A New Computational Approach to Lattice Field Theories

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NO! NOT YET QCD!!
What is exciting about physics is that not only that there is a quantitative theory to understand a phenomena but that there are at least a handful of different approaches to solve the theory and each approach teaches you something new and interesting
Research over the past decade suggests that there exists a new computational approach to solve well known lattice field theories which includes models with chemical potential, models with massless fermions, models with gauge fields.

The potential of the method remains largely unexplored.
Outline

★ Basic Ideas: A Simple Example
  ➡ XY model + Chemical Potential
  ➡ $\text{CP}^{N-1}$ model

★ Bosons as Fermionic Composites
  ➡ XY model
  ➡ A model of pions in $N_f=2$ QCD [ $\text{SU}(2)\times\text{SU}(2)\times\text{U}(1)$ model ]

★ Massless Thirring Model (any dimension)
  ➡ A new fermion algorithm

★ Gauge Theories (?)

★ Conclusions
XY Model + Chemical Potential

In the conventional approach the action is

\[ S = -\frac{\beta}{2} \sum_{x,\alpha} \left\{ \exp \left( i[\phi_x - \phi_{x+\alpha}] + \mu \delta_{\alpha,t} \right) + \exp \left( -i[\phi_x - \phi_{x+\alpha}] - \mu \delta_{\alpha,t} \right) \right\} \]

action is complex!

A complex action is a generic feature of many field theories in the presence of a chemical potential in the conventional formulation.
Solution: World-Line Representation

\[ Z = \int [d\phi] \exp \left[ \sum_{x,\alpha} \left\{ \frac{\beta}{2} \exp \left( i[\phi_x - \phi_{x+\alpha}] + \mu \delta_{\alpha, \xi} \right) + \frac{\beta}{2} \exp \left( -i[\phi_x - \phi_{x+\alpha}] - \mu \delta_{\alpha, \xi} \right) \right\} \right] \]

High temperature expansion

\[ Z = \int [d\phi] \prod_{[x,\alpha]} \sum_{n_{x,\alpha}} \left( \frac{(\beta/2)^{n_{x,\alpha}} e^{n_{x,\alpha}(i[\phi_x - \phi_{x+\alpha}] + \mu \delta_{\alpha, \xi})}}{n_{x,\alpha}!} \right) \sum_{m_{x,\alpha}} \left( \frac{(\beta/2)^{m_{x,\alpha}} e^{-m_{x,\alpha}(i[\phi_x - \phi_{x+\alpha}] + \mu \delta_{\alpha, \xi})}}{m_{x,\alpha}!} \right) \]

\[ Z = \sum_{[k_{x,\alpha}]} \prod_{[x,\alpha]} e^{\mu \delta_{\alpha, \xi} k_{x,\alpha}} I_{k_{x,\alpha}}(\beta) \delta \left( \sum_{\alpha} [k_{x,\alpha} - k_{x-\alpha, \alpha}] \right) \]

World-line representation: no sign problem
Each world line configuration is defined by a set of constrained integers on bonds $[k_{x,\alpha}]$. 

XY-Model world-line configuration (Example)
Can we solve models in the World-Line Approach?

- Have to deal with constraints
- Local algorithms may not be ergodic, efficient
- Correlation functions may be non-diagonal
  - involve introducing defects (Off-diagonal Observables)
The Worm (directed path) Algorithm

Prokof’ev and Svistunov, PRL 87, 160601 (2001)
Syljuasen and Sandvik, PRE 66, 046701 (2002)

The basic idea of an update

- Create a pair of defects and propagate them
- Update ends when the defects meet and can be removed
- Motion of defects satisfies detailed balance
- Complete update is non-local
Performance

• As efficient as the conventional cluster algorithms
• But, more flexible with addition of couplings
• Applicable to more models

Disclaimer
Of course the world-line approach may not be efficient for all problems
$Z = \int \prod_i [dz_i] \exp(\beta \sum_{<ij>} (z_i^a z_i^b)(z_j^b z_j^a))$

$z_i^a, a = 1, 2, \ldots, N$

$\sum_a |z_i^a|^2 = 1$

High Temperature Expansion

$Z = \int \prod_i [dz_i] \prod_{<ij>} \prod_{<ab>} \sum_{n_{ij}} \frac{\beta^{n_{ij}ab}}{n_{ij}!} (z_i^a z_i^b)^{n_{ij}ab} (z_j^b z_j^a)^{n_{ij}ab}$

$Z = \sum_{[n_{ij}]} \left( \prod_i I(q^i) \right) \delta_{q^i,p^i} \left( \prod_{<ij>} \prod_{ab} \beta^{n_{ij}ab} \right)$

$I(k) = \frac{2\pi^{N-k_1-k_2-\ldots-k_N}N!}{\Gamma(k_1 + k_2 + \ldots + k_N + N)}$

$q^i_a = \sum_{\mu} \sum_b \left\{ n_{i(i+\mu)}^{ab} + n_{i(i-\mu)}^{ba} \right\}$,

$p^i_a = \sum_{\mu} \sum_b \left\{ n_{i(i+\mu)}^{ba} + n_{i(i-\mu)}^{ab} \right\}$

World-line Representation
Apparent difficulties of the world-line approach!

Difficult to write in closed form
&
The resulting models difficult to code

Is there a simpler way?
The D-theory Approach
U.-J. Wiese, Lattice 1998

- Formulate field theories using Dimensional reduction of Discrete variables

- Past formulations begin with the Hamiltonian
  - World-line representations are natural
  - Sometimes encounter new sign problems

- Now can formulate directly in Lagrangian approach!
  - use Grassmann variables
  - more flexible, easier and new ways to deal with GV!
Bosons as Composite Fermions

A Fermionic XY Model in d-dimensions (Strongly Coupled QED)

\[ S = - \sum_{x,i=1,2,..,d} \bar{\psi}_x \psi_x \bar{\psi}_{x+i} \psi_{x+i} - T \sum_x \bar{\psi}_x \psi_x \bar{\psi}_{x+i} \psi_{x+i} \]

exact global U(1) symmetry:

\[ \psi_x \rightarrow e^{i\sigma_x \theta} \psi_x \text{ and } \bar{\psi}_x \rightarrow e^{i\sigma_x \theta} \bar{\psi}_x \text{ where } \sigma_x \text{ is } +1 \text{ on even sites and } -1 \text{ on odd sites} \]

World-line representations arise naturally!

four-fermion terms makes the problem easy!

\[ \exp(U \bar{\psi}_x \psi_x \bar{\psi}_{x+\mu} \psi_{x+\mu}) = 1 + U \bar{\psi}_x \psi_x \bar{\psi}_{x+\mu} \psi_{x+\mu} \]

Dimer models of Rossi and Wolff, 1984!
Pions with Quarks

An $SU(2) \times SU(2) \times U(1)$ model of composite fermions

Action

$$S = - \sum_{x,i=1,2,\ldots,d} \text{Tr}[\Sigma_x \Sigma_{x+i}] - T \sum_x \text{Tr}[\Sigma_x \Sigma_{x+i}] - c \sum_x \det \Sigma_x$$

$$\Sigma_x = \begin{pmatrix} u_x & & \\ & \bar{u}_x & \\ d_x & & \bar{d}_x \end{pmatrix} = \begin{pmatrix} u_x \bar{u}_x & u_x \bar{d}_x \\ \bar{u}_x \bar{d}_x & \bar{d}_x \end{pmatrix}$$

Symmetry

$$\Sigma_x \rightarrow L \Sigma_x R^\dagger e^{i\phi} \quad \text{for x even}$$

$$\Sigma_x \rightarrow R \Sigma_x L^\dagger e^{-i\phi} \quad \text{for x odd}$$

No conventional cluster algorithm for this problem
Effects of the Anomaly in two flavor QCD phase transition

Chandrasekharan & Mehta PRL99, 142004 (2007)

![Diagram showing phase transition in QCD with anomaly strengths and critical behavior](image-url)
Anomaly strength at the tricritical point

\[ r = \frac{(M_{\eta'} - M_{\pi})}{\rho_{\eta'}} \sim 7 \]

Strong Anomaly!

C=0.03
Strong-Coupling QCD + $\mu$

- Originally proposed by Karsch & Mutter (local algorithm).
- Sign problem remains unsolved, but milder.
- Fromm & Forcrand use a worm algorithm.
- Talk by Fromm
  - Wednesday 2:30pm
  - Non-zero temp. and density
Massless Thirring Model

Action

\[ S = -\sum_{x,\mu} \eta(x) \bar{\psi}_x [\psi_{x+\mu} - \psi_{x-\mu}] - U \bar{\psi}_x \psi_x \bar{\psi}_{x+\mu} \psi_{x+\mu} \]

Exact U(1) chiral symmetry:

\[ \psi_x \rightarrow e^{i\sigma_x \theta} \psi_x \text{ and } \bar{\psi}_x \rightarrow e^{i\sigma_x \theta} \bar{\psi}_x \text{ where } \sigma_x \text{ is } +1 \text{ on even sites and } -1 \text{ on odd sites} \]

In \( d \geq 3 \) the model contains a chiral phase transition

\[ U < U_c \quad \text{massless fermions;} \quad U > U_c \quad \text{massless pions} \]

Physics connected to QED, graphene,...

Hands & Strouthos, arXiv:0806.4877
Christofi, Hands & Strouthos, PRD75, 101701 (2007)

Massless limit is usually difficult
World-Line Approach

\[ Z = \int [d\psi d\bar{\psi}] \exp \left( \sum_{x,\mu} \left\{ \eta_\mu(x)\bar{\psi}_x [\psi_{x+\mu} - \psi_{x-\mu}] + U\bar{\psi}_x \psi_x \bar{\psi}_{x+\mu} \psi_{x+\mu} \right\} \right) \]

\[ \exp(U\bar{\psi}_x \psi_x \bar{\psi}_{x+\mu} \psi_{x+\mu}) = 1 + U\bar{\psi}_x \psi_x \bar{\psi}_{x+\mu} \psi_{x+\mu} \]

\[ Z = \sum_{n_{x,\mu} = 0,1} \left( \prod_{x,\mu} U^n_{x,\mu} \right) \Det M[n] \]

fermions are free inside certain regions “Bag Model”
Features of the algorithm

• Clearly slower than a bosonic problem
  - each step involve inverting a matrix.

• But efficient for large $U$
  - Matrix expected to be local and small

• Massless fermions not a problem!
  - zero modes if present can be tackled.

Room for improvements!
World-Sheet Algorithm for gauge theories?

- In principle this should be possible
  - rewrite the model as a model of surface
- Abelian gauge theories good place to start
  - easy to write the surface representation
  - In the confined phase even a local algorithm works well.
  - In the coulomb phase best way to update the surfaces is still unclear.
Wilson Loop in Abelian Gauge Theory

\[ \beta = 0.95 \]
Confined Phase

\[ \beta = 1.05 \]
Deconfined Phase

● World-Sheet Algorithm  ☐ Conventional Algorithm
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