0.004</td>
<td>0.02(07)</td>
</tr>
<tr>
<td></td>
<td>4.05</td>
<td>0.003</td>
<td>-0.10(11)</td>
</tr>
<tr>
<td>$a_f^{V}$</td>
<td>3.90</td>
<td>0.004</td>
<td>-0.07(18)</td>
</tr>
<tr>
<td></td>
<td>4.05</td>
<td>0.003</td>
<td>-0.31(29)</td>
</tr>
<tr>
<td>$a_m^{\Delta}$</td>
<td>3.90</td>
<td>0.004</td>
<td>0.022(29)</td>
</tr>
<tr>
<td></td>
<td>4.05</td>
<td>0.003</td>
<td>-0.004(45)</td>
</tr>
</tbody>
</table>

- Isospin splittings compatible with zero
Finite Size Effects

- correct for finite size effects using $\chi$PT

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$m_{PS} L$</th>
<th>meas [%]</th>
<th>GL [%]</th>
<th>CDH [%]</th>
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<tbody>
<tr>
<td>$m_{PS}$</td>
<td>3.9</td>
<td>3.3</td>
<td>+1.8</td>
<td>+0.6</td>
</tr>
<tr>
<td>$f_{PS}$</td>
<td>3.9</td>
<td>3.3</td>
<td>−2.5</td>
<td>−2.5</td>
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<tr>
<td>$m_{PS}$</td>
<td>4.05</td>
<td>3.0</td>
<td>+6.2</td>
<td>+1.8</td>
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<tr>
<td>$f_{PS}$</td>
<td>4.05</td>
<td>3.0</td>
<td>−10.7</td>
<td>−7.3</td>
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<tr>
<td>$m_{PS}$</td>
<td>4.05</td>
<td>3.5</td>
<td>+1.1</td>
<td>+0.8</td>
</tr>
<tr>
<td>$f_{PS}$</td>
<td>4.05</td>
<td>3.5</td>
<td>−1.8</td>
<td>−3.2</td>
</tr>
</tbody>
</table>

- as input for the parameters estimates from CDH were used
- CDH describes our data in general better than GL for the price of more parameters
Continuum Extrapolation of $m_N$ in Finite Volume

- finite volume $L/r_0 \sim 5.0$
- linear interpolation to reference points
  $r_0 m_{PS} = \text{const}$
- constant extrapolation $a \to 0$
  $\beta = 3.8$ not included
  $\Rightarrow$ Only small lattice artifacts (negligible?)!

[ETMC, arXiv:0803.3190]
Description with $\chi$PT

- quark mass dependence of $f_{PS}$, $m_{PS}$ and $m_N$ using $N_f = 2$
  continuum $\chi$PT
  

- simultaneous fit of data at $\beta = 3.9$ and $\beta = 4.05$

- step 1: constant continuum extrapolation
  step 2: continuum $\chi$PT fit

- $r_0/a$ and $Z_P$ are included as data in the fit

- finite size corrections performed using CDH formulae for $f_{PS}$ and
  $m_{PS}$
  
  [Colangelo, Dürr, Haefeli, 2005]
  no FS correction for $m_N$ so far

- statistical error estimated from a bootstrap analysis
Fit Result

- overall $\chi^2$/dof = 21/19
- good quality fit
Estimate Systematic Effects
quark mass dependence in formulae

- for $f_{PS}$ and $m_{PS}$

$$r_0 f_{PS} = r_0 f_0 \left[ 1 - 2\xi \log \left( \frac{\chi_\mu}{\Lambda_4^2} \right) + D_{PS} a^2 / r_0^2 + T_{NNLO} \right] K_f^{CDH}(L)$$

$$(r_0 m_{PS})^2 = \chi_\mu r_0^2 \left[ 1 + \xi \log \left( \frac{\chi_\mu}{\Lambda_3^2} \right) + D_{mPS} a^2 / r_0^2 + T_{NNLO} \right] K_m^{CDH}(L)^2$$

with

$$\xi \equiv \frac{2B_R \mu_R}{(4\pi f_0)^2}, \quad \chi_\mu \equiv 2B_R \mu_R, \quad f_0 = \sqrt{2}F_0$$

and $T_{NNLO}$ stands for continuum NNLO terms

- and for the nucleon using HB\(\chi\)PT

[Jenkins, Manohar, 1991; Becher, Leutwyler, 1999]

$$r_0 m_N = r_0 M_N - \frac{4c_1}{r_0} \chi_\mu r_0^2 - \frac{6g_A^2}{32\pi f_0^2 r_0^2} (\chi_\mu r_0^2)^{3/2} + r_0 M_N D_{mN} a^2 / r_0^2$$
Estimate Systematic Effects

- NNLO fits are not stable: we include priors e.g. for $\ell_1, \ell_2, k_M, k_F$ in the fit
- estimate systematic effects by
  - changing the way the continuum extrapolation is done
  - varying the fit-range
  - including NNLO for $m_{PS}$ and $f_{PS}$
$f_{PS}$: higher order $\chi$PT and fit range

- constant continuum extrapolation
- red: $\beta = 3.90$
- blue: $\beta = 4.05$

overall $\chi^2$:
- NLO fit: $\chi^2/\text{dof} = 21/19$
- NNLO fit: $\chi^2/\text{dof} = 19/19$
- NNLO, extended fit-range $\chi^2/\text{dof} = 50/23$
$f_{PS}$: higher order $\chi$PT and fit range

- constant continuum extrapolation
- red: $\beta = 3.90$
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**overall $\chi^2$:**
- NLO fit: $\chi^2$/dof = 21/19
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- NLO fit: $\chi^2$/dof = 21/19
- NNLO fit: $\chi^2$/dof = 19/19
- NNLO, extended fit-range $\chi^2$/dof = 50/23

for largest mass (N)NLO $\chi$PT presumably not applicable
$f_{PS}$: lattice artifacts

- red: $\beta = 3.90$
- blue: $\beta = 4.05$

overall $\chi^2$:
- NLO fit: $\chi^2$/dof = 21/19
- NLO fit + $a^2$: $\chi^2$/dof = 15/16
$f_{PS}$: lattice artifacts

- red: $\beta = 3.90$
- blue: $\beta = 4.05$

Overall $\chi^2$:
- NLO fit: $\chi^2$/dof = 21/19
- NLO fit + $a^2$: $\chi^2$/dof = 15/16
$f_{PS}$: lattice artifacts

![Graph showing $r_0 f_{PS}$ vs $r_0 \mu_R$]

- red: $\beta = 3.90$
- blue: $\beta = 4.05$

overall $\chi^2$:  
  - NLO fit: $\chi^2$/dof = 21/19  
  - NLO fit + $a^2$: $\chi^2$/dof = 15/16

however, all $D_X$ zero within errors $\Rightarrow$ not significant
$m^2_{PS}/\mu_q$: higher order $\chi$PT and fit range

- constant continuum extrapolation
- red: $\beta = 3.90$
- blue: $\beta = 4.05$

overall $\chi^2$:
- NLO fit: $\chi^2$/dof = 21/19
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$m_{PS}^2/\mu_q$: higher order $\chi$PT and fit range

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- red: $\beta = 3.90$
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Overall $\chi^2$:
- NLO fit: $\chi^2$/dof = 21/19
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$m_{PS}^2/\mu_q$: higher order $\chi$PT and fit range

- constant continuum extrapolation
- red: $\beta = 3.90$
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overall $\chi^2$:
- NLO fit: $\chi^2$/dof = 21/19
- NNLO fit: $\chi^2$/dof = 19/19
- NNLO, extended fit-range $\chi^2$/dof = 50/23
$m_N$: changing the fit range

- constant continuum extrapolation
- red: $\beta = 3.90$
- blue: $\beta = 4.05$

Overall $\chi^2$:
- NLO fit: $\chi^2$/dof = 21/19
- NNLO, extended fit-range $\chi^2$/dof = 50/23
$m_N$: changing the fit range

- constant continuum extrapolation
- red: $\beta = 3.90$
- blue: $\beta = 4.05$

Overall $\chi^2$:
- NLO fit: $\chi^2$/dof = 21/19
- NNLO, extended fit-range $\chi^2$/dof = 50/23
**Fit Results**

mean values and statistical errors come from NLO fit

pion sector

- $\bar{\ell}_3 = 3.43(8)(^{+0}_{-28})(^{+8}_{-0})$
- $\bar{\ell}_4 = 4.60(4)(10)(^{+8}_{-4})$
- $f_0 = 121.7(1)(6)(0) \text{ MeV}$
- $B_0 = 2571(44)(^{+0}_{-100})(^{+200}_{-0}) \text{ MeV}$
- $\Sigma^{1/3} = -267(2)(^{+0}_{-4})(^{+10}_{-0}) \text{ MeV}$
- $f_\pi/f_0 = 1.0740(7)(30)(^{+6}_{-0})$

nucleon sector

- $m_N = 962(45)(10)(3)$
- $c_1 = -1.13(27)(5)(20)$, $g_A = 1.13(21)(5)(10)$

errors: statistical, NNLO, $a^2$
flavour symmetry breaking negligible in many quantities but large in the $\pi^\pm - \pi^0$ mass splitting

finite size effects in $f_{PS}$, $m_{SP}$ describable with CDH formulae

lattice artifacts appear to be small to current statistical accuracy ($\sim 1\%$)

data can be fitted with continuum $\chi$PT
  - extract LEC’s with high precision
  - determine nucleon mass $m_N = 962(45)(10)(3)$ MeV

systematic uncertainties for some quantities larger than statistical error
Sommer Parameter $r_0$

- statistical accuracy of less than 0.5%,
- compatible with $\mu_q^2$ dependence
- $\mu_q$-dependence is rather weak unlike Wilson / Wilson clover

⇒ at $\mu_q \rightarrow 0$:
  \[
  \begin{align*}
  \beta &= 3.8: \quad r_0/a = 4.46(3) \\
  \beta &= 3.9: \quad r_0/a = 5.22(2) \\
  \beta &= 4.05: \quad r_0/a = 6.61(3)
  \end{align*}
\]
Non-perturbative Renormalisation

- RI-MOM renormalisation scheme
  [Martinelli et al., 1995]

- $O(a)$ improved at maximal twist

- compatible with $\mu^2$ dependence

- nicely consistent with WI method / mixed action (MA) approach

- possible alternative: Schrödinger functional
  [Frezzotti, Rossi, 2005; Sint, 2006]
Continuum Extrapolation $f_{PS}$ in Finite Volume

- finite volume $L/r_0 \sim 5.0$
- linear interpolation to reference points
  $r_0 m_{PS} = \text{const}$
- constant extrapolation $a \to 0$
  $\beta = 3.8$ not included

$\Rightarrow$ Only small lattice artifacts (negligible?)!

Finite Size Effects

- our data is compatible with exponential behaviour in $m_{PS} \cdot L$

$$m_{PS}(L) = m_{PS} \left[ 1 + \frac{1}{2} \frac{m_{PS}^2}{(4\pi f_0)^2} \tilde{g}_1(m_{PS}L) \right],$$

$$f_{PS}(L) = f_{PS} \left[ 1 - 2 \frac{m_{PS}^2}{(4\pi f_0)^2} \tilde{g}_1(m_{PS}L) \right],$$

- NNLO known for $m_{PS}$ [Colangelo, Haefeli, 2006]
  - however, resummed asymptotic Lüscher formula provides higher orders easier [Colangelo, Dürr, Haefeli, 2005] (CDH)
  - but depends on many LECs: $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$