Calculation of $B^0 - \bar{B}^0$ Mixing Matrix Elements in 2+1 Lattice QCD

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Outline

• $B$ mixing experimental status and motivation for calculation
• Simulation Details: actions and parameters
• Correlators and fitting
• Perturbative matching
• Chiral Extrapolations
• Results:
  For $\xi$ only
• Outlook
Status of Experimental Measurement

\[ \Delta M_s = 17.77 \pm 0.10 \text{(stat.)} \pm 0.07 \text{(syst.)} \text{ps}^{-1} (CDF \ 2006) \]

\[ \Delta M_d = 0.507 \pm 0.005 \text{ps}^{-1} (PDG\ 2007 \ Average) \]

\[ \sigma \Delta m_s, \sigma \Delta m_d < 1\% \]

\[ |V_{td}/V_{ts}| = \xi \sqrt{\Delta m_d/m_{B_s}} = 0.2060 \pm 0.0007 (\text{exp.})^{+0.0081}_{-0.0060} (\text{theo.}) \]

Theoretical error is from \[ \xi = \frac{f_{B_s} \sqrt{B_{B_s}}}{f_{B_d} \sqrt{B_{B_d}}} = 1.21^{+0.047}_{-0.035}, \]

\[ \sigma_\xi \approx 4\% \]

\[ \xi \] is derived by combining calculations from –

\( f_{B_q} : n_f = 2 + 1, \) HPQCD

\( B_{B_q} : n_f = 2, \) JLQCD (quenched strange)
**B Mixing Hadronic Matrix Element**

\[ \Delta M_q = \frac{G_F^2 M_W^2}{6\pi^2} |V_{tq}^* V_{tb}|^2 \eta_2^B S_0(x_t) M_{B_q}^2 f_{B_q}^2 \hat{B}_{B_q}, \ q = d, s \]

- \( x_t = \frac{m_t^2}{M_W^2} \), \( \eta_2^B \) is a perturbative QCD correction factor and \( S_0(x_t) \) is the Inami-Lim function.

- For \( |V_{tq}^* V_{tb}| \) we need the hadronic matrix element:
  
  \[ -\langle \bar{B}_q|Q_1^1|B_q\rangle = \frac{8}{3} M_{B_q} f_{B_q}^2 B_{B_q} \]
  
  \[ \rightarrow Q_1^1 = \bar{b}\gamma_\mu(1 - \gamma_5)q\bar{b}\gamma_\mu(1 - \gamma_5)q. \]

- \( \frac{|V_{td}|}{V_{ts}} = \frac{f_{B_s}}{f_{B_d}} \sqrt{B_{B_s}} \sqrt{\frac{\Delta M_d M_{B_s}}{\Delta M_s M_{B_d}}} = \xi \sqrt{\frac{\Delta m_d}{\Delta m_s} \frac{m_{B_s}}{m_{B_d}}} \)

  - \( \xi \) has smaller statistical and systematic uncertainties (statistical errors reduced, scale uncertainty reduced etc.)
  - \( |\frac{V_{td}}{V_{ts}}| \) constrains the CKM unitarity triangle (determines the length of one side).
Simulation Details: Configurations and Actions

<table>
<thead>
<tr>
<th>Particle</th>
<th>Action</th>
<th>Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gluons</td>
<td>MILC</td>
<td>$\mathcal{O}(a^2\alpha_s, a^4)$</td>
</tr>
<tr>
<td>Light quarks</td>
<td>Asqtad</td>
<td>$\mathcal{O}(a^2\alpha_s, a^4)$</td>
</tr>
<tr>
<td>Heavy quarks</td>
<td>Fermilab</td>
<td>$\mathcal{O}(\alpha_s\Lambda_{QCD}/M, (\Lambda_{QCD}/M)^2)$</td>
</tr>
</tbody>
</table>

### Details ###

- **Gluons-** MILC 2+1 gauge configurations (Symanzik and Tadpole Improved).
- **Light quarks-** sea quarks: \{u, d, s\} and valence quarks: q.
- **Heavy Quark-** b quark, simulated using clover action with Fermilab Interpretation. Heavy quark “rotated” at source to remove $\mathcal{O}(\Lambda_{QCD}/M)$ errors in $Q_q^1$ and exponentially smeared at sink to improve ground state overlap.
Simulation Details: Lattice Spacings and Masses Used

- Calculation done on 2 lattice spacings.
  - 6 light sea quark masses, lightest $m_{\pi,\text{sea}} \sim 250$ MeV.
  - 6 light valence quark masses, lightest $m_{\pi,\text{val}} \sim 240$ MeV.
- 4 time sources each.

<table>
<thead>
<tr>
<th>$am_l/am_s$</th>
<th>$am_v$</th>
<th>$N_{\text{configs}}$</th>
</tr>
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<tbody>
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<td>534</td>
</tr>
</tbody>
</table>
Correlators Used in Calculation

- Simultaneous fits to two-point and three-point correlator to extract mixing parameters.
- $Q_1^q$ location is fixed with $\bar{B}$ and $B$ positions varying → use same propagator for backward and forward moving quarks.

- Three-point Correlator:
  \[
  C_{Q_1^q} (t_1, t_2) = \sum_{x_1, x_2} \langle \bar{B}_q (t_1, x_1) | Q_1^q (0) | B_q (t_2, x_2) \rangle = \sum_{i,j} ((-1)^{t_1+1})^i ((-1)^{t_2+1})^j \frac{Z_i Z_j O_{ij}}{(2 E_i)(2 E_j)} e^{-E_i t_1 - E_j t_2},
  \]
  \[
  O_{00} = \langle \bar{B}_q | Q_1^q | B_q \rangle = \frac{8}{3} M_{B_q}^2 f_{B_q}^2 B_{B_q}.
  \]

- Two-point Correlators:
  To extract $f_{B_q} \sqrt{M_{B_q} B_{B_q}}$:
  \[
  C_{PS}^q (t) = \sum_{\vec{x}} \langle B_q (t, \vec{x}) | \bar{q} (0) \gamma_5 b (0) \rangle = \sum_i ((-1)^{t+1})^i \frac{Z_i^2}{2 E_i} e^{-E_i t}.
  \]
  To extract $B_{B_q}$:
  \[
  C_{A_4}^q (t) = \sum_{\vec{x}} \langle B_q (t, \vec{x}) | \bar{q} \gamma_0 \gamma_5 b (0) \rangle = \sum_i ((-1)^{t+1})^i \frac{A_{4i} Z_i}{2 E_i} e^{-E_i t},
  A_{40} = f_{B_q} M_{B_q}.
  \]
Example Correlator Fit: 2-D Three-point Correlator

Placing $Q_q^1$ at origin allows fit to be done 2 dimensionally with only two quark inversions, over $t_1$ and $t_2$. (Statistical errors on data not shown)
Data for $\langle \bar{B}_q | Q^1_q | B_q \rangle$: $\beta_q = f_{B_q} \sqrt{M_{B_q} B_{B_q}}$

- Sea mass dependence is mild.
- Lattice spacing dependence is obvious but not extreme.
- Statistical errors vary between 2-5%.

$\beta_q r_{1\frac{3}{2}} = f_{B_q} \sqrt{M_{B_q} B_{B_q}} r_{1\frac{3}{2}}$: Valence mass
Perturbative Matching

Matching coefficient calculation is nearly complete: only preliminary results for the coefficients at present.

- Lattice and continuum matrix elements have different regularizations, must match to obtain physical results.

- \( Q_1^1 \) mixes with \( Q_2^2 = \bar{b}(1 - \gamma_5)s\bar{b}(1 - \gamma_5)s \) at one-loop.
  
  \[ \langle \bar{B}_q | Q_2^2 | B_q \rangle \]  
  
  calculation analogous to \( \langle \bar{B}_q | Q_1^1 | B_q \rangle \). Built from same propagators so cheap to calculate.

-  
  
  \[ \langle \bar{B}_q | Q_1^1 | B_q \rangle^{cont.}(\mu) = (1 + \alpha_S C_1(\mu))\langle \bar{B}_q | Q_1^1 | B_q \rangle^{lat.} + \alpha_S C_2(\mu)\langle \bar{B}_q | Q_2^2 | B_q \rangle^{lat.} \]

- \( \mu \rightarrow m_b \).

- \( \alpha_S = \alpha_V(q^*) \), \( \alpha_V \) determined from lattice measurement (in this case small Wilson loops) and \( q^* \) from typical gluon momentum in loops.
Data for $\langle \bar{B}_q | Q^2 | B_q \rangle$: $\beta S_q = f_{B_q} \sqrt{M_{B_q} B S_{B_q}}$

$\beta S_q r_1^{3/2} = f_{B_q} \sqrt{M_{B_q} B S_{B_q}} r_1^{3/2}$: Valence mass

- Value for $\beta S_q r_1^{3/2}$
- Valence mass
- $r_1^2 m_{qq}$ valence pion mass

Legend:
- 005/050
- 007/050
- 010/050
- 020/050
- 0062/031
- 0124/031
Rooted Staggered Chiral Perturbation Theory (rS$\chi$PT)

-Determine light quark mass dependence using partially quenched data and extrapolate to continuum and physical $d$ mass, interpolate to physical $s$ mass.

- Heavy-Light staggered chiral theory incorporates $O(a^2)$ taste violations.

- $M^2_{ij,\Xi} = \mu(m_i + m_j) + a^2 \Delta_{\Xi}$. 
  $m_i, m_j$ are quark masses, $\Delta_{\Xi}$ is the taste splitting.

- $\langle \bar{B}_q | Q^q_1 | B_q \rangle_{QCD} = \frac{8}{3} m_{Bq}^2 f_{Bq}^2 B_q = m_{Bq} \langle \bar{B}_q | Q^q_1 | B_q \rangle_{HQET} = m_{Bq} \beta [1 + (NLO \ logs) + L_v m_q + L_s (2m_L + m_H) + L_a a^2] + NNLO(\text{analytic})$.

- Central value fit uses all NNLO analytic terms.
  - Light quark discretization and systematic fit errors estimated by including/excluding $NNLO$ terms in fit.

- To extrapolate: $a \rightarrow 0$, $m_L \rightarrow \frac{m_u + m_d}{2}$, $m_H \rightarrow m_s$, and $m_q \rightarrow m_d$ or $m_s$

- $O(a^2)$ taste violations/light quark discretization errors removed.
Chiral Fits-Example: \( \beta_q = f_{Bq} \sqrt{M_{Bq} B_{Bq}} \)

- Fits are done simultaneously to all 6 sea and valence quark masses (36 mass points).
- Data points along fit lines are uncorrelated: sea pion \( m_{LL}^2 = \mu (m_L + m_L) \).
- Continuum/mass extrapolation not shown.

\[
\text{beta}=r_1^{3/2}f_B \sqrt{M_B B_B}: \text{Chiral fit to beta}_d \text{ and beta}_s, \text{ NLO+NNLO(analytic)}
\]

\[\chi^2/\text{d.o.f.}=0.28\]

\[
\text{beta}(m_L,m_L,m_S) \text{ coarse}
\]

\[
\text{beta}(m_S,m_L,m_S) \text{ coarse}
\]

\[
\text{beta}(m_L,m_L,m_S) \text{ fine}
\]

\[
\text{beta}(m_S,m_L,m_S) \text{ fine}
\]
Chiral Fits-Extrapolation for $\xi$

Fit to and Extrapolate $\xi' = f_{B_s} \sqrt{M_{B_s} B_{B_s}} / f_{B_d} \sqrt{M_{B_d} B_{B_d}}$

- Statistical errors reduced
- Many systematic errors cancel. (Perturbative matching corrections are negligible $< 1\%$.)
- Many parameters in chiral fit cancel (simplifies fit and Ansatz)
- Phenomenologically useful quantity.
Chiral Fits-Extrapolation for $\xi$ cont.: $m_{\text{sea}}$ plane

- Fits are done simultaneously to all 6 sea and valence quark masses (36 mass points).
- Errors on extrapolation point are statistical only.

$\chi^2$/d.o.f. = 0.22
Fits are done simultaneously to all 6 sea and valence quark masses (36 mass points).

\[ \xi': \text{Valence plane, NLO+NNLO(analytic)} \]

\[ r_1^2 m_{qq}^2 \text{ valence pion mass} \]
## Results and Uncertainties

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\xi$</th>
<th>$\beta_d$</th>
<th>$\beta_s$</th>
</tr>
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<tbody>
<tr>
<td><strong>Central Value</strong></td>
<td>1.211</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Source of Uncertainty</strong></td>
<td>% Error</td>
<td></td>
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</tr>
<tr>
<td>Statistical</td>
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<tr>
<td>Higher Order Matching</td>
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<td>4</td>
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<tr>
<td>Heavy Quark Discretization</td>
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<td>3.5</td>
<td>3.5</td>
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<tr>
<td>Chiral extrap. errors</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Light Quark Discretization + Chiral Fits</td>
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<td>4.3</td>
<td>1.3</td>
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<tr>
<td>scale uncertainty ($r_1$)</td>
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</tr>
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<td>$g_{BB*\pi}$</td>
<td>0.8</td>
<td>1.4</td>
<td>2.3</td>
</tr>
<tr>
<td>input parameters: $\hat{m}, m_d, m_s$</td>
<td>0.7</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>estimated from FNAL-MILC $f_B$</td>
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<tr>
<td>$\kappa_b$</td>
<td>&lt;0.1</td>
<td>1.1</td>
<td>1.1</td>
</tr>
<tr>
<td>finite volume</td>
<td>0.6</td>
<td>0.6</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>Total Systematic</strong></td>
<td>2.8</td>
<td>7.8</td>
<td>6.8</td>
</tr>
</tbody>
</table>
Comparison of $\xi$ and $f_{B_s}/f_{B_d}$

- $\frac{f_{B_s}}{f_{B_d}}$ determined from separate analysis on 2+1 MILC lattices.
- Ratio $\frac{B_{B_s}}{B_{B_d}} = 1.014(0.015)$ determined from separate correlator and chiral fits.
- $\frac{B_{B_s}}{B_{B_d}}$ is preliminary and uncertainty is statistical only.
- Statistical and systematic uncertainty of other parameters are added in quadrature.

<table>
<thead>
<tr>
<th>$\frac{f_{B_s}}{f_{B_d}} \times \sqrt{\frac{B_{B_s}}{B_{B_d}}}$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.243(0.037) \times 1.007(0.007) = 1.252(0.038)$</td>
<td>$1.211(0.045)$</td>
</tr>
</tbody>
</table>
Summary & Outlook

- The calculation of $\xi, f_{B_d} \sqrt{M_{B_d} B_{B_d}}, \text{and} f_{B_s} \sqrt{M_{B_s} B_{B_s}}$ is nearly complete → likely with total uncertainties of $\sim 4\%$, $\sim 9\%$, and $\sim 8\%$ respectively.

- Increase statistics
  Additional Configurations: $N_{\text{conf}} \sim 600 \rightarrow \sim 2000$.
  Time sources: $N_{ts} = 4 \rightarrow 16$ spatial origin randomized to reduce correlations.
  3-5% correlator errors → 1-2%.

- Matching: Partial non-perturbative determination of coefficients, $4\% \rightarrow 2\%$.

- Super-fine lattice run ($a = 0.06$ fm).

- Most aspects of chiral fits will be improved by smaller correlator errors and super-fine lattice addition.

- Additional mixing matrix elements that arise in extensions to the Standard Model are straightforward to calculate (no additional propagator inversions needed).