Physics results from dynamical overlap fermion simulations

Shoji Hashimoto (KEK)
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JLQCD+TWQCD collaborations

- **JLQCD**
  - H. Ohki, T. Onogi (Kyoto)
  - S. Aoki, N. Ishizuka, K. Kanaya, Y. Kuramashi, Y. Taniguchi, A. Ukawa, T. Yoshie (Tsukuba)
  - K. Ishikawa, M. Okawa (Hiroshima)
  - H. Fukaya (Niels Bohr Inst)

- **TWQCD**
  - T.W. Chiu, T.H. Hsieh, K. Ogawa (National Taiwan Univ)

- **Machines at KEK (since 2006)**
  - SR11000 (2.15 Tflops)
  - BlueGene/L (10 racks, 57.3 Tflops)
Project: dynamical overlap fermions

- Theoretically clean =
  - Respect the symmetry
  - Slow to develop…
Project: dynamical overlap fermions

First large scale simulation with exact chiral symmetry

**Theoretical interest**

- Dirac operator spectrum: Banks-Casher relation, chiral RMT
- Chiral symmetry breaking: chiral condensate and related
- Topology: $\theta$-vacuum, topological susceptibility

**Phenomenological interest**

- Controlled chiral extrapolation with the *continuum* ChPT
- Physics applications: $B_K$, form factors, etc.
- Sum rules, OPE
- Flavor-singlet physics
Publications from the project

Not including conference proceedings

Plan

1. Simulation status
   - Overlap implementation, runs, ...
2. Topology issues
   - Physics from fixed topology
   - Topological susceptibility
3. Physics applications
   - Chiral condensate
   - Convergence of the chiral expansion \((m_\pi, f_\pi)\)
   - Pion form factor, \(B_K\)
   - VV-AA, \(\alpha_s\)
   - Nucleon sigma-term, strange content
1. Simulation status

See also, Matsufuru, poster session
Overlap fermion

- Neuberger-Narayanan (1998)

\[
D = \frac{1}{a} \left[ 1 + \frac{X}{\sqrt{X^\dagger X}} \right], \quad X = aD_w - 1 = \frac{1}{a} \left[ 1 + \gamma_5 \operatorname{sgn}(aH_w) \right], \quad aH_w = \gamma_5 (aD_w - 1)
\]

- Exact chiral symmetry through the Ginsparg-Wilson relation.

\[
D \gamma_5 + \gamma_5 D = aD \gamma_5 D
\]

- Continuum-like Ward-Takahashi identities hold
- Index theorem (relation to topology) satisfied
Sign function

- Rational approximation (Zolotarev)

\[ \varepsilon[x] = x \left( p_0 + \sum_{l=1}^{N_{\text{pole}}} \frac{p_l}{x^2 + q_l} \right) \]

- Problem of near-zero modes of \( H_W \)
  - Their density is non-zero at any finite \( \beta \) (Edwards, Heller, Narayanan, 1998)

Need to subtract before approximate
= potentially \( O(V^2) \)
Near-zero mode suppression

- Near-zero modes are unphysical (associated with a local lump or dislocation = lattice artifact)

- Lattice action to suppress them
  - Introduce unphysical (heavy negative mass) Wilson fermions (Vranas, JLQCD, 2006)

\[
\det \left[ \frac{H_w(-m_0)^2}{H_w(-m_0)^2 + \mu^2} \right]
\]

Plaquette gauge, 
\[\beta=5.83, \mu=0; \beta=5.70, \mu=0.2\]

Completely wash-out the near-zero modes.
Topological freezing

- Suppress the dislocations, e.g. by adding extra Wilson fermions
  \[ \det \left( \frac{H_W(-m_0)^2}{H_W(-m_0)^2 + \mu^2} \right) \]
  - Zero probability to have an exact zero-mode
  - No chance to tunnel between different topological sectors.

- If the MD-type algorithm is used, the global topology never changes.
  - Provided that the step size is small enough.

Property of the continuum QCD: common for all lattice formulations as the continuum limit is approached.
Dynamical overlap

Recent attempts:

  - Reflection/refraction trick
- Cundy et al. (2004)
  - Many algorithmic improvements
- DeGrand-Schaefer (2005)
  - Fat-link
  - Some physics results

Our project:

- Aoki et al., arXiv:0803.3197 [hep-lat]
- Fixed topology
- Large scale simulation with $L \approx 2$ fm, $m_q \sim m_s/6$.
- 2-flavor and 2+1-flavor runs

Broad physics program:

- Pion/kaon physics
- $\varepsilon$-regime
- Nucleon sigma term etc.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>N(_f) = 2 runs</th>
<th>N(_f) = 2+1 runs</th>
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<tbody>
<tr>
<td></td>
<td>many physics analysis completed/on-going.</td>
<td>some physics analysis begun.</td>
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<tr>
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<td>(\beta=2.30) (Iwasaki), (a=0.12) fm,</td>
<td>(\beta=2.30) (Iwasaki), (a=0.11) fm,</td>
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<td></td>
<td>16(^3)x32</td>
<td>16(^3)x48</td>
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<td></td>
<td>6 sea quark masses covering</td>
<td>5 ud quark masses, covering</td>
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<td>(m_s/6\sim m_s)</td>
<td>(m_s/6\sim m_s)</td>
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<td>10,000 HMC traj.</td>
<td>x 2 s quark masses</td>
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<td>(Q=0) sector only, except (Q=-2, -4) runs at (m_q=0.050)</td>
<td>2,500 HMC traj.</td>
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<td>Using 5D solver</td>
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<td>Q=0 sector only</td>
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- Using 5D solver
- Q=0 sector only
Lattice spacing

- $\beta$ fixed ($= 2.30$) with varying $m_q$
- Overlap fermion: close to the mass independent renormalization = no $O(am_q)$ term.
- Sommer scale $r_0$, from the static quark potential
  - $N_f = 2$
    - $a = 0.118(2)$ fm
  - $N_f = 2+1$
    - $a = 0.108(2)$ fm
2. Topology issues

Instanton behind the moon...?
Topology is fixed. Any problem?

- Yes = the real QCD vacuum is the $\theta$-vacuum, a superposition of different topological sectors.
- A serious problem for everyone
  - Topological tunneling occurs through rough gauge configs. If you observe frequent topology change, you are far apart from the continuum.

- A solution: *accept* it and reconstruct the $\theta$-vacuum physics
  - Aoki, Fukaya, SH, Onogi, PRD76, 054508 (2007)
    - Finite volume effect of $O(1/V)$
    - Topological susceptibility calculable on a fixed topology configs.
Cluster decomposition

- In QCD, the real vacuum has a certain distribution of the topological charge = the $\theta$ vacuum.
  - Required to satisfy the cluster decomposition property: topology distribution must satisfy

\[
f(Q_1 + Q_2) = f(Q_1)f(Q_2)
\]

\[
f(Q) = e^{i\theta Q}
\]

- Can one reproduce the physics of the $\theta$ vacuum from the fixed topology simulations?
  - Sum-up the topology! Or, not?
Sum-up the topology!

- Partition function of the vacuum
  \[ Z(\theta) = \exp[-VE(\theta)] \]

- Vacuum energy density \( E(\theta) \)
  \[ E(\theta) = \frac{\chi_t}{2} \theta^2 + \frac{c_4}{12} \theta^4 + \ldots \]

- Partition function for a fixed \( Q \)
  \[ Z_Q = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta Z(\theta)e^{i\theta Q} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \exp[-VE(\theta) + i\theta Q] \]

- Using a saddle point expansion around \( (\theta_c = iQ/V) \), one can evaluate the \( \theta \) integral to obtain
  \[ Z_Q = \frac{1}{\sqrt{2\pi\chi_tV}} \exp\left[-\frac{Q^2}{2\chi_tV}\right]\left[1 - \frac{c_4}{8V\chi_t} + O\left(\frac{1}{(\chi_tV)^2}, \frac{Q^2}{(\chi_tV)^2}\right)\right] \]

- Then, the original partition function can be recovered, if one knows \( \chi_t, c_4, \) etc., as
  \[ Z(\theta) = \sum_Q Z_Q e^{-i\theta Q} \]
Fixing topology = Finite volume effect

When the volume is large enough, the global topology is irrelevant.

Topological charge fluctuate locally, according to $\chi_t$, topological susceptibility.

Physics of the $\theta$-vacuum can be recovered by a similar saddle-point analysis, e.g. Some Green’s function:

$$G_Q^{\text{CP even}} = G(0) + G^{(2)}(0) \frac{1}{2\chi_t V} \left[ 1 - \frac{Q^2}{\chi_t V} - \frac{c_4}{2\chi_t^2 V} \right] + G^{(4)}(0) \frac{1}{8\chi_t^2 V^2} + \ldots$$
Topological susceptibility $\chi_t = \langle Q^2 \rangle / V$

- Applying the same formula for the flavor-singlet PS density, $\chi_t$ can be extracted.

$$\lim_{x \to \infty} \langle mP(x)mP(0) \rangle_Q = -\frac{1}{V} \left( \chi_t - \frac{Q^2}{V} + \ldots \right) + O(e^{-m_\eta x})$$

For $N_f=2$ example

Look at a (negative) constant correlation of the local topological charges.
- Found a clear plateau.
- Results from other topological sectors are consistent.

Talk by Chiu, Tue 2:50, “Chesapeake B”

arXiv:0710.1130 [hep-lat]
Sea quark mass dependence

- $N_f=2$ done; 2+1 on-going
- Disconnected loops constructed from low modes (saturation confirmed)

Clear evidence of the sea quark effects: 2 and 2+1.

Fit with ChPT expectation
- $N_f=2$:
  $$\Sigma = [242(5)(10) \text{ MeV}]^3$$
- $N_f=2+1$:
  $$\Sigma = [240(5)(2) \text{ MeV}]^3$$

Leutwyler-Smilga (1992)

$$
\chi_t = m\Sigma / N_f, \text{ or } \\
\chi_t = \frac{\Sigma}{m_u^{-1} + m_d^{-1} + m_s^{-1}}
$$
3. Physics applications
Chiral condensate
Chiral condensate

- Thanks to the exact chiral symmetry, additive renormalization is prohibited.

\[
(\bar{\psi}\psi)^{cont} = Z_S (\bar{\psi}\psi)^{lat} + Z_{mix} \frac{1}{a^3} (1)^{lat}
\]

- Many ways to extract
  - Banks-Casher relation
  - Low-lying eigenmodes (ChRMT)
  - \( \varepsilon \)-regime correlator
  - Topological susceptibility
  - GMOR relation

\[
m_\pi^2 = (m_u + m_d) \frac{\Sigma}{f^2} [1 + ...]
\]

Banks-Casher relation (\(N_f=2\))
$\varepsilon$-regime

- Matching the low-lying eigenvalue distribution with Chiral Random Matrix Theory (ChRMT)

- Matching the PS and A correlators with $\varepsilon$-regime ChPT

\[ \Sigma = [251(7)(11) \text{ MeV}]^3 \]

\[ F = 87(6)(8) \text{ MeV} \]

\[ \Sigma = [240(4)(7) \text{ MeV}]^3 \]
Two-flavor results

$\Sigma^{1/3} (2 \text{GeV})$

- $\epsilon$-regime eigenvalue
- $p$-regime eigenvalue
- $\epsilon$-regime correlator
- Topological susceptibility
- GMOR

arXiv:0710.1130 [hep-lat]
arXiv:0806.0894 [hep-lat]

In good agreement
$m_\pi$ and $f_\pi$
Convergence of chiral expansion

- Chiral expansion
  - The region of convergence is not known a priori.
  - Test with lattice QCD; conceptually clear with exact chiral symmetry.

\[
\frac{m^2_\pi}{m_q} = 2B \left[ 1 + x \ln x + c_3 x + O(x^2) \right],
\]

\[
f_\pi = f \left[ 1 - 2x \ln x + c_4 x + O(x^2) \right].
\]

- Expand in either

\[
x \equiv \frac{m^2}{(4\pi f)^2}, \quad \hat{x} \equiv \frac{m^2_\pi}{(4\pi f)^2}, \quad \xi \equiv \frac{m^2_\pi}{(4\pi f_\pi)^2}
\]

\(\xi\) extends the region significantly.
Two-loop analysis

- Analysis including NNLO
  - With the $\xi$-expansion
    \[
    \frac{m_\pi^2}{m_q} = 2B \left[ 1 + \xi \ln \xi + \frac{7}{2} (\xi \ln \xi)^2 \right. \\
    \quad + \left. \left( \frac{c_4}{2} - \frac{4}{3} (\tilde{l}^{\text{phys}} + 16) \right) \xi^2 \ln \xi \right] \\
    \quad + c_3 \xi (1 - 9 \xi \ln \xi) + \alpha \xi^2, \\
    \]
    \[
    f_\pi = f \left[ 1 - 2 \xi \ln \xi + 5 (\xi \ln \xi)^2 + \frac{3}{2} (\tilde{l}^{\text{phys}} + \frac{53}{2}) \xi^2 \ln \xi \right] \\
    \quad + c_4 \xi (1 - 10 \xi \ln \xi) + \beta \xi^2. \\
    \]
  - For reliable extraction of the low energy constants, the NNLO terms are mandatory.
2+1 flavors

- Similar analysis including NNLO is on-going for 2+1-flavor data. Preliminary results.

\[
\bar{m}_{ud} (2\text{GeV}) = 3.76(45) \text{ MeV}, \\
\bar{m}_s (2\text{GeV}) = 116(12) \text{ MeV}, \\
\frac{f_K}{f_\pi} = 1.201(30).
\]
Pion form factors
Pion form factors

- Another testing ground of ChPT
  - Vector and scalar
    \[
    \langle \pi(p') | V_\mu | \pi(p) \rangle = i(p_\mu + p'_\mu)F_V(q^2),
    \langle \pi(p') | S | \pi(p) \rangle = F_S(q^2), q_\mu = p_\mu - p_\mu
    \]
  - Charge and scalar radius
    \[
    F_V(q^2) = 1 + \frac{1}{6} \langle r^2 \rangle_v q^2 + O(q^4),
    F_S(q^2) = F_S(0) \left[ 1 + \frac{1}{6} \langle r^2 \rangle_s q^2 + O(q^4) \right]
    \]
  - Calculation using the all-to-all technique.

- q^2 dependence well described by a vector meson pole + corrections.

\[
F_\pi(q^2) = \frac{1}{1 - q^2 / m_v^2} + c_i q^2 + ...
\]
All-to-all

- Disconnected diagram
  - Relevant for the scalar form factor.
  - Calculated using the all-to-all technique.
  - Lowmodes are averaged over space-time.

Disconnected contribution is visible.
Chiral extrapolation

- Fit with NNLO ChPT
  - Data do not show clear evidence of the chiral log. But, it is expected to show up even smaller pion masses.
  - NNLO contribution is significant; necessary to reproduce the phenomenological values.

\[ \left\langle r^2 \right\rangle_\pi \]

\[ \left\langle r^2 \right\rangle_S \]
Vacuum polarization functions
Vector and axial correlators in the momentum space.

\[
\langle J_\mu J_\nu \rangle = \left( g_{\mu\nu} q^2 - q_\mu q_\nu \right) \Pi_{j}^{(1)}(Q^2) - q_\mu q_\nu \Pi_{j}^{(0)}(Q^2)
\]

\[
= \int_0^\infty ds \frac{ds}{s-q^2+i\epsilon} \left[ (g_{\mu\nu} s^2 - s_\mu s_\nu) \text{Im} \Pi_{j}^{(1)}(s) - s_\mu s_\nu \text{Im} \Pi_{j}^{(0)}(s) \right]
\]

- Directly calculable on the lattice for space-like momenta

Weinberg sum rules:

\[
f_\pi^2 = - \lim_{Q^2 \to 0} Q^2 \left[ \Pi_{V}^{(1+0)}(Q^2) - \Pi_{A}^{(1+0)}(Q^2) \right],
\]

\[
S = - \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} Q^2 \left[ \Pi_{V}^{(1+0)}(Q^2) - \Pi_{A}^{(1+0)}(Q^2) \right] \quad \text{or} \quad L_{10}
\]

- Another probe of the chiral symmetry breaking.
- S is relevant for the precision EW test of new strong dynamics.
Pion electromagnetic mass splitting

- Das-Guralnik-Mathur-Low-Young sum rule (1967)

\[
\Delta m_{\pi}^2 = -\frac{3\alpha_{EM}}{4\pi f_{\pi}^2} \int_0^\infty dQ^2 \int_0^\infty Q^2 \left[ \Pi_V^{(1+0)}(Q^2) - \Pi_A^{(1+0)}(Q^2) \right]
\]

- Valid in the chiral limit (soft pion theorem)
- Gives dominant contribution to the \( \pi^\pm - \pi^0 \) splitting.
- Related to the pseudo-NG boson mass in the context of new strong dynamics.
- Exact chiral symmetry is essential.
  - The quantity of interest is obtained after huge cancellation between V and A.
Lattice artifact

- Lorentz violation + currents not conserving

\[ \Pi_{J_{\mu\nu}}(q) = \int d^4x e^{iq\cdot x} \langle 0|T\left[ J_\mu(x)J_\nu(y)\right]|0\rangle = \sum_{n=0}^{\infty} B^{(n)}_J q^{2n}_\mu + \sum_{m,n=1}^{\infty} C^{(m,n)}_J q^{2m-1}_\mu q^{2n-1}_\nu \]

- \( J = V \) or \( A \).
- Only \( B_J(0) \) and \( C_J^{(1,1)} \) are physical.
- Thanks to the exact chiral symmetry, \( B_J \) and \( C_J \) are common (up to \( m_q \)) between \( V \) and \( A \), thus cancel in \( V-A \).

\[ \Delta_J = \sum_{\mu,\nu} \hat{q}_\mu \hat{q}_\nu \left( \frac{1}{q^2} - \frac{\hat{q}_\nu}{\sum_\lambda (\hat{q}_\lambda)^3} \right) \Pi_{J_{\mu\nu}} \]
Lattice results

- Can be fitted with
  - ChPT in the low $q^2$ region
    $$\Pi_{V-A}^{(1)}(q^2) = -\frac{f_\pi^2}{q^2} - 8L_{10}^r(\mu)$$
    $$-\frac{1}{24\pi^2}\left[\ln\frac{m_\pi^2}{\mu^2} + \frac{1}{3} - H(x)\right]$$
    $L_{10}$ is extracted.
    $$L_{10}^r(m_\rho) = -5.2(2)(^{+0}_{-3})(^{+5}_{-0}) \times 10^{-3}$$
  - OPE in the high $q^2$ region. In the massless limit, $1/Q^6$ is the leading.
  - Summing up the two regions, $\Delta m_\pi^2$ is obtained.

$$\Delta m_\pi^2 = 993(12)(^{+0}_{-135})(149) \text{ MeV}^2$$

Exp: $\Delta m_\pi^2 = 1261.2 \text{ MeV}^2$
Matching of $\Pi_j^{(0+1)}(Q^2)$ with its perturbative expansion

Adler function

$$D_j(Q^2) \equiv -Q^2 \frac{d\Pi_j(Q^2)}{dQ^2}$$

is finite, renormalization scheme independent.

Lattice artifacts are non-perturbatively subtracted.

$$\langle J_\mu J_\nu \rangle_{\text{lat}}(Q) = \Pi_j^{(1)}(Q) Q^2 \delta_{\mu\nu} - \Pi_j^{(0+1)}(Q) Q_\mu Q_\nu$$

$$- \sum_{n=0} B_n^J(Q) Q_\mu^{2n} \delta_{\mu\nu} - \sum_{m,n=1} C_{mn}^J(Q) \left\{ Q_\mu^{2m+1} Q_\nu^{2n-1} + Q_\nu^{2m+1} Q_\mu^{2n-1} \right\}$$
Strong coupling constant

- High $Q^2$ region described by OPE.

$$\Pi^{(0+1)}_J(Q^2) = c + C_0(Q^2, \mu^2) + \frac{C_m(Q^2)}{Q^2} + C_{qq}^J(Q^2) \frac{\langle m\bar{q}q \rangle}{Q^4} + C_{GG}^J(Q^2) \frac{\langle \alpha_s \pi \cdot GG \rangle}{Q^4} + \ldots$$

- Perturbative expansions known to $\alpha_s^2$.
- Chiral condensate is an input.

$$\Lambda_{MS}^{(2)} = 234(9)(^{+16}_{-0}) \text{ MeV}$$

cf.) ALPHA: 250(16)(16) MeV
QCDSF: 249(16)(25) MeV
Nucleon structure
Nucleon sigma term

- Finite quark mass effect on the nucleon mass
  \[ \sigma_{\pi N} = m_{ud} \langle N | \bar{u}u + \bar{d}d | N \rangle \]
  - Contains both connected and disconnected contrib.

- Strange quark content
  \[ y = \frac{2 \langle N | ss | N \rangle}{\langle N | \bar{u}u + \bar{d}d | N \rangle} \]
  - Disconnect contrib only.

- Feynman-Hellman theorem:
  - Relates them to derivatives of nucleon mass in terms of \( m_{val} \) and \( m_{sea} \).
    \[ \frac{\partial M_N}{\partial m_{val}} = \langle N | \bar{u}u + \bar{d}d | N \rangle_{\text{conn}}, \]
    \[ \frac{\partial M_N}{\partial m_{sea}} = \langle N | \bar{u}u + \bar{d}d | N \rangle_{\text{disc}} \]
    \[ ( = 2 \langle N | ss | N \rangle) \]
  - Analysis with partially quenched data set (\( m_{val} \neq m_{sea} \))
  - Use PQChPT

Talk by Ohki, Thu 9:50, “Auditorium”
Fit with HBChPT

- **Nucleon mass**
  - Finite volume effect significant.
  - Downward shift observed

\[ \sigma_{\pi N} = 52.2^{+20}_{-2}^{+5} \text{MeV} \]

- **Valence and Sea derivatives**
  - Disconnected contribution relatively small.

\[ y = 0.030(16)^{+6}_{-8}^{+1} \]

Talk by Ohki, Thu 9:50, “Auditorium”
So what?

- **Previous lattice results:**
  - Fukugita et al. (1995)
    \[ y = 0.66(15), \text{quenched} \]
  - Dong-Lagae-Liu (1996)
    \[ y = 0.36(3), \text{quenched} \]
  - SESAM (1999)
    \[ y = 0.59(13), N_f=2 \]
  - UKQCD (2001)
    \[ y = -0.28(33), N_f=2 \]
  - JLQCD (2008)
    \[ y = 0.030(18), N_f=2 \]

- **Problem and solution**
  - Was difficult to calculate due to \(m_{\text{sea}}\) dependence of \(m_{\text{cr}}\) easily spoils the physical effect.
  - Problem persists in the quenched calculation.
  - If subtracted, too large error.
  - Exact chiral symmetry is the key.
Conclusion

**Dynamical overlap fermion**

- clean approach producing interesting physics.
- Feasible with $O(10 \text{ Tflops})$ machines
  - $16^3 \times 48 \rightarrow 24^3 \times 48$: test runs started
- Frozen topology = the property of continuum QCD
  - New strategy successful, e.g. topological susceptibility
- Physics applications (so far)
  - Chiral condensates
  - Test of *continuum* ChPT
  - Sum rules, OPE
  - Nucleon structure, flavor-singlet physics
  - More to come…
Backup slides
Locality

- Lattice Dirac operator must be local in order that a local theory is obtained in the continuum limit.
  - Locality is not obvious for the overlap operator due to \(1/\sqrt{\cdot}\).

\[ D = \frac{1}{a} \left[ 1 + \frac{X}{\sqrt{X^+ X}} \right], \quad X = a D_w - 1 \]

- Locality in the sense that \(|D| < \exp(-\mu x)|, with \(\mu\) a number of order \(1/a\), may be satisfied.

- “Proof” is known for smooth enough gauge configurations (Hernandez, Jansen, Luscher (1999)).

- No mathematical proof in more realistic situations where there is non-zero density of the near-zero modes.
  \(\Rightarrow\) Okay if near-zero modes are always localized.
Locality

- Maybe analyzed by looking at individual eigenmodes of $H_W$.
- Near-zero modes are more localized. Higher modes are extended. There is a critical value above which the modes are extended = “mobility edge” (Golterman, Shamir (2003)).

- An important lesson: do not use the overlap fermion in the Aoki phase (where the near-zero modes are extended).
FAQ on topology fixing

1. **Extra Wilson fermions:**
   - Don’t they spoil the continuum limit or the $O(a^2)$ scaling?
   - No. They are heavy: $m \sim 1/a$. Low-lying modes of $H_W$ are local and irrelevant in the continuum limit.

2. **Ergodicity:**
   - Is the ergodicity maintained?
     - No. HMC visits only the fixed topological sector. It restricts the path integral to a given $Q$. But the same physics can be obtained as explained above.
     - Probably yes, within a given $Q$. A fixed $Q$ manifold is connected in the continuum theory. No proof on the lattice; no counter example, either. Lattice aims at approaching the continuum, anyway.
Measurement techniques

Measurements at every 20 traj ⇒ 500 conf / $m_{\text{sea}}$

- Improved measurements
  - 50 pairs of low modes calculated and stored.
  - Used for low mode preconditioning (deflation)
    ⇒ (multi-mass) solver is then x8 faster
- Low mode averaging (and all-to-all)

\[
D^{-1}_{m}(x, y) = \sum_{k=1}^{N} \frac{u_{k}(x)u_{k}^{\dagger}(y)}{\lambda_{k} + m} + D^{(h)-1}_{m}(x, y)
\]

\[
C(x, y) = C^{ll}(x, y) + C^{hh}(x, y) + C^{hl}(x, y) + C^{lh}(x, y)
\]
All-to-all

To improve the signal

- Usually, the quark propagator is calculated with a fixed initial point (one-to-all)
- Average over initial point (or momentum config) will improve statistics; possible with all-to-all

\[
D^{-1}(x, y) = \sum_{k=1}^{N_v} \frac{1}{\lambda^{(k)}} u^{(k)}(x) u^{(k)\dagger}(y) + \sum_{d=1}^{N_d} \left[ D_{\text{high}}^{-1} \eta^{(d)} \right](x) \eta^{(d)}(y)
\]

Low mode contribution

Random noise

High mode propagation From the random noise
An example: two-point func

Dramatic improvement of the signal, thanks to the averaging over source points

- Similar to the low mode averaging; but all-to-all can be used for any n-point func.
- PP correlator is dominated by the low-modes
First (unquenched) lattice calculation with exact chiral symmetry:

JLQCD collab, arXiv:0801.4186 [hep-lat].

\[ \langle \bar{K}^0 | O_{LL}(\mu) | K^0 \rangle = \frac{8}{3} B_K(\mu) f_K^2 m_K^2 \]

- No problem of operator mixing; otherwise, mixes with \( O_{LR} \), for instance. Enhanced by its wrong chiral behavior.
- Another test of chiral log. Here the data follows the NLO ChPT.

\[
B_p = B_p^Z \left[ 1 - \frac{6m_p^2}{(4\pi f)^2} \ln \frac{m_p^2}{\mu^2} + b m_p^2 + O(m_p^4) \right]
\]

\[ B_K(2\text{GeV}) = 0.534(5)(30) \]