Recent Progress in Lattice QCD at Finite Density

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QCD thermodynamics at $\mu \neq 0$

Heavy-ion experiments (HIC) (low energy RHIC, FAIR)
Properties of QCD at finite density

Important roles of Lattice QCD study

- Critical temperature ($T_c$) and Equation of State (EoS) in low density region
  - Important for Hydrodynamic calculations in HIC.
  - Study by Lattice simulations: available

- Propose interesting observations
  - measureable properties of QCD in HIC
  - Critical point at finite density
    - Large fluctuation in quark number ?
    - Large bulk viscosity ?
Lattice QCD at $\mu \neq 0$

- Many interesting results in QCD thermodynamics at $\mu = 0$
- Study at $\mu \neq 0$: in the stage of development.
- Problem of Complex Determinant at $\mu \neq 0$
  \[
  \left(M(\mu)\right)^\dagger = \gamma_5 M(-\mu)\gamma_5 \quad \text{(}\gamma_5\text{-conjugate)}
  \]
  \[
  \text{(det} M(\mu))^* = \text{det} M(-\mu) \neq \text{det} M(\mu)
  \]
- Boltzmann weight: complex at $\mu \neq 0$
  - Monte-Carlo method is not applicable.
  - Configuration cannot be generated.
- Three approaches
  - Taylor expansion in $\mu$
  - Reweighting method: Simulations at $\mu = 0$, Modify the Boltzmann weight
  - Analytic continuation from imaginary chemical potential simulations
Interesting studies in finite density QCD
(16 parallel talks, 5 posters, and a lot of e-mails)

- Equation of State  - MILC, RBC-Bielefeld, Hot-QCD, WHOT-QCD…
- QCD Critical point – P. de Forcrand (Fri), A. Li (Tue), X. Meng (Tue)
- Stochastic quantization for QCD at finite $\mu$
  - G. Aarts (Tue); G. Aarts, I.-O. Stamatescu, arXiv:0807.1597
- Two Color QCD: Di-quark condensation
  - K. Fukushima (Thu),
  - S. Hands, J. Skullerud and S. Kim
- Isospin chemical potential - Y. Sasai (Tue)
- High temperature effective theory
- Strong coupling limit
  - M. Fromm (Wed), A. Ohnishi (Wed), K. Miura (Pos)
- Chiral perturbation theory - J. Verbaarschot (Tue)
- Chiral fermions (Domain-Wall, overlap)
  - R. Gavai (Tue); arXiv:0803.392, P. Hegde (Pos)
Plan of talk

• Introduction

• Equation of State at finite density
  – Taylor expansion method

• QCD critical point at finite density
  – Quark mass dependence of the critical point
  – Plaquette effective potential
  – Canonical approach

• Summary and Outlook
Equation of State at finite density

- Taylor expansion method
  (Bielefeld-Swansea Collab., ’02–’06, Gavai-Gupta, ’03–’05)
  Systematic studies using p4-imploved staggered action with rather heavy quark masses. (Bielefeld-Swansea Collab., ’02–’06)
  1. Useful for EoS study for Heavy-ion collisions
     • Low density region is important for HIC
  2. Large fluctuations in the quark number at high density
     • Existence of a critical point: suggested

- Recent Progress
  - Simulations near physical mass point
    MILC Collab., RBC-Bielefeld Collab., Hot QCD Collab.
    • Isentropic equation of state, Fluctuations
  - Simulations with a Wilson-type quark action
    WHOT-QCD Collab.,
    • Quark number fluctuations

(Bielefeld-Swansea Collab., ’06)
Taylor expansion method for EoS

• Heavy-ion collisions: low density
  – In the heavy-ion collision at RHIC, the interesting regime of $\mu_q$ is around $\mu_q/T_c \approx 0.1$.

• Taylor expansion in $\mu$ at $\mu=0$.

\[
\frac{p}{T^4}(\mu) = \frac{p}{T^4}(0) + c_2 \left( \frac{\mu_q}{T} \right)^2 + c_4 \left( \frac{\mu_q}{T} \right)^4 + c_6 \left( \frac{\mu_q}{T} \right)^6 + \cdots
\]

\[
\frac{p}{T^4} = \frac{1}{VT^3} \ln Z
\]

\[
c_2 = \frac{N_t^3}{2N_s^3} \frac{\partial^2 \ln Z}{\partial (\mu_q/T)^2}, \quad c_4 = \frac{N_t^3}{4!N_s^3} \frac{\partial^4 \ln Z}{\partial (\mu_q/T)^4}, \cdots
\]

\[
\frac{\partial^n \ln Z}{\partial (\mu_q/T)^n} = 0 \text{ for } n \text{ : odd}
\]

• Simulations at $\mu=0$: Free from the complex determinant problem.
• Calculation of the derivatives: a basic technique for QCD thermodynamics.
  e.g. Energy density, Quark number, Quark number susceptibility

\[
\frac{\varepsilon - 3p}{T^4} = -\frac{N_t^3}{N_s^3} \frac{\partial \ln Z}{\partial \ln a}, \quad n_q = \frac{N_t^3}{N_s^3} \frac{\partial \ln Z}{\partial (\mu_q/T)}, \quad \frac{\chi_q}{T^2} = 9 \frac{\chi_B}{T^2} = \frac{N_t^3}{N_s^3} \frac{\partial^2 \ln Z}{\partial (\mu_q/T)^2}
\]

• Taylor expansion method $\rightarrow$ Useful for EoS study
Isentropic Equation of State

• EoS along lines of constant entropy per baryon number ($S/N_B$)
• Zero-viscosity hydro calculations explain experimental results.
• No entropy production in a heavy-ion collisions (in equilibium)
  $$S/N_B \approx 300$$ (RHIC), $$S/N_B \approx 45$$ (SPS), $$S/N_B \approx 30$$ (AGS)
• MILC Collab. S.Gottlieb’s talk (Monday)
• $N_f=2+1$ Asqtad action, $N_t=4,6$, $m_\pi \approx 220$ MeV

• Lattice discretization error: small. (Open: $N_t=4$, Filled: $N_t=6$)
Isentropic Equation of State

• RBC-Bielefeld Collab. C.Schmidt’s talk (Monday)
• $N_f=2+1$ p4-improved staggered, $N_t=4,6$, $m_\pi \approx 220$MeV
• Consistent with Asqtad results.
• $p/\varepsilon$ vs $\varepsilon$ is important for hydrodynamic calculations in HIC.
• Density dependence: small
• Velocity of sound $c_s$

$$c_s^2 = \frac{dp}{d\varepsilon} = \varepsilon \frac{d(p/\varepsilon)}{d\varepsilon} + \frac{p}{\varepsilon}$$

(Filled: $N_t=4$, Open: $N_t=6$)
Hadronic fluctuations and the QCD critical point

- RBC-Bielefeld Collab. C. Schmidt’s talk (Monday)
- Hadronic fluctuations at $\mu \neq 0$ increase with decreasing mass.

\[ \frac{\partial^2 \langle \chi_B / T^2 \rangle}{\partial (\mu_B / T)^2} / \frac{\chi_B}{T^2} \]

Low T: hadron resonance gas
High T: quark-gluon gas

RBC-Bielefeld Collab.: $N_f=2+1$, $m_\pi=220$ MeV
Bielefeld-Swansea, (’06) : $N_f=2$, $m_\pi=770$ MeV

- Fluctuations for $m_\pi=220$ MeV increase over the hadron resonance gas value at $T_c$. 

\[ \langle B^4 \rangle - 3 \langle B^2 \rangle^2 / \langle B^2 \rangle \]
Equation of state by Wilson quark action

WHOT-QCD Collab. → K. Kanaya’s poster

RG gauge + 2-flavor Clover quark actions, $16^3 \times 4$ lattice, $m_\pi/m_\rho = 0.65$
Hybrid method of Reweighting and Taylor expansion up to $O(\mu_q^4)$

- Large enhancement in the quark number fluctuations at high density.
  → Critical point at finite $\mu$?
QCD critical point in the $(T,\mu)$ plane

- Quark mass dependence of the critical line
- Reweighting method and Sign problem
- Plaquette effective potential
- Canonical approach
Quark mass dependence of the critical point

- **Physical point**
  - 2nd order
  - 1st order
  - Crossover

- **Quark mass dependence near μ=0**
  - Fodor, Katz, ’01-’04; Reweighting method
  - Bielefeld-Swansea Collab., ’02,’03; Reweighting method
  - de Forcrand, Philipsen, ’03-’07; Imaginary chemical potential
  - Kogut, Sinclair, ’05-’07; Phase-quenched approximation

Philipsen’s plenary talk in Lattice 2005
Schmidt’s plenary talk in Lattice 2006
Curvature of the critical surface

- Usual expectation
- Critical point: exists

\[ \frac{\partial^2 m_C}{\partial \mu^2} > 0 \]

- de Forcrand - Philipsen,
  JHEP01(2007)077; PoS(LAT2007)178
- Curvature: slightly negative.
  (3-flavor, 8^3x4 lattice)

New result \( \rightarrow \) de Forcrand’s talk (Friday)

New result by 12^3x4 lattice is consistent with 8^3x4 result.

\[ \frac{\partial^2 m_C}{\partial \mu^2} < 0 \]

\( \rightarrow \) Curvature: Negative.
Imaginary chemical potential approach
(de Forcrand, Philipsen, ’03-’08)

• Binder cumulant:
  – Critical point (Z2 universality): $B_4 = 1.604$
  [Crossover ($m > m_c$): $B_4 = 3$, Strong first order ($m < m_c$): $B_4 = 1$]

• Simulations: possible for imaginary $\mu = i\mu_i$, $\leftarrow \det M(i\mu_i)$: real

• Assumption: $B_4 = 1.604 + b_{10}(m - m_c^0) + b_{01}\mu^2 + b_{02}\mu^4 + \cdots$
  de Forcrand’s talk

• Analytic continuation:
  Fit the simulation results
  $\frac{\partial B_4}{\partial \mu^2} \approx -\frac{b_{01}}{b_{10}} < 0$
  $\iff -b_{01} < 0$

Curvature: Negative $N_t = 4$

Other assumption for analytic continuation,
(D’Elia, Di Renzo, Lombardo, ’ PRD76,114509(2007))
Reweighting method for $\mu \neq 0$ and Sign problem

(Ferrenberg-Swendsen $\rightarrow$ Glasgow group, Fodor-Katz)

- **Reweighting method**
  - **Boltzmann weight**: Complex for $\mu > 0$
    - Monte-Carlo method is not applicable directly.
  - Perform Simulation at $\mu = 0$.

$$\langle O \rangle_{(\beta, \mu)} = \frac{1}{Z} \int DUO (\det M(\mu))^N e^{-S_g(\beta)} = \frac{\langle O e^{i\theta} | \det Nf M(\mu) / \det Nf M(0) \rangle_{(\beta, 0)}}{\langle e^{i\theta} | \det Nf M(\mu) / \det Nf M(0) \rangle_{(\beta, 0)}}$$

- **Sign problem**
  - If $e^{i\theta}$ changes its sign frequently, $\langle O e^{i\theta} \cdots \rangle_{(\beta, 0)}$ and $\langle e^{i\theta} \cdots \rangle_{(\beta, 0)}$ become smaller than their statistical errors.
  - Then $\langle O \rangle_{(\beta, \mu)}$ cannot be computed.
Sign problem and phase fluctuations

• Complex phase of $\det M$  \[ \theta = N_f \text{Im}[\ln \det M(\mu)] \]
  – Taylor expansion: odd terms of $\ln \det M$ (Bielefeld-Swansea, PRD66, 014507 (2002))
  – Good definition (staggered quarks: 4th root trick, $\theta/4$?)

\[
\theta = N_f \text{Im} \left[ \frac{\mu}{T} \frac{d \ln \det M}{d(\mu/T)} + \frac{1}{3!} \left( \frac{\mu}{T} \right)^3 \frac{d^3 \ln \det M}{d^3(\mu/T)} + \frac{1}{5!} \left( \frac{\mu}{T} \right)^5 \frac{d^5 \ln \det M}{d^5(\mu/T)} + \cdots \right]
\]

\[ \theta: \text{NOT in the range of } [-\pi, \pi] \]

• $|\theta| > \pi/2$: Sign problem happens.
  \[ e^{i\theta} \text{ changes its sign.} \]

• Gaussian distribution
  – Results for p4-improved staggered
  – Taylor expansion up to $O(\mu^5)$
  – Dashed line: fit by a Gaussian function

Well approximated
\[ W(\theta) \approx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha \theta^2} \]
Complex phase distribution

- The Gaussian distribution is also suggested by chiral perturbation theory. (K. Splittorff and J. Verbaarschot, Phys.Rev.D77, 014514(2007))

\[ \Rightarrow \text{J. Verbaarschot’s talk (Tuesday)} \]

Assume: Gaussian distribution \( \Rightarrow \) Sign problem is avoided.
(S.E., Phys.Rev.D77, 014508(2008))

- Sign problem: \( \langle (\det M)^N \rangle \equiv \langle e^{i\theta F} \rangle \ll \) (statistical error)

- Gaussian integral:

\[ W(F, \theta) \approx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha \theta^2} W'(F) \]

\[ \langle e^{i\theta F} \rangle = \int dF \int d\theta \ e^{i\theta} FW(F, \theta) \approx \int dF \ e^{-1/(4\alpha)} FW'(F) \]

\[ \langle e^{i\theta F} \rangle \approx \left( e^{-\langle \theta^2 \rangle_F / 2} F \right) \]

- real and positive (No sign problem)
Effective potential of plaquette $V(P)$

Plaquette histogram

- First order phase transition
  Two phases coexists at $T_c$
  e.g. SU(3) Pure gauge theory
- Gauge action $S_g = -6N_{\text{site}}\beta P$
- Partition function
  \[ Z(\beta, \mu) = \int dP \, W(P, \beta, \mu) \]
  \[ W(P', \mu) = \int DU \, (\det M(\mu))^N \, e^{-S_g} \delta(P - P') \]

Effective potential
\[ V(P) \equiv -\ln(W(P)) \]
Distribution function and Effective potential at $\mu \neq 0$
(S.E., Phys.Rev.D77, 014508(2008))

- Distributions of plaquette $P$ (1x1 Wilson loop for the standard action)

\[
Z(\mu) = \int dP \, R(P, \mu) W(P, \beta)
\]

\[
S_g = -6N_{\text{site}} \beta P
\]

\[
W(\overline{P}, \beta) = \int DU \delta(\overline{P}-\overline{P}) \left( \det M(0) \right)^{N_f} \, e^{-S_g}
\]

(Weight factor at $\mu=0$)

\[
R(\overline{P}, \mu) = \frac{\int DU \, \delta(\overline{P}-\overline{P}) \left( \det M(\mu) \right)^{N_f}}{\int DU \, \delta(\overline{P}-\overline{P}) \left( \det M(0) \right)^{N_f}} = \frac{\left\langle \delta(\overline{P}-\overline{P}) \left( \frac{\det M(\mu)}{\det M(0)} \right)^{N_f} \right\rangle_{(\beta, \mu=0)}}{\left\langle \delta(\overline{P}-\overline{P}) \right\rangle_{(\beta, \mu=0)}}
\]

(Reweight factor)

$R(P, \mu)$: independent of $\beta$, $\rightarrow R(P, \mu)$ can be measured at any $\beta$.

Effective potential:

\[
V(P) = -\ln[R(P, \mu) W(P, \beta)] = \sqrt{\mu=0 \text{ crossover}} + \text{non-singular} = 1^{\text{st}} \text{ order phase transition?}
\]

\[
V(P) = -\ln[\overline{W}(P, \beta)] - \ln[R(P, \mu)]
\]
\( \mu \)-dependence of the effective potential

Crossover

\[ -\ln[W(P,\beta)] \]

Critical point

\[ -\ln[W(P,\beta)] - \ln[R(P,\mu)] \]

\( T \)

QGP

hadron

CSC

\[ \mu = 0 \] reweighting

Curvature: Zero

\( 1^{\text{st}} \) order phase transition

\[ -\ln[W(P,\beta)] - \ln[R(P,\mu)] \]

\( \mu = 0 \) reweighting

Curvature: Negative
Effective potential at $\mu \neq 0$

(S.E., Phys.Rev.D77, 014508(2008))

$V(P, \beta, \mu) = -\ln W(P, \beta) - \ln R(P, \mu)$

Results of $N_f=2$ p4-staggage, $m_\pi/m_\rho \approx 0.7$

[data in PRD71,054508(2005)]

- $\det M$: Taylor expansion up to $O(\mu^6)$

- The peak position of $W(P)$ moves left as $\beta$ increases at $\mu=0$.

Solid lines: reweighting factor at finite $\mu/T$, $R(P, \mu)$

Dashed lines: reweighting factor without complex phase factor.
Curvature of the effective potential

Critical point: \[
\frac{d^2 V(P, \beta, \mu)}{dP^2} = - \frac{d^2 \ln W(P, \beta)}{dP^2} - \frac{d^2 \ln R(P, \mu)}{dP^2} = 0
\]

- First order transition for \(\mu_q/T \geq 2.5\)
- Existence of the critical point: suggested
  - Quark mass dependence: large
  - Study near the physical point is important.
Slope of $\ln R(P, \mu)$ at low density

\[-\ln W(P, \beta) - \ln R(P, \mu)\]

\[
\begin{align*}
\beta \Rightarrow \beta_{\text{eff}} &\equiv \beta + \frac{1}{6N_{\text{site}}} \frac{\partial (\ln R)}{\partial P}
\end{align*}
\]

- Minimum point moves, $P \to \text{large}$
- Same effect as
- The phase transition point becomes lower as $\mu$ increases.
Canonical approach

• Canonical partition function (Laplace transformation)

\[ Z_{GC}(T, \mu) = \sum_{N} Z_{C}(T, N) \exp(N\mu/T) \equiv \sum_{N} W(N) \]

• Effective potential as a function of the quark number \( N \).

\[ V(N) = -\ln W(N) = -\ln Z_{C}(T, N) - N\mu/T \]

• At the minimum,

\[ \frac{\partial V(N)}{\partial N} = -\frac{\partial \ln W(N)}{\partial N} = -\frac{\partial \ln Z_{C}(T, N)}{\partial N} - \frac{\mu}{T} = 0 \]

• First order phase transition: Two phases coexist.
First order phase transition line

In the thermodynamic limit,\[ \frac{\partial V(N)}{\partial N} = 0, \]
\[ \frac{\mu^*}{T} = -\frac{\partial \ln Z_c(T, N)}{\partial N} \]

\[ \frac{\mu^*}{T} \rightarrow \frac{\mu}{T} \quad \left( N_s^3 \rightarrow \infty \right) \]

- Mixed state  \[ \rightarrow \]  First order transition
- Inverse Laplace transformation by Glasgow method


\[ N_{\text{f}}=4 \] staggered fermions, \[ 6^3 \times 4 \] lattice
- \[ N_{\text{f}}=4 \]: First order for all \( \rho \).

New results: \( N_{\text{f}}=2 \)
- Direct simulations with fixed \( N \)
- Inverse Laplace transformation in a Saddle point approximation
Simulations with Canonical partition function

\( \chi \)QCD collab. (Kentucky group), A. Li and X. Meng’s talk (Tuesday)

- Canonical partition function with fixed \( N \)
  (Alexandru, Faber, Horvath and Liu, Phys. Rev. D72, 114513 (2005))

\[
Z_C (T, N) = \int DU e^{-S_g} (\det N M)^{N_f}
\]

with Fourier coefficients, \( (\det N M)^{N_f} \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} d(\mu_f/T) e^{-iN\mu_f/T} (\det M(i\mu_f))^{N_f} \)

\[\begin{array}{c}
N_f=4: \text{first order transition} \\
N_f=2: \text{crossover}
\end{array}\]

\( 0.90T_c \) \hspace{2cm} \( 0.92T_c \) \hspace{2cm} \( 0.94T_c \)

\( 0.83T_c \) \hspace{2cm} \( 0.86T_c \)

Wilson quark \( 6^3 \times 4 \) lattice
Inverse Laplace transformation with a saddle point approximation (S.E., arXiv:0804.3227)

- **Approximations:**
  - Taylor expansion: \( \ln \det M \) up to \( O(\mu^6) \)
  - Gaussian distribution: \( \theta \)
  - Saddle point approximation
    - Much easier calculations

- **Two states at the same** \( \mu_q/T \)
  - **First order transition at** \( T/T_c < 0.83 \)

- **Study near the physical point** important

Solid line: multi-\( \beta \) reweighting
Dashed line: spline interpolation
Dot-dashed line: the free gas limit

\( N_f=2 \) p4-staggered, \( m_\pi/m_\rho \approx 0.7 \), \( 16^3 \times 4 \) lattice

Number density
Summary and outlook

• Equation of State at finite density
  – Isentropic EoS for heavy-ion collisions
    • Simulations near physical quark mass point: studied
  – Large hadronic fluctuation near $T_c$: observed
    1. Staggered quark with Small quark mass, 2. Wilson-type quark

• QCD critical point at finite density
  – Technical developments
    • Quark mass dependence of the critical line
    • Avoidance of the Sign problem
    • Plaquette effective potential
    • Canonical approach
  – Existence of the QCD critical point: suggested

• Future studies
  – New Technique for high density: required
  – New phenomena at high density