$\pi N$ scattering in relativistic BChPT revisited

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Part I

*Introduction*
πN scattering is experimentally well known at low energies.

There have been many attempts to study this process using ChPT, but every one has had their own problems:

- Full Covariant ChPT: Power counting problem due to the heavy scale introduced by the nucleon mass 
  \[ \text{[Gasser, Sainio and Svarc, NPB 307:779 (1988)]}. \]

- HBChPT [Jenkins and Manohar, PLB 255 (1991) 558] : Lorentz invariance is lost, does not converge in the subthreshold region 
  \[ \text{[T. Becher and H. Leutwyler, JHEP 0106 (2001) 01]} \Rightarrow \text{We cannot check Chiral symmetry predictions for QCD.} \]

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Part II

Formalism
Formalism

We consider the process \( \pi^a(q)N(p, \sigma; \alpha) \rightarrow \pi^{a'}(q')N(p', \sigma'; \alpha') \) and decomposing the amplitudes in the usual Lorentz and isospin-invariant form:

\[
T_{aa'} = \delta_{aa'} T^+ + \frac{1}{2} [\tau_a, \tau_{a'}] T^-
\]

\[
T^\pm = \bar{u}(p', \sigma') \left[ A^\pm + \frac{1}{2} (\phi + \phi') B^\pm \right] u(p, \sigma)
\]

We assume isospin symmetry and consider the states with definite isospin \( I = 3/2 \) and \( I = 1/2 \), and definite total angular momentum \( J \) and orbital angular momentum \( \ell \):

\[
T_{IJ\ell}(s) = \frac{1}{\sqrt{4\pi(2\ell + 1)(0\sigma\sigma|\ell \frac{1}{2} J)}} \sum_{m, \sigma'} \int d\hat{p}' (m\sigma'\sigma|\ell \frac{1}{2} L) \times Y^m_{\ell}(\hat{p}')^* \langle \pi(-\hat{p}'; a')N(\hat{p}', \sigma'; \alpha') | T | \pi(-\hat{p}; a)N(\hat{p}, \sigma; \alpha) \rangle_I
\]
Formalism

For the calculation of the $\pi N$ amplitude up to $\mathcal{O}(p^3)$, we use the chiral Lagrangian:

$$\mathcal{L}_{\chi PT} = \mathcal{L}_{\pi\pi}^{(2)} + \mathcal{L}_{\pi\pi}^{(4)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)}$$

Where the superscript indicates de chiral order and $\mathcal{L}_{\pi\pi}^{(n)}$ and $\mathcal{L}_{\pi N}^{(n)}$ corresponds to a pure mesonic Lagrangian and a Lagrangian with baryons, respectively, of chiral order $n$. 
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That Lagrangians have the following form:

\[
\mathcal{L}^{(2)}_{\pi\pi} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle
\]

\[
\mathcal{L}^{(4)}_{\pi\pi} = \frac{1}{16} \ell_4 \left( 2 \langle u_\mu u^\mu \rangle \langle \chi_+ \rangle + \langle \chi_+ \rangle^2 \right) + \ldots
\]

Where the ellipsis refers to terms not needed in the calculations given here and \( \langle \ldots \rangle \) refers to the trace over the isospin matrices. \( F \) is the pion weak decay constant in the chiral limit and

\[
u^2 = U, \quad u_\mu = i u^\dagger \partial_\mu U u^\dagger, \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u
\]

\[
U(x) = \sqrt{1 - \frac{\pi(x)^2}{F^2}} + i \frac{\pi(x) \cdot \vec{\tau}}{F} \quad \text{(Non-lineal sigma parametrization)}
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And the $\pi N$ Lagrangians:

$$L^{(1)}_{\pi N} = \bar{\psi}(i \not{D} - \not{m})\psi + \frac{g}{2} \bar{\psi} \not{u} \gamma_5 \psi ,$$

$$L^{(2)}_{\pi N} = c_1 \langle \chi_+ \rangle \bar{\psi} \psi - \frac{c_2}{4m^2} \langle u_\mu u_\nu \rangle (\bar{\psi} D^\mu D^\nu \psi + \text{h.c.}) + \frac{c_3}{2} \langle u_\mu u^\mu \rangle \bar{\psi} \psi - \frac{c_4}{4} \bar{\psi} \gamma^\mu \gamma^\nu [u_\mu, u_\nu] \psi + \ldots ,$$

$$L^{(3)}_{\pi N} = \bar{\psi} \left( - \frac{d_1 + d_2}{4m} ([u_\mu, [D_\nu, u^\mu]] + [D^\mu, u_\nu]] D^\nu + \text{h.c.}) + \frac{d_3}{12m^3} ([u_\mu, [D_\nu, u_\lambda]] (D^\mu D^\nu D^\lambda + \text{sym.}) + \text{h.c.}) + i \frac{d_5}{2m} ([\chi_-, u_\mu] D^\mu + \text{h.c.}) + i \frac{d_{14} - d_{15}}{8m} (\sigma^{\mu\nu} \langle [D_\lambda, u_\mu] u_\nu - u_\mu [D_\nu, u_\lambda] \rangle D^\lambda + \text{h.c.}) + \frac{d_{16}}{2} \gamma^\mu \gamma_5 \langle \chi_+ \rangle u_\mu + \frac{id_{18}}{2} \gamma^\mu \gamma_5 [D^\mu, \chi_-] \right) \psi + \ldots$$
Part III

*Perturbative Calculations*
From the usual power counting, we have the following contributions:

- Tree level diagrams using vertices of $\mathcal{L}_{\pi N}^{(1)}$, $\mathcal{L}_{\pi N}^{(2)}$ and $\mathcal{L}_{\pi N}^{(3)}$.
- Loop diagrams using only $\mathcal{L}_{\pi N}^{(1)}$ and $\mathcal{L}_{\pi \pi}^{(2)}$. 

![Diagram with labeled diagrams](image)
We consider the phase shifts of the partial wave analyses of the Karlsruhe group [Koch, NPA 448 (1986) 707] (KA85) and the current one of the GWU group [R. A. Arndt et al., PRC 74 (2006) 045205. solution SM01] (WI08). Due to the absence of errors in these analyses there is some ambiguity in the calculation of the $\chi^2$ so:

- We assign an error to every point as the sum in quadrature of a systematic plus a relative error.

\[
\text{err}(\delta) = \sqrt{e_s^2 + e_r^2 \delta^2}
\]

- We take $e_r = 2\%$ as a safer estimate for isospin breaking effects (not taken into account in our study).
- And we take $e_s = 0.1$ degrees in order to stabilize fits because an $e_s = 0$ gives too much weight in the threshold region.
- These values of $e_s$ and $e_r$ are not determinant for our conclusions.
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We consider two strategies to fit the KA85 and WI08 data.

- First strategy (KA85-1, WI08-1):
  - Fit phase shifts up to $\sqrt{s_{\text{max}}} = 1.13$ GeV.
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- Second strategy (KA85-2, WI08-2):
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KA85 Fits

WI08 Fits

Solid line: WI08-1. Dashed line: WI08-2.
Results for the LECs:

<table>
<thead>
<tr>
<th>LEC</th>
<th>KA85-1</th>
<th>KA85-2</th>
<th>WI08-1</th>
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<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$-0.71 \pm 0.49$</td>
<td>$-0.79 \pm 0.51$</td>
<td>$-0.27 \pm 0.51$</td>
<td>$-0.30 \pm 0.48$</td>
<td>$-0.52 \pm 0.60$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$4.32 \pm 0.27$</td>
<td>$3.49 \pm 0.25$</td>
<td>$4.28 \pm 0.27$</td>
<td>$3.55 \pm 0.30$</td>
<td>$3.91 \pm 0.54$</td>
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<tr>
<td>$c_3$</td>
<td>$-6.53 \pm 0.33$</td>
<td>$-5.40 \pm 0.13$</td>
<td>$-6.76 \pm 0.27$</td>
<td>$-5.77 \pm 0.29$</td>
<td>$-6.12 \pm 0.72$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$3.87 \pm 0.15$</td>
<td>$3.32 \pm 0.13$</td>
<td>$4.08 \pm 0.13$</td>
<td>$3.60 \pm 0.16$</td>
<td>$3.72 \pm 0.37$</td>
</tr>
<tr>
<td>$d_1 + d_2$</td>
<td>$2.48 \pm 0.59$</td>
<td>$0.94 \pm 0.56$</td>
<td>$2.53 \pm 0.60$</td>
<td>$1.16 \pm 0.65$</td>
<td>$1.78 \pm 1.1$</td>
</tr>
<tr>
<td>$d_3$</td>
<td>$-2.68 \pm 1.02$</td>
<td>$-1.10 \pm 1.16$</td>
<td>$-3.65 \pm 1.01$</td>
<td>$-2.32 \pm 1.04$</td>
<td>$-2.44 \pm 1.6$</td>
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<tr>
<td>$d_5$</td>
<td>$2.69 \pm 2.20$</td>
<td>$1.86 \pm 2.28$</td>
<td>$5.38 \pm 2.40$</td>
<td>$4.83 \pm 2.18$</td>
<td>$3.69 \pm 2.93$</td>
</tr>
<tr>
<td>$d_{14} - d_{15}$</td>
<td>$-1.71 \pm 0.73$</td>
<td>$1.03 \pm 0.71$</td>
<td>$-1.17 \pm 1.00$</td>
<td>$1.27 \pm 1.11$</td>
<td>$-0.145 \pm 1.88$</td>
</tr>
<tr>
<td>$d_{18}$</td>
<td>$-0.26 \pm 0.40$</td>
<td>$-0.07 \pm 0.44$</td>
<td>$-0.86 \pm 0.43$</td>
<td>$-0.72 \pm 0.40$</td>
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</table>

- Following a conservative procedure, the error given in the average is the sum in quarature of the largest statistical error and the one resulting from the dispersion in the central values.
- The average is compatible with those from $\mathcal{O}(p^3)$ HBChPT, except for the $d_{14} - d_{15}$ that differs by more than one standard deviation.
<table>
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</tr>
<tr>
<td>$d_{14} - d_{15}$</td>
<td>$-0.145 \pm 1.88$</td>
<td>$(-5.1, -4.3)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{18}$</td>
<td>$-0.48 \pm 0.58$</td>
<td>$(-1.6, -0.5)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


In order to obtain the scattering lengths and volumes we performed an effective range expansion (ERE) fit to our results in the low energy region, because numerical problems prevent us to take directly the limit:

$$\lim_{|\vec{p}| \to 0} |\vec{p}| \frac{\text{Re} T}{8\pi \sqrt{s} |\vec{p}|^{1+2\ell}}$$

<table>
<thead>
<tr>
<th>Partial Wave</th>
<th>KA85-1</th>
<th>KA85-2</th>
<th>WI08-1</th>
<th>WI08-2</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{531}$</td>
<td>$-0.100 \pm 0.001$</td>
<td>$-0.103 \pm 0.001$</td>
<td>$-0.081 \pm 0.001$</td>
<td>$-0.082 \pm 0.001$</td>
<td>$-0.092 \pm 0.012$</td>
</tr>
<tr>
<td>$a_{511}^+$</td>
<td>$0.171 \pm 0.001$</td>
<td>$0.172 \pm 0.002$</td>
<td>$0.165 \pm 0.002$</td>
<td>$0.167 \pm 0.002$</td>
<td>$0.169 \pm 0.004$</td>
</tr>
<tr>
<td>$a_{0+}^+$</td>
<td>$-0.010 \pm 0.001$</td>
<td>$-0.011 \pm 0.001$</td>
<td>$0.001 \pm 0.001$</td>
<td>$0.001 \pm 0.001$</td>
<td>$-0.005 \pm 0.007$</td>
</tr>
<tr>
<td>$a_{0+}$</td>
<td>$0.090 \pm 0.001$</td>
<td>$0.092 \pm 0.001$</td>
<td>$0.082 \pm 0.001$</td>
<td>$0.083 \pm 0.001$</td>
<td>$0.087 \pm 0.005$</td>
</tr>
<tr>
<td>$a_{p31}$</td>
<td>$-0.052 \pm 0.001$</td>
<td>$-0.051 \pm 0.001$</td>
<td>$-0.048 \pm 0.001$</td>
<td>$-0.051 \pm 0.001$</td>
<td>$-0.051 \pm 0.002$</td>
</tr>
<tr>
<td>$a_{p11}$</td>
<td>$-0.078 \pm 0.001$</td>
<td>$-0.088 \pm 0.001$</td>
<td>$-0.073 \pm 0.001$</td>
<td>$-0.080 \pm 0.001$</td>
<td>$-0.080 \pm 0.006$</td>
</tr>
<tr>
<td>$a_{p33}$</td>
<td>$0.251 \pm 0.002$</td>
<td>$0.214 \pm 0002$</td>
<td>$0.252 \pm 0.002$</td>
<td>$0.222 \pm 0.002$</td>
<td>$0.232 \pm 0.017$</td>
</tr>
<tr>
<td>$a_{p13}$</td>
<td>$-0.034 \pm 0.001$</td>
<td>$-0.035 \pm 0.001$</td>
<td>$-0.032 \pm 0.001$</td>
<td>$-0.035 \pm 0.001$</td>
<td>$-0.034 \pm 0.002$</td>
</tr>
</tbody>
</table>
Threshold parameters comparison

Results for the threshold parameters:

<table>
<thead>
<tr>
<th>Partial Wave</th>
<th>Average</th>
<th>KA85</th>
<th>WI08</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a_{S31}$</td>
<td>$0.092 \pm 0.012$</td>
<td>$-0.100 \pm 0.004$</td>
</tr>
<tr>
<td></td>
<td>$a_{S11}$</td>
<td>$0.169 \pm 0.004$</td>
<td>$0.175 \pm 0.003$</td>
</tr>
<tr>
<td></td>
<td>$a_{0+}$</td>
<td>$-0.005 \pm 0.007$</td>
<td>$-0.008$</td>
</tr>
<tr>
<td></td>
<td>$a_{0-}$</td>
<td>$0.087 \pm 0.005$</td>
<td>$0.092$</td>
</tr>
<tr>
<td></td>
<td>$a_{P31}$</td>
<td>$-0.051 \pm 0.002$</td>
<td>$-0.044 \pm 0.002$</td>
</tr>
<tr>
<td></td>
<td>$a_{P11}$</td>
<td>$-0.080 \pm 0.006$</td>
<td>$-0.078 \pm 0.002$</td>
</tr>
<tr>
<td></td>
<td>$a_{P33}$</td>
<td>$0.232 \pm 0.017$</td>
<td>$0.214 \pm 0.002$</td>
</tr>
<tr>
<td></td>
<td>$a_{P13}$</td>
<td>$-0.034 \pm 0.002$</td>
<td>$-0.030 \pm 0.002$</td>
</tr>
</tbody>
</table>

- None of our fits (KA85-1, KA85-2, WI08-1, WI08-2) is compatible with the value of $a_{P11}$ given by WI08.
The value of $d_{18}$ is important because it is directly related to the violation of the Goldberger-Trieman (GT) relation. One has, up to $\mathcal{O}(M_\pi^3)$:

$$g_{\pi N} = \frac{g_A m}{F_\pi} \left( 1 - \frac{2M_\pi^2 d_{18}}{g_A} \right)$$

We quantify the deviation from the GT relation by:

$$\Delta_{GT} = \frac{g_{\pi N} F_\pi}{g_A m} - 1$$
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For our averaged value of $d_{18}$ we have:

$$\Delta_{GT} = 0.015 \pm 0.018$$

Which is compatible with the values around 2 – 3% obtained from $\pi N$ and $NN$ partial wave analyses [Arndt, Workman and Pavan, PRC 49 (1994) 2729], [Schröder et al], [Swart, Rentmeester and Timmermans, $\pi N$ Newsletter 13 (1997)96]. This value of $\Delta_{GT}$ gives:

$$g_{\pi N} = 13.07 \pm 0.23 \quad \text{or} \quad f^2 = \frac{(g_{\pi N} M_\pi)^2}{\pi} = 0.077 \pm 0.003$$
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But when we implement the loop contribution, we obtain a huge GT relation violation:

- For the fit KA85-1 one has a 22% of violation for $\mu = 1$ GeV (scale) while for $\mu = 0.5$ GeV a 15% stems.

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In order to implement unitarity to the $\pi N$ amplitude and take care of the analyticity properties associated with the right-hand cut we write our unitarized amplitude $T_{IJ\ell}$ by means of an interaction kernel $T_{IJ\ell}$ and the unitary pion-nucleon loop function $g(s)$:

$$T_{IJ\ell} = \frac{1}{T_{IJ\ell}^{-1} + g(s)}$$

- $T_{IJ\ell}$ satisfies unitarity exactly.
- The interaction kernel is determined order by order by matching with the perturbative ChPT result [J. A. Oller and U. G. Meiβner, PLB 500:263-272 (2001)].
- $a_1$ is fixed by requiring $g(m^2) = 0$ (in order to have the $P_{11}$ nucleon pole in its right position).
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We introduce the contribution of the $\Delta(1232)$ in $P_{33}$ through a CDD
[Castillejo, Dalitz and Dyson, PR 101 (1956) 453],
[Oller and Oset, PRD 60, 074023 (1999)]:

- The CDD pole conserves the discontinuities of the partial wave
  amplitude across the cuts.
- The CDD pole corresponds to a zero of the partial wave amplitude
  along the real axis and hence to a pole in the inverse of the amplitude.

$$T_{3\frac{3}{2}3\frac{3}{2}1} = \left( T_{3\frac{3}{2}3\frac{3}{2}1}^{-1} + \frac{\gamma}{s - s_P} + g(s) \right)^{-1}$$
Unitarized Calculations

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\[
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$$T_{3/2\ 3/2\ 1/2}^{-1} = \left(T_{3/2\ 3/2\ 1/2}^{-1} + \frac{\gamma}{s - s_P} + g(s)\right)^{-1}$$
IR regularization introduces unphysical cuts due to the infinite order resummation of the sub-leading $1/m$ kinetic energy when $u = 0$, that correspond to $s = 2(m^2 + M_{\pi}^2) \gtrsim 1.34^2 \text{ GeV}^2$. Consequences:

- Strong violation of unitarity.
- Strong rising of the phase-shifts from energies $\sqrt{s} \gtrsim 1.26 \text{ GeV}$.

$\Rightarrow$ We redo the fits up to $\sqrt{s_{\text{max}}} = 1.25 \text{ GeV}$ for all the partial waves in the same way than in the perturbative case.
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Solid line: Fit to KA85 data. Dashed line: Fit to WI08 data.
We obtain a good agreement with data in the whole energy range from threshold up to 1.25 GeV.

Good reproduction of the raise in the $P_{33}$ phase shifts associated with the $\Delta(1232)$ resonance.

Compared with the perturbative calculation, one observes a drastic increase in the range of energy with globally acceptable description of the data.
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Unitarized Calculations

<table>
<thead>
<tr>
<th>LEC</th>
<th>Fit KA85</th>
<th>Fit WI08</th>
<th>Average (Perturbative)</th>
<th>Partial Wave</th>
<th>Fit KA85</th>
<th>Fit WI08</th>
<th>Average (Perturbative)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$-0.48 \pm 0.51$</td>
<td>$-0.52 \pm 0.60$</td>
<td>$-0.53 \pm 0.48$</td>
<td>$a_{S_{31}}$</td>
<td>$-0.115$</td>
<td>$-0.104$</td>
<td>$-0.092 \pm 0.012$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$4.62 \pm 0.27$</td>
<td>$4.73 \pm 0.30$</td>
<td>$3.91 \pm 0.54$</td>
<td>$a_{S_{11}}$</td>
<td>$0.152$</td>
<td>$0.150$</td>
<td>$0.169 \pm 0.004$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$-6.16 \pm 0.27$</td>
<td>$-6.41 \pm 0.29$</td>
<td>$-6.12 \pm 0.72$</td>
<td>$a_{0+}$</td>
<td>$-0.026$</td>
<td>$-0.020$</td>
<td>$-0.005 \pm 0.007$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$3.68 \pm 0.13$</td>
<td>$3.81 \pm 0.16$</td>
<td>$3.72 \pm 0.37$</td>
<td>$a_{0+}$</td>
<td>$0.089$</td>
<td>$0.085$</td>
<td>$0.087 \pm 0.005$</td>
</tr>
<tr>
<td>$d_1 + d_2$</td>
<td>$2.55 \pm 0.60$</td>
<td>$2.70 \pm 0.65$</td>
<td>$1.78 \pm 1.1$</td>
<td>$a_{P_{31}}$</td>
<td>$-0.050$</td>
<td>$-0.048$</td>
<td>$-0.051 \pm 0.002$</td>
</tr>
<tr>
<td>$d_3$</td>
<td>$-1.61 \pm 1.01$</td>
<td>$-1.73 \pm 1.04$</td>
<td>$-2.44 \pm 1.6$</td>
<td>$a_{P_{11}}$</td>
<td>$-0.080$</td>
<td>$-0.075$</td>
<td>$-0.080 \pm 0.006$</td>
</tr>
<tr>
<td>$d_5$</td>
<td>$0.93 \pm 2.40$</td>
<td>$1.13 \pm 2.18$</td>
<td>$3.69 \pm 2.93$</td>
<td>$a_{P_{33}}$</td>
<td>$0.245$</td>
<td>$0.250$</td>
<td>$0.232 \pm 0.017$</td>
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<td>$d_{14} - d_{15}$</td>
<td>$-0.46 \pm 1.00$</td>
<td>$-0.61 \pm 1.11$</td>
<td>$-0.145 \pm 1.88$</td>
<td>$a_{P_{13}}$</td>
<td>$-0.41$</td>
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</tbody>
</table>
The values of these LECs do not constitute an alternative determination to the perturbative results.

These values only should be employed within UChPT studies.

LECs and threshold parameters compatible with the average values given in the perturbative calculation.

For the threshold parameters we obtain values compatible with the averaged values of the perturbative calculation.

Studying the GT relation deviation we obtain the same value than in the perturbative study.
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Part V

Summary and Conclusions
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- We study $\pi N$ employing ChPT in IR scheme up to $\mathcal{O}(p^3)$.
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  - We used two sets of data (form Karlsruhe and GWU groups) to fit our theoretical result.
  - An accurate reproduction of the phase-shifts was obtained up to 1.14 GeV, similar in quality to that obtained previously with $\mathcal{O}(p^3)$ HBChPT ⇒ Improvement compared with previous works.
  - We obtain a much better reproduction of the $P_{11}$ phase shifts for the Karlsruhe PWA, while IR ChPT is not able to reproduce the $P_{11}$ phase shift for the GWU current solution even at very low energies.
  - The averaged values of the LECs and the threshold parameters resulting from the two strategies are in good agreement with other previous determinations.
  - High GT deviation ($20 - 30\%$) when the full IR ChPT calculation is included.
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- This scheme removes explicitly the power counting breaking terms appearing in the loop integrals in dimensional regularization.
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