Phase diagram and EoS from a Taylor expansion of the pressure

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Universität Bielefeld

RBC-Bielefeld:

See poster by C. Miao
Outline

• Introduction
• Taylor expansion of the pressure
• The radius of convergence and the QCD critical point
• The isentropic equation of state
• Summary
The phase diagram of QCD

- Fluctuations of B, S, Q can be measured experimentally and indicate criticality
- Lattice at $\mu = 0$ → RHIC, LHC
- Lattice at $\mu > 0$ → RHIC at low energies, FAIR@GSI

$T \sim 190\,\text{MeV}$

$\mu_s = \mu_Q = 0$

$\mu \geq 0$

Quark-gluon plasma deconfined, $\chi$-symmetric

Hadron gas confined, $\chi$-broken

Cooling of the fireball:

$\sim$ few times nuclear matter density

$\mu B$
Taylor expansion of the pressure

- **Taylor expansion in $\mu_{B,S,Q}$**

QCD is naturally formulated with quark chemical potentials $\mu_{u,d,s}$

we start from Taylor expansion of the pressure

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V, T, \mu_u, \mu_d, \mu_s) = \sum_{i,j,k} c_{i,j,k}^{u,d,s} \left( \frac{\mu_u}{T} \right)^i \left( \frac{\mu_d}{T} \right)^j \left( \frac{\mu_s}{T} \right)^k$$

- use unbiased, noisy estimators to calculate $c_{i,j,k}^{u,d,s}$
  
  $\longrightarrow$ see C. Miao, CS, PoS (Lattice 2007) 175.

- line of constant physics: $m_q = m_s / 10$
  (physical strange quark mass)

- measure currently up to $O(\mu^8) \longleftrightarrow (N_t = 4)$
  $O(\mu^4) \longleftrightarrow (N_t = 6)$

- action: improved staggered (p4fat3)
Taylor expansion of the pressure

• Taylor expansion in $\mu_B, S, Q$

QCD is naturally formulated with quark chemical potentials $\mu_u, d, s$

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$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V, T, \mu_u, \mu_d, \mu_s) = \sum_{i,j,k} c_{i,j,k}^{u,d,s} \left( \frac{\mu_u}{T} \right)^i \left( \frac{\mu_d}{T} \right)^j \left( \frac{\mu_s}{T} \right)^k$$

• expansion coefficients $c_{i,j,k}^{u,d,s}$ are related to B,S,Q-fluctuations

\[
\begin{align*}
n_B &= \frac{\partial(p/T^4)}{\partial(\mu_B/T)} = \frac{1}{3}(n_u + n_d + n_s) & \mu_u &= \frac{1}{3} \mu_B + \frac{2}{3} \mu_Q \\
n_S &= \frac{\partial(p/T^4)}{\partial(\mu_S/T)} = -n_s & \mu_d &= \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q \\
n_Q &= \frac{\partial(p/T^4)}{\partial(\mu_Q/T)} = \frac{1}{3}(2n_u - n_d - n_s) & \mu_s &= \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q - \mu_S
\end{align*}
\]

• choice of $\mu_u \equiv \mu_d$ is equivalent to $\mu_Q \equiv 0$
Taylor expansion of the pressure

- **Current statistics**

\[ N_\tau = 4 \]

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>#Conf.</th>
<th>#Sep.</th>
<th>#Ran.</th>
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\[ N_\tau = 6 \]

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→ work in progress!
Results for expansion coefficients $c_{i,j,k}^{u,d,s}$

- $c_2^u$
  - $n_{f}=2+1$, $m_\pi=220$ MeV
  - $n_{f}=2$, $m_\pi=770$ MeV
  - filled: $n_f=4$
  - open: $n_f=6$

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- $c_6^u$

Cut-off dependance:
  - Small cut-off effects in the transition region (similar to p, e-3p, ...)

Mass dependance:
  - $T_c$ decreases with decreasing mass
  - Fluctuations increase with decreasing mass

- red: RBC-Bielefeld, preliminary
Hadronic fluctuations and the QCD critical point

**Baryon number fluctuations ($\mu_B = 0$)**

- $2c_2^B = \langle B^2 \rangle$
- $24c_4^B = \langle B^4 \rangle - 3 \langle B^2 \rangle^2$

- Fluctuations increase with decreasing mass
- Fluctuations increase over the resonance gas value

---

- $n_f=2+1, m_\pi=220$ MeV
- $n_f=2, m_\pi=770$ MeV

---

- red: RBC-Bielefeld, preliminary
• Consequences for the phase diagram: the radius of convergence

The radius of convergence can be estimated from the Taylor coefficients of the pressure:

\[ \rho = \lim_{n \to \infty} \rho_n \]

with

\[ \rho_n = \sqrt{\frac{c_n^B}{c_{n+2}^B}} \]

• for \( T > T_c \), \( \rho_n \to \infty \)

• for \( T < T_c \), \( \rho_n \) is bound by the transition line

The Resonance gas limit:

\[ \frac{p}{T^4} = G(T) + F(T) \cosh \left( \frac{\mu_B}{T} \right) \]

\[ \longrightarrow \rho_n = \sqrt{\frac{1}{(n + 2)(n + 1)}} \]

→ look for non-monotonic behavior in the radius of convergence
- Consequences for the phase diagram: the radius of convergence

- non monotonic behavior in the radius of convergence?

<table>
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<th>$\rho_2$</th>
<th>$N_\tau = 4$</th>
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<td>$m_\pi \approx 220$ MeV</td>
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→ first hint for a critical region at small masses?

- higher order approximations are needed to locate the critical point
• Consequences for the phase diagram: the radius of convergence

• non monotonic behavior in the radius of convergence?

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\[
\begin{array}{c|cc}
& \rho_2 & \rho_4 \\
\hline
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→ first hint for a critical region at small masses? (only at \( N_\tau=4 \)?)

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→ first hint for a critical region at small masses? (only at \( N_t=4 \)?)

• higher order approximations are needed to locate the critical point

\( \rho_4 \) (and maybe \( \rho_6 \)) are needed in higher precision
• Consequences for the phase diagram: the radius of convergence

• non monotonic behavior in the radius of convergence?

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→ $\rho_4$ (and maybe $\rho_6$) are needed in higher precision

→ higher order approximations are needed to locate the critical point.
Consequences for the phase diagram: the radius of convergence

- unexpected behavior in the radius of convergence of the pressure expansion in $\mu_I/T$?

→ pion-condensation phase should show up in the $\rho$ profile
• Pressure, Energy and Entropy: the 0th-order

\[ p/T^4 \text{ from integrating over } (\epsilon - 3p)/T^5 \]

- systematic error from starting the integration at \( T_0 = 100\,\text{MeV} \) with \( p(T_0) = 0 \)
- use HRG to estimate systematic error: \( [p(T_0)/T_0^4]_{HRG} \approx 0.265 \)

M. Cheng et al. [RBC-Bielefeld], PRD 77 (2008) 014511.
The EoS at non zero density

- Taylor expansion of the trace anomaly

\[
\frac{\epsilon - 3p}{T^4} = \sum_{n=0}^{\infty} c'_n B(T, m_l, m_s) \left( \frac{\mu_B}{T} \right)^n
\]

Coefficients are defined by

\[
c'_n B(T, m_l, m_s) = T \frac{dc^n_B(T, m_l, m_s)}{dT}
\]

Perform T-derivative numerically: discretization error

- "local version" is work in progress

\[
c'_n B(T, \hat{m}_l, \hat{m}_s) = -a \frac{d\beta}{da} \frac{dc^n_B(T, \hat{m}_l, \hat{m}_s)}{d\beta} - a \frac{d\hat{m}_l}{da} \frac{dc^n_B(T, \hat{m}_l, \hat{m}_s)}{d\hat{m}_l} - a \frac{d\hat{m}_s}{da} \frac{dc^n_B(T, \hat{m}_l, \hat{m}_s)}{d\hat{m}_s}
\]

- Taylor expansion of energy and entropy densities

\[
\frac{\epsilon}{T^4} = \sum_{n=0}^{\infty} \left( 3c^n_B(T, m_l, m_s) + c'_n B(T, m_l, m_s) \right) \left( \frac{\mu_B}{T} \right)^n \equiv \sum_{n=0}^{\infty} \epsilon^n_B \left( \frac{\mu_B}{T} \right)^n
\]

\[
\frac{s}{T^3} = \sum_{n=0}^{\infty} \left( (4 - n)c^n_B(T, m_l, m_s) + c'_n B(T, m_l, m_s) \right) \left( \frac{\mu_B}{T} \right)^n \equiv \sum_{n=0}^{\infty} s^n_B \left( \frac{\mu_B}{T} \right)^n
\]
• Coefficients of the $\mu_B$-expansion

\[
\epsilon_n \approx 10\% \\
\text{pattern of } \epsilon_n \text{ and } s_n \text{ is that of } c_{n+2}
\]
**Isentropic trajectories**

- Solve numerically for
  \[ S(T, \mu_B) / N_B(T, \mu_B) = \text{const.} \]
- 6th order is small at S/N = 30 (trajectories inside estimated radius of convergence)
- Calculate pressure and energy density along isentropic trajectories
- Pressure and energy density increase by \( \approx 10\% \) for S/N = 30.

\[ N_T = 4 \]

\[ N_T = 6 \]
• Isentropic trajectories
  
  → solve numerically for
  
  \[ S(T, \mu_B)/N_B(T, \mu_B) = \text{const.} \]

  → 6th order is small at S/N=30
  (trajectories inside estimated radius of convergence)

  → calculate pressure and energy density
     along isentropic trajectories

  → pressure and energy density increase
     by \( \approx 10\% \) for S/N=30.
• **Isentropic trajectories**

- solve numerically for
  \[ S(T, \mu_B)/N_B(T, \mu_B) = \text{const.} \]
- 6th order is small at S/N=30 (trajectories inside estimated radius of convergence)
- calculate pressure and energy density along isentropic trajectories
- pressure and energy density increase by \( \approx 10\% \) for S/N=30.

The EoS along isentropic trajectories is fairly independent on S/N.

Leading order corrections:

\[
\frac{p}{\epsilon} = \frac{1}{3} - \frac{1}{3} \left( \frac{\epsilon_0 - 3p_0}{\epsilon_0} \right) \left( 1 + \frac{c'_2}{\epsilon_0 - 3p_0} - \frac{\epsilon_2}{\epsilon_0} \right) \left( \frac{\mu_B}{T} \right)^2
\]
• Cut-off effect for Taylor expansion coefficients are small and sizable only in the transition region (similar to the interaction measure e-3p)

• We find non-monotonic behavior in the radius of convergence for \( N_{\tau} = 4 \) which could be a first hint for a critical region in the \( T, \mu_B \) - plane. This needs to be confirmed by \( N_{\tau} = 6 \).

• Isentropic trajectories show non-monotonic behavior for \( N_{\tau} = 4 \). This needs to be confirmed by \( N_{\tau} = 6 \).

• Finite density correction for EoS are small, pressure and energy density increase by \( \approx 10\% \) for S/N=30 (AGS/FAIR), corrections cancel to large extent in \( p/\epsilon \).

• Taylor expansion method will provide valuable input for HIC phenomenology.