Tuning improved anisotropic actions in lattice perturbation theory

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Motivation

- Part of the LHPC hadron spectrum effort
- Try to identify all the low-lying excitations predicted by QCD
- Interested in the spectrum beyond ground states
- Strongly constrains the choice of lattice action (well defined single-timeslice transfer operator)
- The use of $3 + 1$ anisotropic lattices ($a_t < a_s$) greatly assists in the extraction of excited states
Actions

- Use a 3+1 anisotropic Sheikholeslami-Wohlert quark action and a tree-level Symanzik and tadpole-improved gauge action

\[
M_{\text{quark}} = m_0 + \gamma_t \nabla_t - \frac{a_t}{2} \triangle_t + \nu_s \sum_k \left( \gamma_k \nabla_k - \frac{a_s}{2} \triangle_k \right) 
+ \frac{1}{2} \left[ c_t a_s \sum_k \sigma_{tk} F_{tk} + c_s a_s \sum_{k<l} \sigma_{kl} F_{kl} \right] 
\tag{1}
\]

\[
S_{\text{gauge}} = -\beta \left[ \frac{4\xi_g}{3} \sum_i P_{it} - \frac{\xi_g}{12} \sum_i R_{ti} + \frac{5}{3\xi_g} \sum_{i<j} P_{ij} - \frac{1}{12\xi_g} \sum_{i<j} (R_{ij} + R_{ji}) \right] 
\]

- To further reduce lattice artefacts all spatial link variables in the quark action are stout smeared.
To obtain the correct continuum limit, \( \nu_s \) and \( \xi_g \) must be tuned such that measurements of the ratio \( a_s/a_t \) using different physical probes agree at a fixed target value.

This can be done non-perturbatively (R. Edwards - this conference).

However, in principle, new tuning runs are required for each new parameter set.

Lattice perturbation theory can provide precision results at high \( \beta \).

Real progress is made when both approaches are combined.

The ultimate goal is to combine results from these complementary methods to obtain functional forms for the action parameters which hold over much of parameter space.
The input parameters in this study are the quark mass, the (target) anisotropy and the smearing parameters.

The stout smearing algorithm is

\[
U_{\mu}^{(n+1)}(x) = \exp\left(iQ_{\mu}^{(n)}(x)\right) U_{\mu}^{(n)}(x)
\]

\[
Q_{\mu}(x) = \frac{i}{2} \left( \Omega_{\mu}^{\dagger}(x) - \Omega_{\mu}(x) \right) - \frac{i}{2N} \text{Tr} \left( \Omega_{\mu}^{\dagger}(x) - \Omega_{\mu}(x) \right)
\]

\[
\Omega_{\mu}(x) = \sum_{\nu \neq \mu} \rho_{\mu\nu} \left( U_{\nu}(x) U_{\mu}(x + \hat{\nu}) U_{\nu}^{\dagger}(x + \hat{\mu}) \right.
\]

\[
\quad + \left. U_{\nu}^{\dagger}(x - \hat{\nu}) U_{\mu}(x - \hat{\nu}) U_{\nu}(x - \hat{\nu} + \hat{\mu}) \right) U_{\mu}^{\dagger}(x)
\]

(3)

In our simulations \( \rho_{ij} = \rho \) and \( \rho_{t\mu} = \rho_{\mu t} = 0 \).
Quantities of interest

- The parameter $\nu_s$ appearing in the quark action is fixed by demanding that the anisotropy measured from the quark dispersion relation takes a predefined target value.

- $\xi_g$ is determined from the gluon dispersion relation.

- $c_t$ and $c_s$ are tuned by matching lattice scattering amplitudes to their continuum counterparts.
Methodology

- Smeared vertex functions quickly become complicated
- Automated generation of vertex functions
- HiPPy (Hart et al. \(^1\)) and independent C++ code
- Can handle any number of gluons and any level of smearing (1,2,3,100)
- Suite of C++ code used to evaluate integrands
- All spin manipulations are handled by the code
- Automatic differentiation used to evaluate derivatives with respect to external momenta
- Significantly reduces the chances of human error

Tree-level values

- Already at tree-level the coefficients in the quark action are mass-dependent
- $\mathcal{O}(a_t, a_s)$ improvement requires that $c_s^{(0)} = \nu_s^{(0)}$
- Expressions agree with Fermilab formulae
Get $\nu_s^{(1)}$ (but not $c_s^{(1)}$ or $c_t^{(1)}$) from the one-loop quark propagator

Solve for the pole and expand the energy in powers of the spatial momentum

Energy and momenta are measured in lattice units $(1/a_t, 1/a_s)$

At fixed anisotropy, tune $\nu_s$ such that $E^2(\vec{p}) = E^2(\vec{0}) + |\vec{p}|^2$ for small $|\vec{p}|$

At one loop just two diagrams contribute
Quark Masses

- Critical quark mass and one-loop rest mass appear in the calculation
- Smearing parameters $N_\rho = 2, \rho = 0.14$ minimise the critical quark mass (and maximise the spatial plaquettes) at $\xi = 3.5$
Corrections to $\nu_s^{(1)}$ are small over a range of quark masses

The choice of smearing parameters which minimises $a_t m_{\text{crit}}^{(1)}$ does not minimise $\nu_s^{(1)}$

Tadpole improvement has a small effect
Gauge anisotropy

- The gauge anisotropy is fixed by requiring that a gluon obey a relativistic dispersion relation at small momentum
- Determined the gluon propagator to one-loop order and solve for poles
- Seven diagrams contribute to the gluon propagator at one-loop order
At one-loop order the measured anisotropy is

\[ \xi_R = \left( 1 + g_0^2 \eta \right) \xi_g \]

\[ \eta = \frac{\xi_g^2}{4} \left[ \frac{d^2}{d (a_s p_i)^2} a_t^2 \Sigma_j^{(1)} (p) \right] \vec{p} = (p_i, 0, 0) \]

Corrections to the gauge anisotropy are additive.

At one-loop order the sea quark contribution to the gauge anisotropy is independent of the gauge action.

Pure gluonic part agrees with the calculation Drummond et al. \(^2\)

Gauge anisotropy - results

![Graph showing the gluonic and quark contributions with parameters $a_t M_{\text{pole}} = 0.1$, $N_\rho = 2$, $\rho = 0.14$.]

- Gluonic contribution
- Quark contribution

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Mass/Smearing dependence

\[ a_t M_{\text{pole}}^{(0)} \]

\[ \eta_{\text{quark}} \]

Unsmeared
\[ N_\rho = 2, \rho = 0.14 \]

\[ \xi_g = 4 \]

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Future directions

- Need to explore parameter space fully
- Monte Carlo comparison
- Use P.T. to guide the choice of smearing parameters

$O(\alpha_s a_t, \alpha_s a_s)$ improvement