Exact Chiral Fermions and Finite Density on Lattice

Debasish Banerjee, Rajiv V. Gavai & Sayantan Sharma*

T. I. F. R., Mumbai

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Introduction: Why Exact Chiral Fermions?

Overlap and Domain Wall Fermions

Our Results

Summary

Introduction: Why Exact Chiral Fermions?

- The finite temperature transition in our world, i.e., QCD with $2 + 1$ flavours of dynamical quarks, is widely accepted to be governed by chiral symmetry.

- Staggered fermions have dominated the area of nonzero temperatures and densities.
Introduction: Why Exact Chiral Fermions?

• The finite temperature transition in our world, i.e., QCD with 2 + 1 flavours of dynamical quarks, is widely accepted to be governed by chiral symmetry.

• Staggered fermions have dominated the area of nonzero temperatures and densities.

• As I presented in Lattice 2006, hadronic screening lengths, advocated by DeTar & Kogut (PRD ’87) to explore the large scale composition of QGP, illustrate their deficiency in the pionic screening length.

• Overlap fermions appear to do better.
Overlap Compared with Staggered Fermions

Local masses \[ \sim \ln(C(r)/C(r+1)) \] show nice plateau behaviour for pi & rho for Overlap (left) unlike staggered (right) fermions.

Gavai, Gupta, Lacaze PRD 2008  
Gavai, Gupta PRD 2002
The pionic screening length shows significant $a^2$ corrections for staggered (left) unlike Overlap (right) fermions.

Gavai, Gupta PRD 2002

Gavai, Gupta, Lacaze PRD 2008
QCD Phase diagram

♠ Another fundamental aspect – Critical Point in $T - \mu_B$ plane;
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Expected QCD Phase Diagram

From Rajagopal-Wilczek Review
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McLerran-Pisarski 2007

From Rajagopal-Wilczek Review
Exact chiral invariance for a lattice fermion operator $D$ is assured if it satisfies the Ginsparg-Wilson relation: $\{\gamma_5, D\} = aD\gamma_5 D$. 
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In particular, the chiral transformations (Lüscher, PLB 1999) $\delta\psi = \alpha\gamma_5(1 - \frac{a}{2}D)\psi$ and $\delta\bar{\psi} = \alpha\bar{\psi}(1 - \frac{a}{2}D)\gamma_5$, leave the action $S = \sum \bar{\psi}D\psi$ invariant:

$$\delta S = \alpha \sum_{x,y} \bar{\psi}_x \left[ \gamma_5D + D\gamma_5 - \frac{a}{2}D\gamma_5D - \frac{a}{2}D\gamma_5D \right]_{xy} \psi_y = 0$$

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Overlap fermions, and Domain Wall fermions in the limit of large fifth dimension satisfy this relation.
Introducing Chemical Potential

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- Simpler alternative: $D_w \rightarrow D_w(a\mu)$ by $K(a\mu) = \exp(a\mu)$ and $L(a\mu) = \exp(-a\mu)$ in positive/negative time direction respectively. (Bloch and Wettig, PRL 2006; PRD 2007).
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- Note $\gamma_5 D_w(a\mu)$ is no longer Hermitian, requiring an extension of the sign function. B & W proposal: For complex $\lambda = (x + iy)$, $\text{sign}(\lambda) = \text{sign}(x)$. 

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- Gattringer-Liptak, PRD 2007, showed for $M = 1$ numerically that no $\mu^2$ divergences exist for the free case ($U = 1$).
• We show this to be true analytically and for all $M$ as well. Furthermore, this holds for all functions such that $K(a\mu) \cdot L(a\mu) = 1$ for Overlap (Banerjee, Gavai, Sharma, PRD 2008) and Domain Wall Fermions (Gavai, Sharma 2008).
We show this to be true analytically and for all $M$ as well. Furthermore, this holds for all functions such that $K(a\mu) \cdot L(a\mu) = 1$ for Overlap (Banerjee, Gavai, Sharma, PRD 2008) and Domain Wall Fermions (Gavai, Sharma 2008).

We claim that chiral invariance is lost for nonzero $\mu$. Note that

$$\delta S = \alpha \sum_{x,y} \bar{\psi}_x \left[ \gamma_5 D(a\mu) + D(a\mu) \gamma_5 - \frac{a}{2} D(0) \gamma_5 D(a\mu) - \frac{a}{2} D(a\mu) \gamma_5 D(0) \right]_{xy} \psi_y,$$

under Lüscher’s chiral transformations.
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• However, the sign function definition above merely ensures

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which is not sufficient to make \( \delta S = 0 \).
• We show this to be true analytically and for all $M$ as well. Furthermore, this holds for all functions such that $K(a\mu) \cdot L(a\mu) = 1$ for Overlap (Banerjee, Gavai, Sharma, PRD 2008) and Domain Wall Fermions (Gavai, Sharma 2008).

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$$\gamma_5 D(a\mu) + D(a\mu) \gamma_5 - a \ D(a\mu) \gamma_5 D(a\mu) = 0 ,$$

which is not sufficient to make $\delta S = 0$. True for both Overlap and Domain Wall fermions and any $K,L$. 

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Consequences

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• Only smooth chiral condensates : No (clear) chiral transition for any (large) $\mu$ possible. How small $a$, or large $N_T$ may suffice ?

• All coefficients of a Taylor expansion in $\mu$ do have the chiral invariance but the series will be smooth and should always converge.
What if . . .

♠ the chiral transformations were \( \delta \psi = \alpha \gamma_5 (1 - \frac{a}{2} D(a\mu))\psi \) and \\
\( \delta \bar{\psi} = \alpha \bar{\psi} (1 - \frac{a}{2} D(a\mu)) \gamma_5 \) ?
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• Symmetry transformations should not depend on “external” parameter \( \mu \).
Chemical potential is introduced for charges \( N_i \) with \([H, N_i] = 0\). At least the symmetry should not change as \( \mu \) does.
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• Moreover, symmetry groups different at each \( \mu \). Recall we wish to investigate \( \langle \bar{\psi}\psi \rangle(a\mu) \) to explore if chiral symmetry is restored.

• The symmetry group remains same at each \( T \) with \( \mu = 0 \)

\( \Rightarrow \langle \bar{\psi}\psi \rangle(am = 0, T) \) is an order parameter for the chiral transition.
Our Results

• We investigated thermodynamics of free overlap and domain wall fermions with an aim to examine the continuum limit analytically and numerically.

• Analytically, we prove the absence of $\mu^2$-divergences for general $K$ and $L$. Our numerical results were for tuning the irrelevant parameter $M$ to obtain small deviations from continuum limit on coarse lattices.
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• Energy density and pressure can be obtained from $\ln Z = \ln \det D_{ov}$ by taking $T$ and $V$, or equivalently $a_4$ and $a$, partial derivatives.
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- Energy density and pressure can be obtained from $\ln Z = \ln \det D_{ov}$ by taking $T$ and $V$, or equivalently $a_4$ and $a$, partial derivatives.

- Dirac operator is diagonal in momentum space. Use its eigenvalues to compute $Z$:
  \[ \lambda_\pm = 1 - \left[ \text{sgn} \left( \sqrt{h^2 + h_5^2} \right) h_5 \pm i \sqrt{h^2} \right] / \sqrt{h^2 + h_5^2}, \]
  with
  \[ h_i = -\sin a p_i, \quad i = 1, 2 \text{ and } 3, \quad h_4 = -a \sin(a_4 p_4) / a_4 \text{ and } \]
  \[ h_5 = M - \sum_{i=1}^{3} [1 - \cos(a p_i)] - a[1 - \cos(a_4 p_4)] / a_4. \]
• Easy to show that $\epsilon = 3P$ for all $a$ and $a_4$. 
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• Hiding $p_i$-dependence in terms of known functions $g$, $d$ and $f$, the energy density on an $N^3 \times N_T$ lattice is found to be

$$\epsilon a^4 = \frac{2}{N^3 N_T} \sum_{p_i,n} F(\omega_n) = \frac{2}{N^3 N_T} \sum_{p_i,n} \left[ (g + \cos \omega_n) + \sqrt{d + 2g \cos \omega_n} \right]$$

$$\times \left[ \frac{(1 - \cos \omega_n)}{d + 2g \cos \omega_n} + \frac{\sin^2 \omega_n (g + \cos \omega_n)}{(d + 2g \cos \omega_n)(f + \sin^2 \omega_n)} \right](4)$$

where $\omega_n$ are the Matsubara frequencies.
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• Can be evaluated using the standard contour technique or numerically.
Analytic Evaluation: $\mu = 0$. 
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- Poles at $\omega = \pm i \sinh^{-1} \sqrt{f}$
- and Poles (branch points) at $\pm i \cosh^{-1} \frac{d}{2g}$. 

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- Poles at $\omega = \pm i \sinh^{-1} \sqrt{f}$ and Poles (branch points) at $\pm i \cosh^{-1} \frac{d}{2g}$.

- Evaluating integrals, $\epsilon a^4 = 4N^{-3} \sum_{p_j} \left[ \sqrt{f/(1 + f)} \left[ \exp(N_T \sinh^{-1} \sqrt{f}) + 1 \right]^{-1} + \epsilon_3 + \epsilon_4 \right]$, where $f = \sum_i \sin^2(a p_i)$. 

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- Can be seen to go to \( \epsilon_{SB} \) as \( a \to 0 \) for all \( M \).
More Details: $T = 0, \mu \neq 0$

- Defining $K(\mu) + L(\mu) = 2R \cosh \theta$ and $K(\mu) - L(\mu) = 2R \sinh \theta$, the same treatment as above goes through by substituting $\sin \omega_n \rightarrow R \sin (\omega_n - i\theta)$ and $\cos \omega_n \rightarrow R \cos (\omega_n - i\theta)$. 
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- Energy density is also functionally the same with \( F(1, \omega_n) \rightarrow F(R, \omega_n - i\theta) \).

- Additional observable, number density: Has the same pole structure so similar computation.
Divergence Cancellation at $T = 0, \mu \neq 0$

- Doing the contour integral, the energy density turns out to be:

$$\epsilon a^4 = (\pi N^3)^{-1} \sum_{p_j} [2\pi \text{Res} \, F(R, \omega) \Theta (K(a\mu) - L(a\mu) - 2\sqrt{f})$$

$$+ \int_{-\pi}^{\pi} F(R, \omega) d\omega - \int_{-\pi}^{\pi} F(1, \omega) d\omega].$$
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- $R = K(a\mu) \cdot L(a\mu) = 1$ ensures cancellation of the last two terms and the canonical result in the continuum limit $a \rightarrow 0$.

- If $R \neq 1$, one has a $\mu^2$ divergence in the continuum limit as well as violation of Fermi surface since $\epsilon \neq 0$ for any $\mu$. 
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- $K$ and $L$ should be such that $K(a\mu) - L(a\mu) = 2a \mu + O(a^3)$ with $K(0) = 1 = L(0)$. 
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- $K$ and $L$ should be such that $K(a\mu) - L(a\mu) = 2a \mu + \mathcal{O}(a^3)$ with $K(0) = 1 = L(0)$.

- Similar derivation goes through for Domain Wall Fermions ($a_5 = 1$) as well.
Numerical Evaluation

◊ Two Observables: \( \Delta \epsilon(\mu, T) = \epsilon(\mu, T) - \epsilon(0, T) \) and Susceptibility, \( \sim \partial^2 \ln Z / \partial \mu^2 \). Done for both Overlap and Domain Wall Fermions \((a_5 = 1)\).
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♦ For odd \( N_T \) and large enough \( \mu \) the sign function is undefined as an eigenvalue becomes pure imaginary (Overlap).
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◊ Former computed for two $r = \mu / T = 0.5$ and $0.8$ while latter for $\mu = 0$. 
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Overlap Fermions

Domain Wall Fermions

❤️ Susceptibility too behaves the same way as the energy density.
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\[ 1.50 \leq M \leq 1.60 \] seems optimal, with 2-3 \% deviations already for \( N_T = 12 \) for overlap, while \( 1.40 \leq M \leq 1.50 \) seems optimal for Domain Wall Fermions, with similar deviations for \( N_T = 12 \).
$\chi(0)/\chi_{SB}$ vs. $N_T$ for $\zeta = 5$. 

- $M=1.30$ represented by $+$ symbols.
- $M=1.55$ represented by $x$ symbols.
- $M=1.60$ represented by $*$ symbols.
- $M=1.65$ represented by $+$ symbols.
- $M=1.70$ represented by $v$ symbols.
- $M=1.75$ represented by $o$ symbols.
- $M=1.85$ represented by $\triangle$ symbols.

Summary

- Exact chiral symmetry without violation of flavour symmetry important for many studies on lattice, especially for the critical point and the QCD phase diagram in $\mu$–$T$ plane.

- Overlap and Domain wall fermions lose their chiral invariance on introduction of chemical potential in the Bloch-Wettig method and its generalizations.
Summary

- Exact chiral symmetry without violation of flavour symmetry important for many studies on lattice, especially for the critical point and the QCD phase diagram in $\mu$–$T$ plane.

- Overlap and Domain wall fermions lose their chiral invariance on introduction of chemical potential in the Bloch-Wettig method and its generalizations.

- However, no $\mu^2$-divergence exists in the continuum limit for both Overlap and Domain Wall Fermions for B & W and an associated general class of functions $K(\mu)$ and $L(\mu)$ with $K(\mu) \cdot L(\mu) = 1$.

- For the choice of $1.5 \leq M \leq 1.6$ ($1.4 \leq M \leq 1.5$), both the energy density and the quark number susceptibility at $\mu = 0$ exhibited the smallest deviations from the ideal gas limit for $N_T \geq 12$ for Overlap (Domain Wall) Fermions.
Numerical Evaluation

♣ Zero temperature contribution: as $N_T \to \infty$, $\omega$ sum becomes integral which we estimated numerically.
♣ Continuum limit by holding $\zeta = N/N_T = LT$ fixed and increasing $N_T$. 
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![Graph showing data points and lines for different values of \( \zeta \).]
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Approach to SB-Limit
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$\frac{\varepsilon}{\varepsilon_{SB}}$ vs. $\frac{1}{N_T^2}$

$\zeta = 5$

$p/p_{SB}$ vs. $(\pi/N_T)^2$

$N_T$

1-link actions
- Wilson
- staggered
- overlap

$O(1/N_T^2)$

Banerjee, Gavai & Sharma, arXiv:0803.3925

Hegde, Karsch, Laermann & Shcheredin, arXiv:0801.4883

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Approach to SB-Limit

-love

Results for \( M = 1 \) agree with Hegde et al. (free energy); Smaller corrections than for Staggered or Wilson fermions.
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Domain Wall Fermions

Rajiv V. Gavai and Sayantan Sharma, in preparation.
Domain Wall Fermions

\[ \epsilon / \epsilon_{SB} \]

\( \frac{1}{N_T^2} \)

\( M=1.55, \zeta = 4 \)

\( L_5 = 14, \zeta = 4 \)

\( L_5 \geq 14 \) seems to be large enough to get \( L_5 \)-independent results.

Rajiv V. Gavai and Sayantan Sharma, in preparation.
Domain Wall Fermions

\[ \frac{\varepsilon}{\varepsilon_{SB}} \]

\[ \frac{\varepsilon}{\varepsilon_{SB}} \]

\[ \frac{1}{NT} \]

\[ L_5 = 14, \zeta = 4 \]

\[ M = 1.00, 1.45, 1.50, 1.55, 1.60, 1.65 \]

\[ L_5 \geq 14 \] seems to be large enough to get \( L_5 \)-independent results.

\[ \text{Optimal range again seems to be } 1.50 \leq M \leq 1.60; \ M = 1.9 \text{ used by Chen et al. (PRD 2001) in their study of order parameters of FTQCD.} \]