Approximate forms of the density of states in pure gauge theory

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Content of the talk

• Motivations

• The density of states (definition and properties)

• Numerical study of finite size effects

• Apparent convergence of series expansions

• Conclusions and perspectives

See arXiv:0807.0185 [hep-lat]
Motivations

Problems that can be addressed using the density of states:

• How to combine weak and strong coupling expansions

• Study of finite size effects for small lattices

• Large order behavior of perturbative series

• Location of Fisher’s zeros for large lattices (poster)
Figure 1: Fisher’s zeros from the density of states with a numerical interpolation (left) and a polynomial approximation (right).
The density of states

Focus: $SU(2)$, Wilson’s action, $L^4$ lattice, periodic b. c.

$N_p = 6 \times L^4$ is the number of plaquettes

$Z(\beta)$ is the Laplace transform of $n(S)$, the density of states

$$Z(\beta) = \int_0^{2N_p} dS \ n(S) \ e^{-\beta S},$$

with

$$n(S) = \prod_l \int dU_l \delta(S - \sum_p (1 - (1/N) ReTr(U_p)))$$

$\ln(n(S))$ is a ”color entropy” (extensive).
A $SU(2)$ duality ($g^2 \rightarrow -g^2$ means $S \rightarrow 2N_p - S$)

For cubic lattices with even number of sites in each direction and a gauge group that contains $-1$, it is possible to change $\beta \text{Re} Tr U_p$ into $-\beta \text{Re} Tr U_p$ by a change of variables $U_l \rightarrow -U_l$ on a set of links such that for any plaquette, exactly one link of the set belongs to that plaquette (Li, YM PRD71 016008). This implies

$$Z(-\beta) = e^{2\beta N_p} Z(\beta)$$

$$n(2N_p - S) = n(S)$$

Thanks to this symmetry, we only need to know $n(S)$ for $0 \leq S \leq N_p$. (Note $< S > = N_p$ means $< Tr U_p > = 0$)
The one plaquette case (Li, YM, PRD71 054509)

\[ n_{1pl.}(S) = \frac{2}{\pi} \sqrt{S(2 - S)} \]

\[ n(S) \propto \sqrt{S} \text{ for small } S \text{ implies } Z \propto \beta^{-3/2} \text{ for large } \beta \]

1/\beta corrections can be calculated by expanding the remaining factor \( \sqrt{2 - S} \)

Series with finite radius of convergence \( \rightarrow \) asymptotic series if we integrate over \( S \) from 0 to \( \infty \) (instead of 0 to 2).

It is easier to approximate \( n(S) \) than the corresponding partition function. Does this survive the infinite volume limit?

\( n(S) \) near \( S = 2 \) can be probed by taking \( \beta \rightarrow -\infty \) in agreement with the common wisdom that the large order behavior of weak coupling series can be understood in terms of the behavior at small negative coupling.
**Volume dependence**

\[ f(x, \mathcal{N}_p) \equiv \ln(n(x\mathcal{N}_p, \mathcal{N}_p)) / \mathcal{N}_p \]

The \( SU(2) \) duality symmetry implies that

\[ f(x, \mathcal{N}_p) = f(2 - x, \mathcal{N}_p) \]

The existence of the infinite volume limit requires that

\[ \lim_{\mathcal{N}_p \to \infty} f(x, \mathcal{N}_p) = f(x) , \]

In the same limit, the integral can be evaluated by the saddle point method. The maximization of the integrand requires

\[ f'(x) = \beta \]
Numerical calculation

Figure 2: Results of patching $P_\beta(S)e^{\beta S}$ for $4^4$ and $6^4$. 
Finite Volume Effects

Figure 3: The difference between $\ln(n(S))/N_p$ for $4^4$ and $6^4$. The noise on the right is consistent with our understanding of the volume (in)dependence of the strong coupling expansion.
Volume dependence of the leading log

Figure 4: The difference between $\ln\left(\frac{n(S)}{N_p}\right)$ (left) and $\frac{\ln\left(\frac{n(S)}{N_p}\right)}{\ln\left(\frac{S}{N_p}\right)}$ (right) for $4^4$ and $6^4$. Predicted value of the plateau is -0.0013.
Figure 5: Average plaquette (left) and $\ln(n(S))/\mathcal{N}_p$ (right) compared to weak and strong coupling expansions ($x = S/\mathcal{N}_p$).
Strong coupling expansion

\[ P(\beta) \simeq 1 + \sum_{m=1} a_{2m-1} \beta^{2m-1} \]

(From Balian et al.). With periodic b.c., topologically trivial graphs have volume independent contributions.

\[ f(1 + y) = g(y) \simeq \sum_{m=0} g_{2m} y^{2m} \]

\[ h(y) \equiv g(y) - A(ln(1 - y^2)) \]

In the infinite volume limit, we have \( A = 3/4 \). Expanding

\[ h(y) \simeq \sum_{m=0} h_{2m} y^{2m} \]
Figure 6: Logarithm of the absolute value of $g_{2m}$ and $h_{2m}$
Evidence for finite radius of convergence

Figure 7: Logarithm of the absolute value of the difference between the numerical data and the strong coupling expansion of \( P \) (left) and \( f \) (right) at successive orders. For reference, we also show the numerical errors.
Weak coupling expansion

\[ P(\beta) \simeq \sum_{m=1} b_m \beta^{-m} \]

From Karsch, Heller, Alles et al. + dilogarithm model for order 4 and higher; We assume the behavior

\[ f(x) \simeq A \ln(x) + \sum_{m=0} f_m x^m \]

Using the saddle point, \( \beta \simeq A/x \simeq A/(b_1/\beta) \) At finite volume, the saddle point calculation of \( P \) should be corrected in order to include \( 1/V \) effects \( (V = L^D) \). If we perform the Gaussian integration of the quadratic fluctuations, and use the \( V \) dependent value of \( b_1 \) given below,

\[ A = (3/4) - (5/12)(1/V) \]
This leading coefficient correction, predicts a difference of \(-0.0013 \ln(x)\) for the difference between \(f(x)\) for a \(4^4\) and \(6^4\). A closed form expression can be found using the zero mode contribution (Coste et al.) for \(b_1\). For the case \(N_c = 2\) and \(D = 4\),

\[ b_1 = (3/4)(1 - 1/(3V)) \]

Assuming that \(\partial P / \partial \beta\) has a logarithmic singularity in the complex \(\beta\) plane and integrating (very successful for \(SU(3)\), YM PRD74:096005)

\[ \sum_{m=1} b_m \beta^{-k} \approx C(Li_2(\beta^{-1}/(\beta_m^{-1} + i\Gamma))) + h.c, \]

with

\[ Li_2(x) = \sum_{k=1} x^k / k^2. \]
Figure 8: Logarithm of the absolute value of the difference between the numerical data and the weak coupling expansion of $P$ at successive orders (left) and without the zero mode (right).
Figure 9: Numerical value of $f(x)$ compared to the weak coupling expansion at successive orders (left). Logarithm of the absolute value of the difference between the numerical data and the weak coupling expansion of $f$ at successive orders (right).
Expansion in Legendre polynomials

\[ h(y) \equiv g(y) - A(\ln(1 - y^2)) . \]

\[ f(1 + y) = g(y) \simeq \sum_{m=0} g_{2m} y^{2m} \]

\[ h(y) = \sum_{m=0} q_{2m} P_{2m}(y) \]

Coefficients decay exponentially.

Approximations improve uniformly with the order.
Figure 10: Legendre polynomial coefficients $q_{2m}$ with the three methods described in the text.
Figure 11: $h(y)$ together with the expansion in Legendre polynomials up to order 20 (left). Logarithm of the absolute value of the difference between the numerical data for $h(y)$ and expansions in Legendre polynomials at successive orders (right). $y = S/N_p - 1 = - \sum_p TrU_p/N_p$
Figure 12: $P$ together with the expansion in Legendre polynomials up to order 20 (left). Logarithm of the absolute value of the difference between the numerical data for $P$ and expansions in Legendre polynomials at successive orders (right).
Conclusions

• Good overlap of weak and strong coupling at low orders (but large orders similar to the plaquette)

• Finite size effects in the leading logarithm under control

• Apparent convergence of polynomial approximations after subtracting log. singularities (this allows us to work in the complex $S$ plane).

• Application: Fisher’s zeros (in progress)

• Plans: decimation in a multicoupling generalization of $n(S)$, finite size effects on asymmetric lattices, $U(1)$, first order PT, ....
Consider a lattice model in $D$ dimensions, with lattice spacing $a$, linear size $N$, volume $V = N^D$ and nonlinear scaling variables $u_i$.

Under a RG transformation

$$a \rightarrow \ell a; \ N \rightarrow N/\ell; \ u_i \rightarrow \ell^{y_i}u_i$$

with $\ell$ a fixed value (e.g. 2) that cannot be shrunk to 1

For scalar models with average magnetization $m$

$$V_{\text{eff}}(\ell^y m, \ell^{y_i}u_i, N/\ell) = \ell^D V_{\text{eff}}(m, u_i, N)$$
For gauge models ($SU(2)$ hereafter) with $N_p = \frac{D(D-1)}{2}V$ plaquettes

$$Z(\beta, \{\beta_i\}) = \int_0^{2N_p} dS \ n(S, \{\beta_i\}) e^{-\beta S},$$

$$n(S, \{\beta_i\}) = \prod_l \int dU_l \delta(S - \sum_p (1 - (1/N)ReTr(U_p))) e^{-\sum_i \beta_i (1 - \chi_i(U_p)/d_i)}$$

$$f(s, \{\beta_i\}, N_p) \equiv \ln(n(sN_p, \{\beta_i\}, N_p))/N_p$$

can be used as the effective potential if we can find a RG transformation for the $\{\beta_i\}$ associated with the characters $\chi_i$ (e.g. Migdal-Kadanoff)

$$\lim_{N_p \to \infty} f(s, \{\beta_i\}, N_p) = f(s, \{\beta_i\})$$