Multi-meson systems in lattice QCD

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arXiv:0803.2728
arXiv:0807.1856 + ...
• n<14 pion and kaon systems in QCD

• Two and three body interactions

• Investigate pion and kaon condensates
Why?

• Poor man’s nuclear physics
  • Many body physics
  • Exponentially bad signal/noise ameliorated
• Testing ground
• Meson condensates
• EOS in neutron stars (kaons [Kaplan&Nelson])
Scattering states

• Maiani-Testa: extracting multi-hadron S-matrix elements from Euclidean lattice calculations of corresponding Green functions is impossible

• Lüscher: volume dependence of two-particle energy levels $\Rightarrow$ scattering phase-shift up to inelastic threshold

• Exact relation provided $r \ll L$

• Used for $\pi\pi$, $KK$, $NN$, $\Lambda N$

\[ p \cot \delta(p) = \frac{1}{\pi L} S \left[ \left( \frac{pL}{2\pi} \right)^2 \right] \]

\[ S[x] = \sum_{|\vec{j}|<\Lambda} \frac{1}{|\vec{j}|^2 - x} - 4\pi \Lambda \]
Multi-boson energies
[Beane, WD & Savage; WD & Savage ]

- Large volume expansion of GS energy of $n$ meson system to $1/L^7$
- $2 \& 3$ body interactions ($N$ body: $L^{-3(N-1)}$)
- Derived in low energy EFT, but relativistic
- $n=2$: reproduces expansion of Lüscher
- Can include higher PW, higher body, excited states

[See talk of T Luu for fermion systems]
Multi-boson energies

[WD+Savage arXiv:0801.0763]

- $1/L^7$: Relativistic effects

\[
\Delta E_n = \frac{4\pi \bar{a}}{M L^3} n C_2 \left\{ 1 - \left( \frac{\bar{a}}{\pi L} \right) \mathcal{I} + \left( \frac{\bar{a}}{\pi L} \right)^2 \left[ \mathcal{I}^2 + (2n - 5) \mathcal{J} \right]
- \left( \frac{\bar{a}}{\pi L} \right)^3 \left[ \mathcal{I}^3 + (2n - 7) \mathcal{I} \mathcal{J} + (5n^2 - 41n + 63) \mathcal{K} \right]
+ \left( \frac{\bar{a}}{\pi L} \right)^4 \left[ \mathcal{I}^4 - 6\mathcal{I}^2 \mathcal{J} + (4 + n - n^2) \mathcal{J}^2 + 4(27 - 15n + n^2) \mathcal{I} \mathcal{K}
+ (14n^3 - 227n^2 + 919n - 1043) \mathcal{L} + 16(n - 2) (\mathcal{I}_0 + n \mathcal{I}_1) \right] \right\}
+ n C_3 \frac{1}{L^6} \hat{\eta}_3^L
+ n C_3 \frac{6\pi \bar{a}^3}{M^3 L^7} (n + 3) \mathcal{I}
+ \mathcal{O} \left( L^{-8} \right)
\]

$\bar{a} = a + \frac{2\pi}{L^3} a^3 r$

$\hat{\eta}_3^L = \bar{\eta}_3^L \left[ 1 - \frac{6\bar{a}}{\pi L} \mathcal{I} \right] + \frac{72\pi \bar{a}^4 r}{ML} \mathcal{I}$

$\mathcal{I}, \mathcal{J}, \ldots$: geometric constants
Many mesons correlators

- Eg: $n \pi^+$ correlator ($m_u = m_d$)

$$C_n(t) = \langle 0 | \left[ \sum_x \bar{d}\gamma_5 u(x, t) \bar{u}\gamma_5 d(0, 0) \right]^n | 0 \rangle$$

$$\rightarrow Ae^{-E_nt}$$

- $n!^2$ Wick contractions: $(13!) \sim 10^{19}$

$$C_3(t) = \text{tr} [\Pi]^3 - 3 \text{tr} [\Pi] \text{tr} [\Pi^2] + 2 \text{tr} [\Pi^3]$$

$$\Pi = \sum_x \gamma_5 S(x, t; 0)\gamma_5 S^\dagger(x, t; 0)$$

- Maximal isospin states: one quark propagator
Lattice details

- Calculations use MILC gauge configurations
  - \( L = 2.5 \text{ fm}, a = 0.12 \text{ fm}, \text{rooted staggered} \)
  - also \( L = 3.5 \text{ fm} \) and \( a = 0.09 \text{ fm} \)
- NPLQCD: domain-wall quark propagators
  - \( m_\pi \sim 291, 318, 352, 358, 491, 591 \text{ MeV} \)
  - 24 propagators / lattice in best case
- \( l_z = n = 1, \ldots, 12 \) pion and \((S = n)\) kaon systems
n-meson energies

- Effective energy plots: $\log\left[\frac{C_n(t)}{C_n(t+1)}\right]$
n-meson energies

- Effective energy plots: \( \log[C_n(t)/C_n(t+1)] \)

\[ m_\pi = 352 \text{ MeV} \]
n-meson energies

- Clean signals for $n=1,\ldots,12$

\[ \delta E_n/E_n \]

- $m_\pi = 291$ MeV
- $m_\pi = 352$ MeV
- $m_\pi = 491$ MeV
- $m_\pi = 591$ MeV

$\sim 8$ GeV!
$n$ correlations

- Fit effective energies to extract two parameters: $a$, $\eta_3$ from $1/L$ expansion

- Use 12 eff. energies in $n$-$t$-correlated analysis
  - Large correlation matrix: correlated $\chi^2$
  - Reduces uncertainties as $n$ pion correlators “explore more of the lattice”
n correlations

Three-body

Two body
$2\pi^+$ and $2K^-$ interaction

curves: Weinberg

[See talk of M Savage]
3π⁺ and 3K⁻ interaction

Naïve dimension analysis: 1
$3\pi^+$ and $3K^-$ interaction

Naïve dimension analysis: 1
Chemical Potentials

- Chemical potential
  \[ \mu = \left. \frac{d E}{d n} \right|_{V_{\text{const}}} \]
- Analytic form: EOS
- Numerically using finite difference
Isospin Chemical Potential

\( m_\pi \sim 291 \text{ MeV} \)
\( m_K \sim 580 \text{ MeV} \)

\( \frac{\mu_{\pi^-}}{m_\pi} \)

\( (2.5 \text{ fm})^3 \rho_{\pi^-} \)

\( m_\pi \sim 352 \text{ MeV} \)
\( m_K \sim 597 \text{ MeV} \)

\( \frac{\mu_{\pi^-}}{m_\pi} \)

\( (2.5 \text{ fm})^3 \rho_{\pi^-} \)

\( m_\pi \sim 491 \text{ MeV} \)
\( m_K \sim 640 \text{ MeV} \)

\( \frac{\mu_{\pi^-}}{m_\pi} \)

\( (2.5 \text{ fm})^3 \rho_{\pi^-} \)

\( m_\pi \sim 591 \text{ MeV} \)
\( m_K \sim 678 \text{ MeV} \)

\( \frac{\mu_{\pi^-}}{m_\pi} \)

\( (2.5 \text{ fm})^3 \rho_{\pi^-} \)

\( \text{2+3 body fit} \)
\( \text{No 3 body} \)
\( \text{LO}\chi\text{PT} \)
Kaon Chemical Potential

\[ \frac{\mu_{K^-}}{m_K} - 1 \]

\( m_\pi \approx 291 \text{ MeV} \)
\( m_K \approx 580 \text{ MeV} \)

\( m_\pi \approx 352 \text{ MeV} \)
\( m_K \approx 597 \text{ MeV} \)
\( m_\pi \approx 491 \text{ MeV} \)
\( m_K \approx 640 \text{ MeV} \)
\( m_\pi \approx 591 \text{ MeV} \)
\( m_K \approx 678 \text{ MeV} \)

\( \rho_{K^-} \)

\( \text{LO} \chi PT \)

\( 2+3 \text{ body fit} \)
\( \text{No 3 body} \)
Summary

• Explored $n<14$ meson systems
• Clean signals for all $n$
• Two and three-body interactions
• $\pi^-$ and $K^-$ chemical potentials
• Important contribution from $\pi\pi\pi$
• Results consistent with $\text{LO}\chi\text{PT}$: Kaplan/Nelson analysis
Numerical precision

- Double precision is not enough: not gauge invariant

- Need arbitrary precision contraction code

- Propagator precision??
Numerical precision

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Numerical precision

• Double precision is not enough: not gauge invariant

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• Propagator precision??
n=13 & matrix identities

- Pauli principle: $C_{13}(t)$ correlator vanishes
- Matrix Identity: vanishes $\forall$ 12x12 matrices
- Numerically
\[ C_{13}(t) = T_1^{13} - 78T_2T_1^{11} + 572T_3T_1^{10} + 2145T_2^2T_1^9 - 4290T_4T_1^9 - 25740T_2T_3T_1^8 + 30888T_5T_1^8 \]

\[-25740T_2^3T_1^7 + 68640T_3^2T_1^7 + 154440T_2T_4T_1^7 - 205920T_6T_1^7 + 360360T_2^2T_3T_1^6 \]

\[-720720T_3T_4T_1^6 - 864864T_2T_5T_1^6 + 1235520T_7T_1^6 + 135135T_4^4T_1^5 - 1441440T_2T_3^2T_1^5 \]

\[+1621620T_4^3T_1^5 - 1621620T_2^2T_4T_1^5 + 3459456T_3T_5T_1^5 + 4324320T_2T_6T_1^5 - 6486480T_8T_1^5 \]

\[+1601600T_3^3T_4T_1^4 - 1801800T_3^2T_7T_1^4 + 10810800T_2T_3T_4T_1^4 + 6486480T_2^2T_5T_1^4 - 12972960T_4T_5T_1^4 \]

\[ -14414400T_3T_6T_1^4 - 18532800T_2T_7T_1^4 + 28828800T_9T_1^4 - 270270T_2^5T_1^3 + 7207200T_2^2T_3^2T_1^3 \]

\[-16216200T_2^2T_4^2T_1^3 + 20756736T_5^2T_1^3 + 5405400T_2^3T_4T_1^3 - 14414400T_3^2T_4^2T_1^3 - 34594560T_2T_3T_5T_1^3 \]

\[-21621600T_2^2T_6T_1^3 + 43243200T_4T_6T_1^3 + 49420800T_3T_7T_1^3 + 6486480T_2T_8T_1^3 - 103783680T_1^3T_1^3 \]

\[-9609600T_2^3T_3^2T_1^2 + 32432400T_3T_4^2T_1^2 + 2702700T_4^2T_3^2T_1^2 - 32432400T_2^2T_3^2T_4T_1^2 \]

\[-12972960T_2^3T_5T_1^2 + 34594560T_3^2T_5T_1^2 + 77837760T_2T_4T_5T_1^2 + 86464600T_2T_3T_6T_1^2 \]

\[-103783680T_5T_6T_1^2 + 55598400T_2^2T_7T_1^2 - 111196800T_4T_7T_1^2 - 129729600T_3T_8T_1^2 \]

\[-172972800T_2T_9T_1^2 + 28304600T_{11}T_1^2 + 135135T_6^2T_1 + 3203200T_4^4T_1 - 16216200T_4^3T_1 \]

\[-7207200T_2^3T_3^2T_1 + 24324300T_2^2T_4^2T_1 - 62270208T_2^2T_5T_1 + 86464600T_6^2T_1 \]

\[-4054050T_4^2T_4T_1 + 43243200T_2^2T_4^2T_1 + 51891840T_2^2T_3T_5T_1 - 103783680T_3T_4T_5T_1 \]

\[+21621600T_2^3T_6T_1 - 57657600T_3^2T_6T_1 - 129729600T_2T_4T_6T_1 - 148262400T_2T_3T_7T_1 \]

\[+177914880T_5T_7T_1 - 97297200T_2^2T_8T_1 + 194594400T_4T_8T_1 + 230630400T_3T_9T_1 \]

\[+311351040T_2T_1T_1 - 518918400T_{12}T_1 + 4804800T_2^2T_3T_1 - 32432400T_2T_3^2T_1 \]

\[+41513472T_3T_2^2T_5 - 540540T_5^3T_3 - 9609600T_3^3T_4 + 10810800T_3^3T_4 \]

\[+3243240T_4^2T_5 - 34594560T_2T_3^2T_5 + 38918880T_4^2T_5 - 38918880T_2^2T_4T_5 \]

\[-43243200T_2^2T_3T_6 + 86486400T_3T_4T_6 + 103783680T_2T_5T_6 - 18532800T_3^2T_7 \]

\[+49420800T_3^2T_7 + 111196800T_2T_4T_7 - 148262400T_6T_7 + 129729600T_2T_3T_8 \]

\[-155675520T_5T_8 + 86486400T_2^2T_9 - 172972800T_4T_9 - 207567360T_3T_{10} \]

\[-283046400T_2T_{11} + 479001600T_3 \]

\[(A13)\]
Pion scattering

$\pi^2 P_{\pi\pi}$

$m_{\pi} \alpha_{\pi\pi}$

$\text{Exact two-body}$

$m=3,\ldots,8$

$\text{LO}$

$\text{NLO}$

$\text{NNLO}$

$\text{N}^3\text{LO}$
Pion scattering

$m_\pi = 291$ MeV

$m_\pi = 352$ MeV

$m_\pi = 491$ MeV

$m_\pi = 591$ MeV
Expansion shows no sign of breakdown
Pion scattering

- Extractions of $m_{\pi}a$ from four orders in L

Lüscher
exact
two-body
Pion scattering

- Extractions of $m_\pi a$ from four orders in $L$
Pion scattering

- Extractions of $m_\pi a$ from four orders in $L$

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**Diagram**: Graph showing the dependence of $m_\pi a$ on $t/b$, with data points and error bars. The graph includes a horizontal line labeled as "exact two-body" and a shaded region indicating the Lüscher NNLO result. The y-axis is labeled $m_\pi a[I=2][NNLO]$, and the x-axis is labeled $t/b$. The graph includes data points for different $n$ values, with $n=9$ highlighted.
\( N^3\text{LO} \)

- Two energies to cancel 3-body: 45 combinations
Three meson interactions

- At 1/L⁶, point-like three-boson interaction must occur [Braaten, Nieto ‘95]
- RGI 3BI: $\eta_3^{(L)}$ physically meaningful
- Depends logarithmically on L
- Naive dimensional-analysis $m_\pi f_\pi^4 \eta_3^{(L)} \sim 1$
- Combinations of energy shifts isolates the RGI interaction
$\pi^+\pi^+\pi^+$ interaction

$m_\pi = 352$ MeV