$B_K$ for 2+1 flavour domain wall fermions from 24$^3$ and 32$^3 \times 64$ lattices

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The neutral kaon mixing amplitude $B_K$
Outline

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Ensemble details
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Measurement of $B_K$
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The chiral extrapolation of $B_K$
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Conclusions and Outlook
The neutral kaon mixing amplitude $B_K$
Kaon mixing

- Indirect CP violation in neutral kaon sector
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- Neutral kaon mixing amplitude:

\[ A(K^0 \to \bar{K}^0) = \frac{G_F}{2} \sum_i V^i_{\text{CKM}} C_i(\mu) \langle K^0 \mid Q_i(\mu) \mid \bar{K}^0 \rangle \]
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scheme dependent perturbative factor summarising contributions from scales \( \gg \mu \)
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- Scheme dependent hadronic matrix element at scale \( \mu \sim M_K \) obtainable from lattice
- Scheme dependent perturbative factor summarising contributions from scales \( \gg \mu \)
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  - relation contains unknown direct CP violating parameters.

- $\epsilon_K$ known experimentally to high precision $\Rightarrow B_K$ constrains unknown direct CP violating parameters.
Ensemble details
Details of ensembles

\[ 24^3 \times 64 \quad 32^3 \times 64 \]
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- 2+1f domain wall fermion ensemble with $L_s = 16$

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Up/down sea quark masses

<table>
<thead>
<tr>
<th></th>
<th>$24^3 \times 64$</th>
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<tbody>
<tr>
<td>latt. units</td>
<td>$m_\pi$ (MeV)</td>
<td>latt. units</td>
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</tr>
<tr>
<td>0.03</td>
<td>626</td>
<td>0.008</td>
<td>∼ 420</td>
</tr>
<tr>
<td>0.02</td>
<td>558</td>
<td>0.006</td>
<td>∼ 360</td>
</tr>
<tr>
<td>0.01</td>
<td>345</td>
<td>0.004</td>
<td>∼ 300</td>
</tr>
<tr>
<td>0.005</td>
<td>331</td>
<td></td>
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</tr>
</tbody>
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Highly preliminary data as datasets only partially complete.
Measurement of $B_K$
Method comparison

24³ × 64

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$B_K$ example plateaux

$24^3 \times 64 \ m_l = 0.005$

Preliminary $32^3 \times 64 \ m_l = 0.004$
The chiral extrapolation of $B_K$
We use NLO $SU(2) \times SU(2)$ partially-quenched chiral perturbation theory (PQChPT) for maximum use of ensembles.
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Kaon sector is coupled to $SU(2)$ soft pion loops at lowest order in non-relativistic expansion

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$24^3$ analysis [Allton et al arXiv:0804.0473] indicated $SU(3) \times SU(3)$ PQChPT has large higher order corrections and doesn’t fit data well up to physical strange quark mass (R. Mawhinney).
$SU(2) \times SU(2)$ PQChPT fit form for $B_K$

$$B_K = B_K^0 \left[ 1 + \frac{2B(m_d+m_{\text{res}})c_0}{f^2} + \frac{2B(m_y+m_{\text{res}})c_1}{f^2} ight.$$ 

$$\left. - \frac{2B(m_y+m_{\text{res}})}{32\pi^2 f^2} \log \left( \frac{2B(m_y+m_{\text{res}})}{\Lambda^2} \right) \right]$$
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► 5 free parameters:
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- 5 free parameters: $B_K^0$, $B$, $f$, $c_0$, $c_1$

- Use simultaneous pure SU(2) × SU(2) PQChPT fit (no coupling to Kaon sector) to $F_{PS}$ and $M_{PS}$ to determine $B$ and $f$ (E. Scholz)
$SU(2) \times SU(2)$ PQChPT fit form for $B_K$

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$\rightarrow$ perform frozen 3-parameter fit to $B_K$
Simultaneous PQChPT fits to $F_{PS}$ and $M_{PS}$: $f_{PS}$

24$^3 \times 64$

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$32^3 \times 64$

$$f_{xy}$$

$m_l = 0.005, m_s = 0.04$  

fit: $m_{avg} \leq 0.01$

$m_x = 0.001$

$m_x = 0.005$

$m_x = 0.01$

$m_x = 0.02$

$m_x = 0.03$

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$$f_{xy}$$

$m_l = 0.004, m_s = 0.03$  

fit: $m_{avg} \leq 0.008$

$m_x = 0.004$

$m_x = 0.006$

$m_x = 0.008$

$m_x = 0.025$

$m_x = 0.03$

$m_{res}$
Simultaneous PQChPT fits to $F_{PS}$ and $M_{PS}$: $f_{PS}$

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![Graph showing $f_{xy}$ vs. $m_y + m_{res}$ with different $m_x$ values]
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- Unitary curve is finite valued at chiral limit.
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32$^3 \times 64$

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\end{align*}$

$m_l = 0.008$

$m_l = 0.006$

$m_l = 0.004$

$m_x = m_l$

$m_x + m_{\text{res}}$

Stat err?

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$24^3 \times 64 \ B_K$ chiral limit results – Allton et al
[arXiv:0804.0473]

$24^3 \times 64 \ B^K_{\text{lat}} = 0.565(10)$. 
$24^3 \times 64 \, B_K$ chiral limit results – Allton et al [arXiv:0804.0473]

- $24^3 \times 64 \, B_K^{lat} = 0.565(10)$.
- $32^3 \times 64$ extrapolation not yet available, dataset only partially complete
  - Stat uncertainties in data sets, unknown physical quark masses
The non-perturbative renormalisation of $B_K$
Why NPR?

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- Use Rome-Southampton RI/MOM scheme
Bilinear vertices

- $Z_V = Z_A$ due to good chiral symmetry
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- However, even at high momenta we find $\Lambda_A \neq \Lambda_V$ at 2% level.
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- However, even at high momenta we find $\Lambda_A \neq \Lambda_V$ at 2% level
  → difference caused by kinematic choice: Exceptional momentum configuration
  → Gives weak $1/p^2$ suppression of low energy chiral symmetry breaking.
Chiral symmetry breaking and exceptional momenta

- Generic bilinear vertex graph

\[ q \Gamma^{\mu}(0) q \]

\[ p_1 \quad p_2 \]
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  - Low energy subgraph not contained within circled subgraph
Chiral symmetry breaking and exceptional momenta

- Generic bilinear vertex graph
- $p^2 \to \infty$ behaviour governed by subgraph with least negative degree of divergence through which we can route all hard external momenta.
- For $p_1 \neq p_2$ this is the entire graph.
- Can connect to low-energy subgraphs which are affected by spont. chiral symmetry breaking, but:
  - Low energy subgraph not contained within circled subgraph
  - Adding extra external legs to circled subgraph increases suppression of the graph
However in case $p_2 - p_1 = 0$ then high momenta do not enter internal subgraphs.
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Graph free to couple to low-energy chiral symmetry breaking subgraphs with no further suppression.
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This is an exceptional momentum configuration.
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Graph free to couple to low-energy chiral symmetry breaking subgraphs with no further suppression.

This is an exceptional momentum configuration.

Chiral symmetry breaking induces difference between $\Lambda_A$ and $\Lambda_V$ → use $\frac{1}{2}(\Lambda_A + \Lambda_V) \approx \frac{Z_q}{Z_A}$.
$B_K$ NPR with RI/MOM and exceptional momenta

- Calculate four-quark vertex matrix element in Landau gauge.
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Amputate vertex with ensemble averaged unrenormalised propagator, giving $\Lambda_{O_{VV+AA}}$. 
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Amputate vertex with ensemble averaged unrenormalised propagator, giving $\Lambda_{O_{VV+AA}}$

Renormalisation condition: Fix to tree level value at $\mu^2 = p^2$

$$\frac{Z_{VV+AA}}{Z_q^2} \Lambda_{O_{VV+AA}} = O_{VV+AA}^{tree}$$
Calculate four-quark vertex matrix element in Landau gauge.

Amputate vertex with ensemble averaged unrenormalised propagator, giving $\Lambda_{\mathcal{O}_{VV+AA}}$

Renormalisation condition: Fix to tree level value at $\mu^2 = p^2$

\[
\frac{Z_{VV+AA}}{Z_q^2} \Lambda_{\mathcal{O}_{VV+AA}} = \mathcal{O}_{VV+AA}^{\text{tree}}
\]

Define

\[
Z_{BK}^{RI/MOM} = \frac{Z_{VV+AA}}{Z_A^2} = \left(\frac{Z_q^2}{Z_A^2}\right) \frac{Z_{VV+AA}}{Z_q^2}
\]
BK NPR with RI/MOM and exceptional momenta

- Calculate four-quark vertex matrix element in Landau gauge.
- Amputate vertex with ensemble averaged unrenormalised propagator, giving $\Lambda_{O_{VV+AA}}$
- Renormalisation condition: Fix to tree level value at $\mu^2 = p^2$

$$\frac{Z_{VV+AA}}{Z_q^2} \Lambda_{O_{VV+AA}} = O_{VV+AA}^{tree}$$

- Define

$$Z_{RI/MOM}^{BK} = \frac{Z_{VV+AA}}{Z_q^2} \frac{Z_A^2}{Z_A^2} = \left(\frac{Z_q^2}{Z_A^2}\right) \frac{Z_{VV+AA}}{Z_q^2}$$

- Use $\frac{1}{2}(\Lambda_A + \Lambda_V) \approx \frac{Z_q}{Z_A}$
Method comparison

$24^3 \times 32$

$32^3 \times 64$
Method comparison

\[ 16^3 \times 32 \]

- Use point sources, 4 quark vertex formed at source location.

\[ 32^3 \times 64 \]
Method comparison

$16^3 \times 32$  $32^3 \times 64$

- Use point sources, 4 quark vertex formed at source location.
- Average over 4 source locations on 75 configurations on our $m_l = 0.03, 0.02$ and $0.01$ ensembles.
Method comparison

\[ 16^3 \times 32 \]

- Use point sources, 4 quark vertex formed at source location.

- Average over 4 source locations on 75 configurations on our \( m_\ell = 0.03, 0.02 \) and 0.01 ensembles.

- Momentum applied by applying phase difference between propagator source and sink. Solution can be given arbitrary momentum.

\[ 32^3 \times 64 \]
Method comparison

$16^3 \times 32$

- Use point sources, 4 quark vertex formed at source location.
- Average over 4 source locations on 75 configurations on our $m_f = 0.03, 0.02$ and 0.01 ensembles.
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$32^3 \times 64$

- Use lattice volume sources, vertex formed at propagator sink. (D. Broemmel)
**Method comparison**

\[ 16^3 \times 32 \]

- Use point sources, 4 quark vertex formed at source location.
- Average over 4 source locations on 75 configurations on our \( m_l = 0.03, 0.02 \) and 0.01 ensembles.
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\[ 32^3 \times 64 \]

- Use lattice volume sources, vertex formed at propagator sink. (D. Broemmel)
- Average over all sink locations, lattice volume factor gain over point approach.
Method comparison

16^3 \times 32

- Use point sources, 4 quark vertex formed at source location.
- Average over 4 source locations on 75 configurations on our \( m_{1/2} = 0.03, 0.02 \) and 0.01 ensembles.
- Momentum applied by applying phase difference between propagator source and sink. Solution can be given arbitrary momentum.

32^3 \times 64

- Use lattice volume sources, vertex formed at propagator sink. (D. Broemmel)
- Average over all sink locations, lattice volume factor gain over point approach.
- Volume source has fixed momentum as phase must be applied to source lattice sites before inversion.
Method comparison

$16^3 \times 32$
- Use point sources, 4 quark vertex formed at source location.
- Average over 4 source locations on 75 configurations on our $m_l = 0.03, 0.02$ and 0.01 ensembles.
- Momentum applied by applying phase difference between propagator source and sink. Solution can be given arbitrary momentum.

$32^3 \times 64$
- Currently calculated 5 independent momenta (10 total) on 10 configurations on our $m_l = 0.006$ and 0.004 ensembles.
$Z_{BK}^{RI/MOM}(\mu)$

$16^3 \times 32$

$32^3 \times 64$

Graph showing $Z_{\Phi_k}$ vs. $(\mu^2)$ with different values of $m_1$: $m_1 = 0.01$, $m_1 = 0.02$, $m_1 = 0.03$, and the chiral limit.
$Z_{BK}^{RI/MOM}(\mu)$

$16^3 \times 32$

$32^3 \times 64$

$Z_{\mu_k}$

$(\alpha \mu)^2$

- $m_1 = 0.01$
- $m_1 = 0.02$
- $m_1 = 0.03$
- chiral limit

- $m_1 = 0.006$
- $m_1 = 0.004$
- chiral limit
- Point $m_1 = 0.006$
\[ Z^{RI/MOM}_{BK}(\mu) \]

\[ 16^3 \times 32 \]

\[ 32^3 \times 64 \]
For each $Z_{BK}(\mu)$, perform a linear chiral extrapolation to $m = -m_{\text{res}}$
Chiral extrapolation – $32^3 \times 64$

For each $Z_{BK}(\mu)$, perform a linear chiral extrapolation to $m = -m_{\text{res}}$

- $32^3$ lever-arm for extrapolation small compared to extrapolation distance
For each $Z_{BK}(\mu)$, perform a linear chiral extrapolation to $m = -m_{\text{res}}$

32$^3$ lever-arm for extrapolation small compared to extrapolation distance

→ Future: Add $m_1 = 0.008$ dataset
Exceptional momenta systematic error

- $32^3$ stat errors small compared to systematic error from exceptional momenta.
Exceptional momenta systematic error

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Exceptional momenta systematic error

- $32^3$ stat errors small compared to systematic error from exceptional momenta.
- On $24^3$ we attributed a 1.5% sys error to this alone.
- Difference greatly reduced by using non-exceptional momentum configuration $p_1 \neq p_2$
- Unfortunately no perturbative calculation available for non-exceptional (Y. Aoki)
Removal of lattice artefacts

- Divide out perturbative running: Quantity is scale invariant up to lattice artefacts
Removal of lattice artefacts

- Divide out perturbative running: Quantity is scale invariant up to lattice artefacts
- Expect quadratic dependence of lattice artefacts on lattice spacing
  → fit to form $Z_{BK}^{SI} + B(a\mu)^2$
Extrapolation of $Z_{BK}^{SI}$

$16^3 \times 32$  \hspace{1cm} $32^3 \times 64$
Extrapolation of $Z^{SI}_{BK}$

$16^3 \times 32$

$32^3 \times 64$
Reapply RI/MOM perturbative running to $Z_{BK}^{SI}$ and scale to conventional $\mu = 2$ GeV.
Reapply RI/MOM perturbative running to $Z_{BK}^{SI}$ and scale to conventional $\mu = 2$ GeV.

Apply conversion factor $Z_{BK}^{RI/MOM} \rightarrow Z_{BK}^{\overline{MS}}$
Reapply RI/MOM perturbative running to $Z_{BK}^{SI}$ and scale to conventional $\mu = 2$ GeV.

Apply conversion factor $Z_{BK}^{RI/MOM} \rightarrow Z_{BK}^{MS}$

$Z_{BK}^{MS}$ (2 GeV) $= 0.9276 \pm 0.0052$ (stat) $\pm 0.0220$ (sys).
Reapply RI/MOM perturbative running to \( Z_{BK}^{SI} \) and scale to conventional \( \mu = 2 \) GeV.

Apply conversion factor \( Z_{BK}^{RI/MOM} \rightarrow Z_{BK}^{\overline{MS}} \)

\[ Z_{BK}^{\overline{MS}} (2 \text{ GeV}) = 0.9276 \pm 0.0052(\text{stat}) \pm 0.0220(\text{sys}). \]

Sys errors:
Reapply RI/MOM perturbative running to $Z_{BK}^{SI}$ and scale to conventional $\mu = 2$ GeV.

Apply conversion factor $Z_{BK}^{RI/MOM} \rightarrow Z_{BK}^{\overline{MS}}$.

$Z_{BK}^{\overline{MS}} (2 \text{ GeV}) = 0.9276 \pm 0.0052(\text{stat}) \pm 0.0220(\text{sys})$.

Sys errors:
- $O(\alpha_s) \Rightarrow 0.0177$ corrections due to truncation of perturbative analysis
Reapply RI/MOM perturbative running to $Z_{BK}^{SI}$ and scale to conventional $\mu = 2$ GeV.

Apply conversion factor $Z_{BK}^{RI/MOM} \rightarrow Z_{BK}^{MS}$

$Z_{BK}^{MS}(2 \text{GeV}) = 0.9276 \pm 0.0052(\text{stat}) \pm 0.0220(\text{sys})$.

Sys errors:
- $O(\alpha_s)$ \Rightarrow 0.0177 corrections due to truncation of perturbative analysis
- \Rightarrow 0.0007 unphysical strange mass correction
Reapply RI/MOM perturbative running to $Z_{BK}^{SI}$ and scale to conventional $\mu = 2$ GeV.

Apply conversion factor $Z_{BK}^{RI/MOM} \rightarrow Z_{BK}^{MS}$

$Z_{BK}^{MS} (2$ GeV$) = 0.9276 \pm 0.0052(\text{stat}) \pm 0.0220(\text{sys})$.

Sys errors:

$\mathcal{O}(\alpha_s) \Rightarrow 0.0177$ corrections due to truncation of perturbative analysis

$\Rightarrow 0.0007$ unphysical strange mass correction

$\Rightarrow 0.0131$ correction for use of exceptional momenta
Reapply RI/MOM perturbative running to $Z_{BK}^{SI}$ and scale to conventional $\mu = 2$ GeV.

Apply conversion factor $Z_{BK}^{RI/MOM} \rightarrow Z_{BK}^{\overline{MS}}$

$Z_{BK}^{\overline{MS}} (2 \text{ GeV}) = 0.9276 \pm 0.0052(\text{stat}) \pm 0.0220(\text{sys})$.

Sys errors:

- $O(\alpha_s) \Rightarrow 0.0177$ corrections due to truncation of perturbative analysis
- $\Rightarrow 0.0007$ unphysical strange mass correction
- $\Rightarrow 0.0131$ correction for use of exceptional momenta

Current $32^3 Z_{BK}^{\overline{MS}}$ stat error $\sim 0.0013$. 
Conclusions and Outlook
$24^3 \times 64$ final value and $32^3$ outlook

- Combining chirally extrapolated $B_K$ with aforementioned $Z_{BK}$ result
Combining chirally extrapolated $B_K$ with aforementioned $Z_{BK}$ result: $B_K^{\text{MS}}(2 \text{ GeV}) = 0.524(10)_{\text{stat}}(13)_{\text{ren}}(25)_{\text{sys}}$

[arXiv:0804.0473]
Combining chirally extrapolated $B_K$ with aforementioned $Z_{BK}$ result

$B_K^{\text{MS}}(2 \text{ GeV}) = 0.524(10)_{\text{stat}}(13)_{\text{ren}}(25)_{\text{sys}}$

[arXiv:0804.0473]

Improved techniques for $32^3$ in use; results expected soon:
Watch this space!