Generalisations of the Ginsparg-Wilson relation and a remnant of supersymmetry on the lattice

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1. Introduction: blocking the continuum

2. Blocking induced symmetry relations

3. Solution of the additional constraint for SUSY

4. Solutions for supersymmetric quantum mechanics

5. Conclusions and outlook
Introduction

continuum path integral

continuum limit $N \to \infty$

symmetry $\{\gamma_5, D\} = 0$

should generate fine tuning

lattice $N$ integrations

? broken by the lattice?; anomalies?; realisation?
Introduction

continuum path integral

RG inspired

continuum limit \( N \to \infty \)

symmetry \( \{\gamma_5, \mathcal{D}\} = 0 \)

relation \( \{\gamma_5, \mathcal{D}\} = a\mathcal{D}\gamma_5\mathcal{D} \)
defines a “symmetry”

\( \{\gamma_5, \text{def}, \mathcal{D}\} = 0 \)
\( \gamma_5, \text{def} = \gamma_5(\mathbb{1} - a\mathcal{D}) \)
The blocking transformation

- averaging of the continuum field $\varphi(x)$ around the lattice point $x_n = an$:

$$\Phi_n[\varphi] := \int dx \ f(x - x_n)\varphi(x)$$

- define a blocked lattice action $S[\phi]$ depending on lattice fields $\phi_n$ for a given continuum action $S_{cl}[\varphi]$

$$e^{-S[\phi]} := \frac{1}{\mathcal{N}} \int d\varphi \ e^{-\frac{1}{2}(\phi - \Phi[\varphi])_n \alpha_{nm}(\phi - \Phi[\varphi])_m} \ e^{-S_{cl}[\varphi]}$$

- simple interpretation if $f(x - x_n) \rightarrow \delta(x - x_n)$ and $\alpha \rightarrow \infty$ as $a \rightarrow 0$ since $S \rightarrow S_{cl}$; more generally

$$\int d\phi \ e^{-S[\phi] + J\phi} = e^{\frac{1}{2} \frac{1}{\alpha} J^{-1} J} \int d\varphi \ e^{-S_{cl}[\varphi] + J\Phi[\varphi]}$$
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$$\int d\phi \ e^{-S[\phi] + J\phi} = e^{\frac{1}{2}J\alpha^{-1}J} \int d\varphi \ e^{-S_{cl}[\varphi] + J\Phi[\varphi]}$$
A lattice symmetry

- Continuum action is invariant under infinitesimal continuum symmetry transformations:
  \[ S_{\text{cl}}[\phi + \delta\phi] = S_{\text{cl}}[(1 + \varepsilon \tilde{M})^{ij}\phi^j] = S_{\text{cl}}[\phi] \]

- To translate the continuum symmetry transformations \( \tilde{M} \) into naive lattice transformations \( M \):
  \[ \Phi^i_n[\tilde{M}\phi] = \int dx \ f_n(x) \ \tilde{M}^{ij}\phi^j(x) = M_{nm}^{ij}\Phi^j_m[\phi] \]

- Can not be found for every \( \tilde{M} \) and \( f \) → additional constraint:

- Naive lattice symmetry transformations:
  \[ (\delta\phi)_m^i = \varepsilon M_{nm}^{ij}\phi_m^j \]

- Naive invariance:
  \[ S[\phi + \delta\phi] = S[\phi] \]
A lattice symmetry

- continuum action is invariant under infinitesimal continuum symmetry transformations:
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  \]

- can not be found for every \( \tilde{M} \) and \( f \) \( \leftrightarrow \) additional constraint

- naive lattice symmetry transformations: \((\delta \phi)_m^i = \varepsilon M_{nm}^{ij} \phi_m^j\)

- naive invariance: \( S[\phi + \delta \phi] = S[\phi] \)

A Remnant of Susy. on the Lattice

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A lattice symmetry

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- Can not be found for every \( \tilde{M} \) and \( f \) \( \xrightarrow{\text{additional constraint}} \)

- Naive lattice symmetry transformations: \((\delta \phi)^i_m = \varepsilon M^{ij}_{nm} \phi^j_m\)

- Naive invariance: \[ S[\phi + \delta \phi] = S[\phi] \]
A lattice symmetry

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- can not be found for every \( \tilde{M} \) and \( f \) \( \rightarrow \) additional constraint

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- naive invariance: \( S[\varphi + \delta \varphi] = S[\varphi] \)
**Inherited symmetry of the blocked action**

\[ e^{-S[\phi]} = \frac{1}{N} \int d\phi \ e^{-S_{cl}[\phi]} \ e^{-\frac{1}{2}(\phi - \Phi[\phi])\alpha(\phi - \Phi[\phi])} \]

- infinitesimal naive transformation of the blocked action:
- infinitesimal continuum transformation of \( \varphi \); use additional constraint: \( \Phi[\tilde{M}\varphi] = M\Phi[\varphi] \)
- express \( (\phi - \Phi) \) in terms of \( \frac{\delta}{\delta\phi} \) and \( \alpha^{-1} \)

\[
M_{nm}^{ij} \frac{\delta S}{\delta \phi_{n}^{i}} = (M\alpha^{-1})_{nm}^{ij} \left( \frac{\delta S}{\delta \phi_{m}^{j}} \frac{\delta S}{\delta \phi_{n}^{i}} - \frac{\delta^2 S}{\delta \phi_{m}^{j} \delta \phi_{n}^{i}} \right) + (\text{STr}M - \text{STr}\tilde{M})
\]

\( \text{STr} \tilde{M} \) accounts for infinitesimal change of the measure \( \rightarrow \) anomaly
Inherited symmetry of the blocked action

\[
M_{nm}^{ij} \phi_m^{j} \frac{\delta}{\delta \phi_n^{i}} e^{-S[\phi]} = \frac{1}{\mathcal{N}} \int d\varphi \ e^{-S_{cl}[\varphi]} \ M_{nm}^{ij} \phi_m^{j} \frac{\delta}{\delta \phi_n^{i}} e^{-\frac{1}{2}(\phi - \Phi[\varphi])}\alpha(\phi - \Phi[\varphi])
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M_{nm}^{ij} \frac{\delta S}{\delta \phi_i^j} \ e^{-S[\phi]} = \frac{1}{N} \int d\varphi \ e^{-S_{cl}[\varphi]} \ M_{nm}^{ij} (\phi - \Phi)_{m}^{j} \frac{\delta}{\delta \phi_i^j} \ e^{-\frac{1}{2}(\phi - \Phi[\varphi]) \alpha (\phi - \Phi[\varphi])}
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M_{nm}^{ij} \frac{\delta}{\delta \phi^i_n} e^{-S[\phi]} = \frac{1}{N} \int d\phi \ e^{-S_{cl}[\phi]} \ M_{nm}^{ij} (\phi - \Phi)^j_m \frac{\delta}{\delta \phi^i_n} e^{-\frac{1}{2} (\phi - \Phi[\varphi]) \alpha (\phi - \Phi[\varphi])}
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\]

\( S\text{Tr} \tilde{M} \) accounts for infinitesimal change of the measure \( \rightarrow \) anomaly
**Symmetry relation for the lattice action**

\[
M_{nm}^{ij} \phi_j^m \frac{\delta S}{\delta \phi_n^i} = (M \alpha^{-1})_{nm}^{ij} \left( \frac{\delta S}{\delta \phi_m^j} \frac{\delta S}{\delta \phi_n^i} - \frac{\delta^2 S}{\delta \phi_m^j \delta \phi_n^i} \right) + (\text{STr} M - \text{STr} \tilde{M})
\]

- $\alpha S^{-1}$ drops out if $(\alpha S^{-1} M)^T + M \alpha S^{-1} = 0$ (supertransposed $\alpha = \alpha^T$)
  \[\Rightarrow\] same relations for $\alpha^{-1}$ and $\alpha^{-1} + \alpha S^{-1}$

- for a quadratic action, $S = \frac{1}{2} \phi_n^j K_{nm}^{ij} \phi_m^i$, the relation turns into
  
  \[
  M^T K + (M^T K)^T = K^T \left[ (M \alpha^{-1})^T + M \alpha^{-1} \right] K
  \]
  and can be rewritten as
  
  \[
  M_{\text{def}}^T K + K^T M_{\text{def}} = 0; \quad M_{\text{def}} = M(\mathbb{1} - \alpha^{-1} K)
  \]

- conditions for $M_{\text{def}}$ to define a deformed symmetry
  - $M_{\text{def}}$ local
  - $M_{\text{def}}$ approaches continuum counterpart (excludes $M_{\text{def}} = 0$)
  \[\Rightarrow\] restricts possible choices of $\alpha$ and $K$
Symmetry relation for the lattice action

\[ M_{nm}^{ij} \phi_{m}^{j} \frac{\delta S}{\delta \phi_{n}^{i}} = (M \alpha^{-1})_{nm}^{ij} \left( \frac{\delta S}{\delta \phi_{m}^{j}} \frac{\delta S}{\delta \phi_{n}^{i}} - \frac{\delta^2 S}{\delta \phi_{m}^{j} \delta \phi_{n}^{i}} \right) + (\text{STr} M - \text{STr} \tilde{M}) \]

- \( \alpha^{-1}_S \) drops out if \( (\alpha^{-1}_S M)^T + M \alpha^{-1}_S = 0 \) (supertransposed \( \alpha = \alpha^T \))
- \( \Rightarrow \) same relations for \( \alpha^{-1} \) and \( \alpha^{-1} + \alpha^{-1}_S \)
- for a quadratic action, \( S = \frac{1}{2} \phi_{n}^{j} K_{nm}^{ij} \phi_{m}^{j} \), the relation turns into
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  \[ M_{\text{def}}^T K + K^T M_{\text{def}} = 0; \quad M_{\text{def}} = M(1 - \alpha^{-1} K) \]

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- \( \alpha^{-1} \) drops out if \((\alpha^{-1}M)^T + \alpha^{-1}M\) (supertransposed \(\alpha = \alpha^T\))
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\[ M^T K + (M^T K)^T = K^T \left[ (M\alpha^{-1})^T + M\alpha^{-1} \right] K \]

\[ M_{\text{def}}^T K + K^T M_{\text{def}} = 0; \quad M_{\text{def}} = M(1 - \alpha^{-1}K) \]

GW: \(\{\gamma_5, D\} = 0; \gamma_5,\text{def} = \gamma_5(1 - \alpha^{-1}D)\)

1. \(M_{\text{def}}\) local
2. \(M_{\text{def}}\) approaches continuum counterpart

GW: excludes Wilson fermions

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Solution of the additional constraint for SUSY

\[
\int dx \ f(x - an) \tilde{M}^{ij} \varphi^j(x) = M^{ij}_{nm} \Phi^j_m[\varphi] = M^{ij}_{nm} \int dx \ f(x - am) \varphi^j(x)
\]

- trivial if \(\tilde{M}^{ij}\) merely acts on multiplet index \(j\); but for SUSY derivative operators appear in the continuum transformations
- must hold for all \(\varphi\); in Fourier space

\[
[\nabla(p_k) - ip_k]f(p_k) = 0
\]

for \(p_k = \frac{2\pi}{L} k, \ k \in \mathbb{Z}\) and \(\nabla(p + \frac{2\pi}{a}) = \nabla(p)\)

- solutions: nonlocal SLAC-derivative; otherwise effective cutoff below \(\frac{2\pi}{a}\) is introduced by \(f(p)\)
Solution of the additional constraint for SUSY

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Setting for supersymmetric quantum mechanics

- transformations in the continuum, \( \varphi^i(x) = (\chi(x), F(x), \psi(x), \bar{\psi}(x)) \):

  \[
  \begin{align*}
  \delta \chi &= -\bar{\epsilon} \psi + \epsilon \bar{\psi} \\
  \delta F &= -\bar{\epsilon} \partial \psi - \epsilon \partial \bar{\psi} \\
  \delta \psi &= -\epsilon \partial \chi - \epsilon F \\
  \delta \bar{\psi} &= \bar{\epsilon} \partial \bar{\phi} - \bar{\epsilon} F
  \end{align*}
  \]

- naive transformations on the lattice, \( \phi^i_n = (\chi_n, F_n, \psi_n, \bar{\psi}_n) \):

  \[
  \delta \begin{pmatrix}
  \chi \\
  F \\
  \psi \\
  \bar{\psi}
  \end{pmatrix}
  = \begin{pmatrix}
  0 & 0 & -\bar{\epsilon} & \epsilon \\
  0 & 0 & -\bar{\epsilon} \nabla & -\epsilon \nabla \\
  -\epsilon \nabla & -\epsilon & 0 & 0 \\
  \bar{\epsilon} \nabla & -\bar{\epsilon} & 0 & 0
  \end{pmatrix}
  \begin{pmatrix}
  \varphi \\
  F \\
  \psi \\
  \bar{\psi}
  \end{pmatrix}
  = (\epsilon M + \bar{\epsilon} \bar{M}) \phi
  \]

  \( \nabla \) solution of additional constraint (SLAC-derivative)
Setting for supersymmetric quantum mechanics

- invariant quadratic action in the continuum:
  \[ S_{\text{cl}} = \int dx \left[ \frac{1}{2} (\partial_x \chi) + \bar{\psi} \partial_x \psi - \frac{1}{2} F^2 + \bar{\psi} W'(\chi) \psi - FW(\chi) \right] \]
  \[ = \int dx \left[ \frac{1}{2} (\partial_x \chi) + \bar{\psi} \partial_x \psi - \frac{1}{2} F^2 + m \bar{\psi} \psi - m F \chi \right] \]

- ansatz for the lattice action \( S = \frac{1}{2} \phi K \phi \):
  \[
  K_{ij} = \left( \begin{array}{cccc}
  -\Box_{nm} & -m_{b,nm} & 0 & 0 \\
  -m_{b,nm} & -l_{nm} & 0 & 0 \\
  0 & 0 & 0 & (\hat{\nabla} - m_f)_{nm} \\
  0 & 0 & (\hat{\nabla} + m_f)_{nm} & 0 \\
  \end{array} \right)
  
  I, \Box, m_b, m_f \text{ symmetric; } \hat{\nabla} \text{ antisymmetric}
  
  \text{translation invariance: all circulant matrices (\( \rightarrow \text{ commute} \))} \]
Solutions for a quadratic action

- solve $M_{\text{def}}^T K + K^T M_{\text{def}} = 0$ with $M_{\text{def}} = M(1 - \alpha^{-1} K)$
- diagonal blocking matrix (as for overlap: $\alpha \sim \delta_{nm}$) leads to nonlocal action (use freedom to choose $\alpha^{-1}_S$ to reduce matrix elements)

$$a(\alpha^{-1})_{nm} = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & 0 & -a_1 \\ 0 & 0 & a_1 & 0 \end{pmatrix} \delta_{nm}; \quad \hat{\nabla} + m_f = \frac{\nabla + m_b}{1 + a_0 + a_1 mb + (a_1 + a_2 mb)\nabla}$$

- local solutions like $\hat{\nabla}$ symmetric derivative, $\Box = \hat{\nabla}^2$, $I = 1$, $m_b = m_f = m + m_w$ generically lead to nonlocal $\alpha^{-1}$
- demand $M_{\text{def}}$ and $K$

$$M_{\text{def}} = \begin{pmatrix} 0 & 0 & 0 & 0 & / \\ 0 & 0 & 0 & 0 & -/\nabla \\ -\nabla & -/\nabla & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \hat{\nabla} = /\nabla$$

$\rho \rightarrow 1$, $/\nabla \rightarrow \partial_x$ cont. limit

$\rho$ and $/\nabla$ must be local

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$$-\Box + m_b^2 = \frac{-\nabla^2 + m_b^2}{1 + a_0 - a_2 \nabla^2}$$

$$I = 1$$

- local solutions like $\hat{\nabla}$ symmetric derivative, $\Box = \hat{\nabla}^2$, $I = 1$, and $m_b = m_f = m + m_w$ generically lead to nonlocal $\alpha^{-1}$

- demand $M_{\text{def}}$ and $K$

$$M_{\text{def}} = \begin{pmatrix} 0 & 0 & 0 & \ell \\ 0 & 0 & 0 & -\ell \nabla \\ -\nabla & -\ell \nabla & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \hat{\nabla} = \ell \nabla$$

$I \rightarrow 1$, $\ell \nabla \rightarrow \partial_x$ cont. limit

$I$ and $\ell \nabla$ must be local
Solutions for a quadratic action

- solve $M_{\text{def}}^T K + K^T M_{\text{def}} = 0$ with $M_{\text{def}} = M(1 - \alpha^{-1}K)$
- diagonal blocking matrix (as for overlap: $\alpha \sim \delta_{nm}$) leads to nonlocal action (use freedom to choose $\alpha^{-1}$ to reduce matrix elements)

\[a(\alpha^{-1})_{nm} = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & 0 & -a_1 \\ 0 & 0 & a_1 & 0 \end{pmatrix}\delta_{nm}; \quad \nabla + m_f = \frac{\nabla + m_b}{1 + a_0 + a_1 m_b + (a_1 + a_2 m_b)\nabla}
\]

- local solutions like $\hat{\nabla}$ symmetric derivative, $\Box = \hat{\nabla}^2$, $I = 1$, and $m_b = m_f = m + m_w$ generically lead to nonlocal $\alpha^{-1}$
- demand $M_{\text{def}}$ and $K$

\[M_{\text{def}} = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & -I\nabla \\ -\nabla & -I\nabla & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \hat{\nabla} = I\nabla \quad \text{cont. limit} \quad I \to 1, \quad I\nabla \to \partial_x \quad \text{must be local} \]
Beyond the quadratic action

- final goal: construct a supersymmetric local interacting lattice action
- the given relation extends beyond the quadratic case
- it connects different orders of the field \(\rightarrow\) generically nonpolynomial solutions
- not unexpected since blocked action is comparable to the effective action
- under special conditions a truncation can be achieved
Conclusions and outlook

- symmetry of a continuum action implies the fulfilment of certain relations for the lattice action which ensure a symmetric continuum limit and define deformed lattice symmetry operators
- requirement: definition of a naive lattice transformation by the “averaged” continuum symmetry transformation (additional constraint) $\leftrightarrow$ SLAC-derivative for SUSY
- severe restriction: $M_{\text{def}}$ and the action must be local; can be fulfilled under special conditions
- although the relation couples different orders of the fields, even for interacting theories a polynomial solution can be achieved
- from the GW point of view: more careful investigations of the conditions for lattice SUSY is needed: compare with other symmetries; use the knowledge from ERG studies for interacting case; generalise the setup