Fermionic correlation functions from the Staggered SF

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Schrödinger functional

- Schrödinger functional:
  \[ Z[C, C', \rho, \bar{\rho}, \rho', \bar{\rho}'] = \int D[A, \psi, \bar{\psi}] e^{-S[A, \psi, \bar{\psi}]} \]

- Boundary conditions:
  \[
  \begin{align*}
  A_k(y) \big|_{y_0=0} &= C_k \quad & A_k(y) \big|_{y_0=T} &= C_k', \\
  P_+ \psi(y) \big|_{y_0=0} &= \rho \quad & P_- \psi(y) \big|_{y_0=T} &= \rho', \\
  \bar{\psi}(y) P_- \big|_{y_0=0} &= \bar{\rho} \quad & \bar{\psi}(y) P_+ \big|_{y_0=T} &= \bar{\rho}'
  \end{align*}
  \]
  with \( P_\pm = \frac{1}{2} (1 \pm \gamma_0) \).

- Expectation value of \( \mathcal{O} \):
  \[
  \langle \mathcal{O} \rangle = \left\{ \frac{1}{Z} \int D[A] D[\psi] D[\bar{\psi}] \mathcal{O} e^{-S[A, \bar{\psi}, \psi]} \right\}_{\bar{\rho}'=\rho'=\bar{\rho}=\rho=0}
  \]

- \( \mathcal{O} \) may involve the “boundary fields”,
  \[
  \begin{align*}
  \zeta(y) &= \frac{\delta}{\delta \rho(y)}, \quad & \bar{\zeta}(y) &= -\frac{\delta}{\delta \bar{\rho}(y)}, \\
  \zeta'(y) &= \frac{\delta}{\delta \rho'(y)}, \quad & \bar{\zeta}'(y) &= -\frac{\delta}{\delta \bar{\rho}'(y)}
  \end{align*}
  \]
Correlation functions

- **Notation:** $\lambda^a$ flavour matrices in a theory with $N_f$ flavours.

- **Axial current:** $A_{\mu}^a(y) = \bar{\psi}(y)\gamma_{\mu}\gamma_5\frac{1}{2}\lambda^a\psi(y)$.

- **Axial density:** $P^a(y) = \bar{\psi}(y)\gamma_5\frac{1}{2}\lambda^a\psi(y)$.

- **Creation of a $q\bar{q}$ pair at $y_0 = 0$, $T$:**
  $$\mathcal{O}^a = \int d^3y' d^3y'' \bar{\zeta}(y')\gamma_5\frac{1}{2}\lambda^a\zeta(y''), \quad \mathcal{O}'^a = \int d^3zd^3z' \bar{\zeta}'(z)\gamma_5\frac{1}{2}\tau^a\zeta'(z')$$

- **Correlation functions:**
  $$\delta^{ab} f_A(y_0) = -\int d^3y' d^3y'' \langle A_0^a(y)\mathcal{O}^b \rangle,$$
  $$\delta^{ab} f_P(y_0) = -\int d^3y' d^3y'' \langle P^a(y)\mathcal{O}^b \rangle,$$
  $$\delta^{ab} f_1 = -\int d^3y' d^3y'' d^3zd^3z' \langle \mathcal{O}^a\mathcal{O}'^b \rangle.$$
Staggered fermions and continuum limit

- Technical problem with staggered fermions (Miyazaki and Kikukawa ’94 & Heller ’97): $T/a$ odd and $L/a$ even.

- Modified conventions: take the continuum limit at fixed $T'/L$ where $T' = T + sa$ is the extent of the dual lattice ($s = \pm 1$).

- This modifies the $O(a)$ effects in the pure gauge theory even at tree level. This has been studied in previous works up to one loop in perturbation theory.

- The reconstruction of the four component spinors is different for the two cases $T' = T + sa$, with $s = \pm 1$. This is to be discussed here.
Interpretation of the reconstruction for $T' = T - a$

- Four component spinors reside in a coarse lattice, $\bar{a} = 2a$.

- Set $x = 2y + a\xi$, $\xi_{\mu} \in \{0, 1\}$.
  \[
  \chi_{\xi}(y) = \chi(x), \quad \bar{\chi}_{\xi}(x) = \bar{\chi}(x)
  \]

- Four component spinors being:
  \[
  \psi_{\alpha a}(y) = \frac{1}{4} \sum_{\xi} (\Gamma_{\xi})_{\alpha a} \chi_{\xi}(y),
  \]
  \[
  \bar{\psi}_{a\alpha}(y) = \frac{1}{4} \sum_{\xi} \bar{\chi}_{\xi}(y)(\Gamma_{\xi}^\dagger)_{a\alpha}.
  \]

- The transformation matrices read:
  \[
  \Gamma_{\xi} = \frac{1}{2} \gamma_0 \gamma_1 \gamma_2 \gamma_3.
  \]
Reconstructed action for $T' = T - a$

- **Notation:**
  - Flavour matrices: $\tau_\mu = \gamma^T_\mu$, $\tau_{\mu 5} = i(\gamma_\mu\gamma_5)^T$, ...
  - Symmetric derivative: $\tilde{\partial}_\mu$.
  - Second derivative: $\Delta_\mu$.

- **b.c.’s:** $Q_\pm = \frac{1}{2}(1 \pm i\gamma_0\gamma_5\tau_{05})$, project onto the boundary fields,

$$
Q_+\psi(0, y) = \hat{\rho}(y), \quad Q_-\psi(T', y) = \hat{\rho}'(y),
\bar{\psi}(0, y)Q_+ = \hat{\rho}(y), \quad \bar{\psi}(T', y)Q_- = \hat{\rho}'(y).
$$

- **Reconstructed action** (homogeneous b.c.’s):

$$
S_{SQ}^{(-1)} = \bar{a}^4 \sum_{y_0=0}^{T'} \sum_{y_\mu} \bar{\psi}(y) \left[ \gamma_\mu \tilde{\partial}_\mu + i \frac{\bar{a}}{2} \gamma_5 \tau_{\mu 5} \Delta_\mu \right] \psi(y).
$$

Fields outside the cylinder have been set to 0.

- **Define:** $D_\mu = \tilde{\partial}_\mu + i \frac{\bar{a}}{2} \gamma_\mu \gamma_5 \tau_{\mu 5} \Delta_\mu$. 

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Chiral rotation to the standard SF for $\mathcal{T}' = \mathcal{T} - \alpha$

- The usual SF b.c.'s can be obtained by performing a chiral rotation,

$$\psi'(y) = R(\alpha)\psi(y), \quad \bar{\psi}'(y) = \bar{\psi}(y)R(\alpha), \quad R(\alpha) = \exp\left(i\frac{\alpha}{2}\gamma_5\tau_0\right).$$

- Set $\alpha = \frac{\pi}{2} \Rightarrow R\left(\frac{\pi}{2}\right)Q \pm R^{-1}\left(\frac{\pi}{2}\right) = P \pm$.

  The b.c.'s will be the usual ones with,

  $$\rho(y) = R\left(\frac{\pi}{2}\right)\hat{\rho}(y), \quad \bar{\rho}(y) = \hat{\rho}(y)R\left(\frac{\pi}{2}\right).$$

- For **homogeneous** b.c.'s, the action in this basis reads,

$$S_{SQ}^{(-1)} = \bar{a}^4 \sum_{y=0}^{\mathcal{T}'} \sum_y \bar{\psi}'(y) \left[ \sum_k \gamma_k D_k + \gamma_0 \tilde{\partial}_0 + \frac{\bar{a}}{2} \Delta_0 \right] \psi'(y).$$

- **Remark:** $P_- \psi'(0, y), \ldots$, are dynamical fields.
Interpretation and reconstruction for $T' = T + a$

**Figure:** Reconstruction for $s = 1^+$. 

$x_0 = 2y_0 - a + a\xi_0, \quad x = 2y + a\xi$

$\psi_{\alpha\alpha}(y) = \frac{1}{4} \sum_\xi (\tilde{\Gamma}_\xi)_{\alpha\alpha} \chi_\xi(y)$,

$\bar{\psi}_{\alpha\alpha}(y) = -\frac{1}{4} \sum_\xi \bar{\chi}_\xi(y)(\tilde{\Gamma}_\xi^\dagger)_{\alpha\alpha}$

$\tilde{\Gamma}_\xi = \frac{1}{2}(-1)^{\xi_0} \gamma_0^{\xi_0} \gamma_1^{\xi_1} \gamma_2^{\xi_2} \gamma_3^{\xi_3}$

**Figure:** Reconstruction for $s = 1^-$. 

$x_0 = 2y_0 - a\xi_0, \quad x = 2y + a\xi$

$\psi_{\alpha\alpha}(y) = \frac{1}{4} \sum_\xi (\Gamma_\xi)_{\alpha\alpha} \chi_\xi(y)$,

$\bar{\psi}_{\alpha\alpha}(y) = \frac{1}{4} \sum_\xi \bar{\chi}_\xi(y)(\Gamma_\xi^\dagger)_{\alpha\alpha}$. 

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Rotating back to the standard SF, \( T' = T + a \)

**Case \( s = 1^+ \)**

- **b.c.'s:**
  \[
  \begin{align*}
  Q_\psi(0, y) &= \hat{\rho}, & Q_+\psi(0, y) &= \hat{\rho}' \\
  \bar{\psi}(0, y)Q_- &= \hat{\rho} & \bar{\psi}'(0, y)Q_+ &= \hat{\rho}'.
  \end{align*}
  \]

- **chiral rotation:**
  \[
  \begin{align*}
  \psi'(y) &= R(-\frac{\pi}{2})\psi(y), \\
  \bar{\psi}'(y) &= \bar{\psi}(y)R(-\frac{\pi}{2}).
  \end{align*}
  \]

**Case \( s = 1^- \)**

- **b.c.'s:**
  \[
  \begin{align*}
  Q_+\psi(0, y) &= \hat{\rho}, & Q_-\psi(0, y) &= \hat{\rho}' \\
  \bar{\psi}(0, y)Q_- &= \hat{\rho} & \bar{\psi}'(0, y)Q_+ &= \hat{\rho}'.
  \end{align*}
  \]

- **chiral rotation:**
  \[
  \begin{align*}
  \psi'(y) &= R(\frac{\pi}{2})\psi(y), \\
  \bar{\psi}'(y) &= \bar{\psi}(y)R(\frac{\pi}{2}).
  \end{align*}
  \]

**SF basis**

- **usual b.c.'s:** \( P_+\psi'(0, y) = \rho(y) \ldots \)

- **action** (homogeneous b.c.'s):
  \[
  S_{SQ}^{(1)} = \bar{a}^4 \sum_{y_0, y} \bar{\psi}'(y) \left[ \sum_k \gamma_k D_k + \gamma_0 \tilde{\partial}_0 - \frac{\bar{a}}{2} \Delta_0 \right] \psi'(y). \]
Staggered symmetries of the Schrödinger functional

1. Rotations, reflections, fermion number, charge conjugation.

2. Chiral symmetry:
   - Standard staggered basis: \( \psi(y) \rightarrow e^{i\beta \gamma_5 \tau_5} \psi(y), \quad \bar{\psi}(y) \rightarrow \bar{\psi}(y)e^{i\beta \gamma_5 \tau_5}. \)
   - SF basis: \( \psi'(y) \rightarrow e^{i\beta \tau_0} \psi'(y_0), \quad \bar{\psi}'(y) \rightarrow \bar{\psi}'(y)e^{-i\beta \tau_0}. \)

   **FLAVOUR SYMMETRY!**

3. Shift symmetry: Set \( Q^{(k)}_{\pm} = \frac{1}{2} (1 \pm i \gamma_k \gamma_5 \tau_5), \)

   \[
   \psi(y) \rightarrow \tau_k \psi(y) + \bar{a} \tau_k Q^{(k)}_+ \partial_k \psi(y),
   \quad
   \bar{\psi}(y) \rightarrow \bar{\psi} \tau_k \psi(y) + \bar{a} \bar{\psi}(y) \partial_k \tau_k Q^{(k)}_+.
   \]

   **DISCRETE FLAVOUR SYMMETRY IN THE CONTINUUM LIMIT!**
Quark propagation

- Integrate over the quark fields: \( \langle O \rangle = \langle [O]_F \rangle_G \).
- Quark field average: \([O]_F = \left\{ \frac{1}{Z_F} O Z_F \right\}_{\vec{p}'=\rho=0} \).
- Two point functions: \([\psi'(y)\bar{\psi}'(y')]_F = S(y,y') \).
- Chiral Symmetry: Forbids disconnected diagrams in computation of \( f_A, f_P, f_1 \) for flavour matrices \( \{\tau^a, \tau_0\} = 0 \).
- \( f_A, f_P, f_1 \) on the lattice read:

\[
f^{ab}_A(y_0) = \bar{a}^6 \sum_{y',y''} \frac{1}{8} \left\langle \text{tr} \left( [\zeta(y'')\bar{\psi}'(y)]_F \gamma_0 \gamma_5 \tau^a [\psi'(y)\bar{\zeta}(y')]_F \gamma_5 \tau^b \right) \right\rangle_G,
\]

\[
f^{ab}_P(y_0) = \bar{a}^6 \sum_{y',y''} \frac{1}{8} \left\langle \text{tr} \left( [\zeta(y'')\bar{\psi}'(y)]_F \gamma_5 \tau^a [\psi'(y)\bar{\zeta}(y')]_F \gamma_5 \tau^b \right) \right\rangle_G,
\]

\[
f^{ab}_1 = \bar{a}^{12} \sum_{y',y''} \sum_{z',z''} \frac{1}{8} \left\langle \text{tr} \left( [\zeta(y'')\bar{\zeta}'(z')]_F \gamma_0 \gamma_5 \tau^a [\zeta'(z'')\bar{\zeta}(y')]_F \gamma_5 \tau^b \right) \right\rangle_G.
\]
Results

• Continuum values of \( f_X \) at tree level, zero background fields:

\[
\begin{align*}
    f^c_A \left( \frac{T'}{2} \right) &= -\frac{N_c}{\cosh^2(\sqrt{3}\theta)}, \\
    f^c_P \left( \frac{T'}{2} \right) &= \frac{N_c}{\cosh(\sqrt{3}\theta)}, \\
    f^c_1 &= \frac{N_c}{\cosh^2(\sqrt{3}\theta)}.
\end{align*}
\]

\( \theta \) is a phase factor coming from the generalised boundary conditions,

\[
\psi(y + L\hat{k}) = e^{i\theta} \psi(y), \quad \bar{\psi}(y + L\hat{k}) = \bar{\psi}(y)e^{-i\theta}.
\]

• Computed from the one component staggered action, including \( \tilde{c}_{s1}^{(0)} \) to be discussed in the next section.

• Computed using the analytic expression of the staggered propagator with insertions corresponding to corrections related to \( \tilde{c}_{s1}^{(0)} \).

• Results obtained:

\[
\begin{align*}
    f_A \left( \frac{T'}{2} \right) &= f^c_A \left( \frac{T'}{2} \right) + O(a), \\
    f_P \left( \frac{T'}{2} \right) &= f^c_P \left( \frac{T'}{2} \right) + O(a^2), \\
    f_1 &= f^c_1 + O(a^2).
\end{align*}
\]
$O(a) \text{ improvement. Infinite volume}$

- Next to the continuum limit, Symanzik 83':
  
  $$S_{\text{eff}} = S_0 + aS_1 + a^2 S_2 + \ldots, \quad S_k = \int d^4 y \mathcal{L}_k(y).$$

- $\bar{\psi}\gamma_\mu D_\mu \psi$ invariant under shift symmetry. $O(a)$ volume effects fixed.

- Luo ’97: no dimension 5 operators for staggered fermions.

- $O(a)$ improvement implemented by the use of improved staggered fields.
  
  $$\psi^I(y) = \psi(y) + \frac{a}{4} \sum_\nu (Q_+^{(\nu)} - Q_-^{(\nu)}) \partial_\nu \psi(y),$$

  $$\bar{\psi}^I(y) = \bar{\psi}(y) + \frac{a}{4} \sum_\nu \bar{\psi}(y) (Q_+^{(\nu)} - Q_-^{(\nu)}).$$

- Action with improved fields,
  
  $$S_{SQ} = \bar{\psi}^I(y) \gamma_\mu \partial_\mu \psi^I(y) + O(a^2).$$
\( O(a) \) improvement. Boundaries

- **Dimension 3**: \( \mathcal{O}_1 = \bar{\psi} \psi' \quad \implies \text{Renormalisation of the quark boundary fields} \, \bar{\zeta} \zeta. \)

- **Dimension 4**: Same counterterms as Wilson. Choice

\[
\delta S_{F,b}[U, \bar{\psi}, \psi] = \tilde{a}^4 \sum_{y} \left\{ (\tilde{c}_s - 1)[\hat{\mathcal{O}}_s + \hat{\mathcal{O}}'_{s1}] + (\tilde{c}_s - 1)[\hat{\mathcal{O}}_{s2} + \hat{\mathcal{O}}'_{s2}] \right\},
\]

\[
\hat{\mathcal{O}}_s = \bar{\psi}'(0, y) P_+ \gamma_k \mathcal{D}_k \psi'(0, y), \quad \hat{\mathcal{O}}'_s = \bar{\rho}(y) \gamma_k \mathcal{D}_k \rho(y),
\]

\[
\hat{\mathcal{O}}_{s1} = \bar{\psi}'(T, y) P_+ \gamma_k \mathcal{D}_k \psi'(T, y), \quad \hat{\mathcal{O}}'_{s1} = \bar{\rho}'(y) \gamma_k \mathcal{D}_k \rho'(y),
\]

- Perturbation expansion of the improvement coefficients:

\[
\tilde{c}_s = \tilde{c}_s^{(0)} + \tilde{c}_s^{(1)} g_0^2 + \ldots,
\]

- Tree level value:

\[
\tilde{c}_{s1}^{(0)} \bigg|_{T' = T \mp a} = 1 \mp \frac{1}{4}.
\]
Summary and outlook

- We have reconstructed the four component spinors in the Schrödinger functional framework, for the cases $T' = T \mp a$.

- The computation of the tree level correlation functions $f_A, f_P, f_1$ for staggered fermions has been done.

- The implementation of the $O(a)$ improvement is being done. WORK IN PROGRESS.

- Once it is fully understood, we will begin to run simulations to determine the running of the coupling and the quark mass.