Non-perturbative renormalization of $N_f = 2 + 1$ QCD with Schrödinger functional scheme

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Our ultimate purpose

- Determine the fundamental parameter of \( N_f = 2 + 1 \) QCD

\[
\mathcal{L} = -\frac{1}{4g^2} F^{a}_{\mu \nu} F^{a}_{\mu \nu} + \bar{\psi}_i (\gamma_\mu D_\mu + m_i) \psi_i
\]

- Strong coupling \( g \): target of this talk
  - Low energy input \( r_0 \) is measured by PACS-CS (Namekawa)
- Quark masses \( m_i \)
  - Bare quark masses are measured by PACS-CS (Kadoh, Kuramashi, Ukita)
  - NP renormalization factor will be needed.

- We adopt input of low energy experimental values.
  - Comparison with high energy input (estimation of systematic error)
  - Need calculation from weak to strong coupling region.
Plan of this project

- Evaluate $\alpha_S(M_Z)$ by an input of low energy observable ($r_0$).
- NP renormalization factor of quark mass.
  - as a by product of $\alpha_S(M_Z)$ in this talk
    (inhomogeneous BC at $t=0, T$)

Method

- Non-perturbative renormalization with Schrödinger functional
  - Finite volume of $L^4$
  - Appropriate boundary condition
  - Renormalization scale $\sim 1/L$
  - Good compatibility with lattice.
  - Covers from low to high energy region.
Schrödinger functional scheme (Lüscher et al, Alpha)

- Dirichlet boundary condition at $t = 0, T$.

$$U_k(x)\big|_{x_0=0} = \exp(\alpha C_k), \quad C_k = \frac{i}{L} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

- Unique global minimum at background field $B_{\mu}$.
- Mass gap in fermionic mode
  (quark mass can be set to zero).
- Renormalized coupling

$$S_0 = \frac{1}{2} g_0^2 F_{\mu\nu}^2 \Rightarrow \Gamma_0[B_{\mu}] = \Gamma[B_{\mu}] = \frac{1}{g_R^2(L)} k[B_{\mu}]$$

- Mass renormalization factor

$$Z_m(L) = \frac{\langle P(t = L/2) \cdot \mathcal{O}_{\text{boundary}} \rangle_{\text{lattice}}}{\langle P(t = L/2) \cdot \mathcal{O}_{\text{boundary}} \rangle_{\text{tree}}}$$
Step Scaling Function

- Renormalization group flow $g(L) \rightarrow g(2L)$ when one changes the renormalization scale $L \rightarrow 2L$

implemented easily on lattice

- Follow the renormalization group flow in discretized way.
Step scaling function

- The point is:
  - To take continuum limit for every step of RG flow.
    - \( L \to 2L, \quad g(L, a) \to g(2L, a) \)

- Obtain the RG flow \( g(L) \to g(2L) \) in the continuum.
Numerical setup

- Iwasaki gauge action $\beta = 2.1 \sim 6.2$
  - tree level boundary improvement
  - inhomogeneous DBC, $\theta = \pi/5$
- Wilson fermion with clover term.
  - non-perturbative $c_{SW}$
  - one loop boundary improvement
- RHMC/HMC algorithm for 3rd/two flavour(s)
- CPS++ code
- Machines
  - PC cluster kaede at Tsukuba: ($\sim180$ PU)
  - SR11000 at Tokyo: ($\sim64$ PU)
  - PACS-CS: ($256$ PU)$\times$ 1 month
  - RSCC at Riken ($128$ PU)
  - T2K at Tsukuba: ($2560$ cores)$\times$ 10 days
  - T2K at Tokyo: ($128$ cores)
Current status

- Take the continuum limit by three box sizes.

<table>
<thead>
<tr>
<th>$L/a$</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2L/a$</td>
<td>8</td>
<td>12</td>
<td>16</td>
</tr>
</tbody>
</table>

- Tuning of $\beta$ and $\kappa$ is finished for fixed physical box size.

<table>
<thead>
<tr>
<th>$\bar{g}^2$</th>
<th>1.001</th>
<th>1.249</th>
<th>1.524</th>
<th>1.840</th>
<th>2.129</th>
<th>2.632</th>
<th>3.418</th>
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<tbody>
<tr>
<td>$4^4$</td>
<td>110K</td>
<td>170K</td>
<td>230K</td>
<td>170K</td>
<td>210K</td>
<td>320K</td>
<td>100K</td>
</tr>
<tr>
<td>$8^4$</td>
<td>40K</td>
<td>40K</td>
<td>86K</td>
<td>134K</td>
<td>50K</td>
<td>74K</td>
<td>308K</td>
</tr>
<tr>
<td>$6^4$</td>
<td>153K</td>
<td>150K</td>
<td>50K</td>
<td>170K</td>
<td>110K</td>
<td>144K</td>
<td>120K</td>
</tr>
<tr>
<td>$12^4$</td>
<td>42K</td>
<td>51K</td>
<td>42K</td>
<td>35K</td>
<td>28K</td>
<td>21K</td>
<td>38K</td>
</tr>
<tr>
<td>$8^4$</td>
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<td>86K</td>
<td>122K</td>
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<td>74K</td>
<td>122K</td>
<td>122K</td>
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<tr>
<td>$16^4$</td>
<td>16.2K</td>
<td>18K</td>
<td>116K</td>
<td>3.2K</td>
<td>4K</td>
<td>5.1K</td>
<td>3.2K</td>
</tr>
</tbody>
</table>

- Now performing simulation for larger box of $2L$

  - $L/a = 8 \rightarrow L/a = 16$: Now going on
Distribution of data

- Distribution of $\partial S/\partial \eta \propto 1/g^2$ ($12^4$ at strong coupling)

$\partial S/\partial \eta$ vs $\tau$

Its distribution
\[ \sigma(u) = \bar{g}^2(2L) \bigg|_{u=g^2(L)} \]
SSF (preliminary)

\[ \sigma(u)/u \]
SSF (preliminary)

- $\sigma(u)/u$

![Graph showing SSF for coupling with different L/a values and polynomial fit](image)

- Polynomial fit

\[ \sigma(u) = u + s_0 u^2 + s_1 u^3 + s_2 u^4 + f_3 u^5 + f_4 u^6 \]

\[ \chi^2/dof \sim 0.8 \]
Introduction of scale

- $r_0$ at vanishing PCAC mass ($m_u = m_d = m_s \to 0$)

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$(r_0/a)_{m=0}$</th>
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</thead>
<tbody>
<tr>
<td>1.83</td>
<td>4.342(26)</td>
<td></td>
</tr>
<tr>
<td>1.90</td>
<td>6.45(27)(34)</td>
<td></td>
</tr>
<tr>
<td>2.05</td>
<td>7.787(66)</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$L^4$</th>
<th>$\beta$</th>
<th>$\bar{g}^2$</th>
<th>$m_{AWT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4^4$</td>
<td>1.90</td>
<td>4.695(23)</td>
<td>$-0.00039(28)$</td>
</tr>
<tr>
<td>$4^4$</td>
<td>2.05</td>
<td>3.808(17)</td>
<td>0.00010(31)</td>
</tr>
<tr>
<td>$6^4$</td>
<td>2.05</td>
<td>4.763(70)</td>
<td>$-0.00810(20)$</td>
</tr>
</tbody>
</table>

$$\Lambda = \mu (b_0 \bar{g})^{-\frac{b_1}{2b_0^2}} \exp \left( -\frac{1}{2b_0 \bar{g}} \right) \exp \left( - \int_{\bar{g}} \exp \left( \frac{1}{\beta} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right) \right)$$

$$\Lambda_{\overline{MS}} = 2.612 \Lambda_{\overline{SF}}$$
Scaling behaviour is not good for $L/a = 4$.

May be able to take the continuum limit by two data points.

PT improvement may not work for $4^4$ but may be for $6^4$. 
Conclusion

- Calculation for the running coupling is going on
- Scaling behaviour seems to be good.
- Scaling behaviour of quark mass SSF is not so good.
  - We may need perturbative improvement.

Future work

- Take the continuum limit
- Take data for $L/a = 16$
  - Much expectation on T2K machine!
- Adopt appropriate setup for $Z_m$ and repeat the calculation
  - Homogeneous BC, $\theta = 0.5$
Distribution of data

- $(8^4 \text{ at strong coupling})$

- $(16^4 \text{ at strong coupling})$
Scaling of SSF

- For one loop improved gauge boundary term (condition B)