# Pion–kaon final–state interactions in heavy-meson decays

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JLab, February 15th 2018

# Outline

**What's not to like about the isobar model?**

**Form factors and final-state interactions**

**Dispersion relations for three-body decays**

- an ideal test case: $\omega/\phi \to 3\pi$ — Niecknig, BK, Schneider 2012
- $D^+ \to \bar{K}\pi\pi^+$ — Niecknig, BK 2015, 2017

**Summary / Outlook**

# CP violation in weak interactions

**CP violation in partial widths** $\Gamma(P \to f) \neq \Gamma(\bar{P} \to \bar{f})$

- at least two interfering decay amplitudes
- different weak (CKM) phases
- different strong (final-state-interaction) phases

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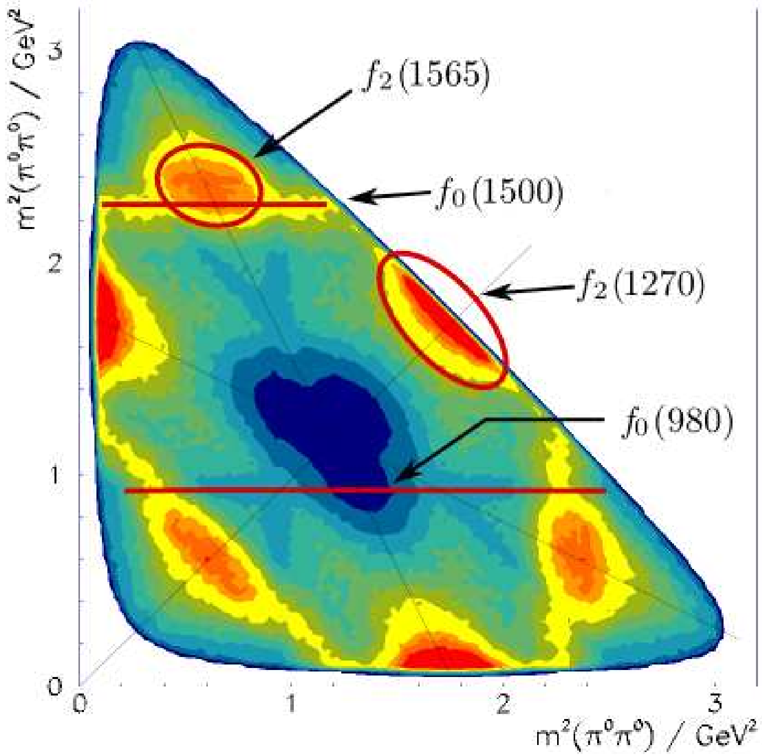
three-body decays: $D \to 3\pi,\ \pi\pi K$

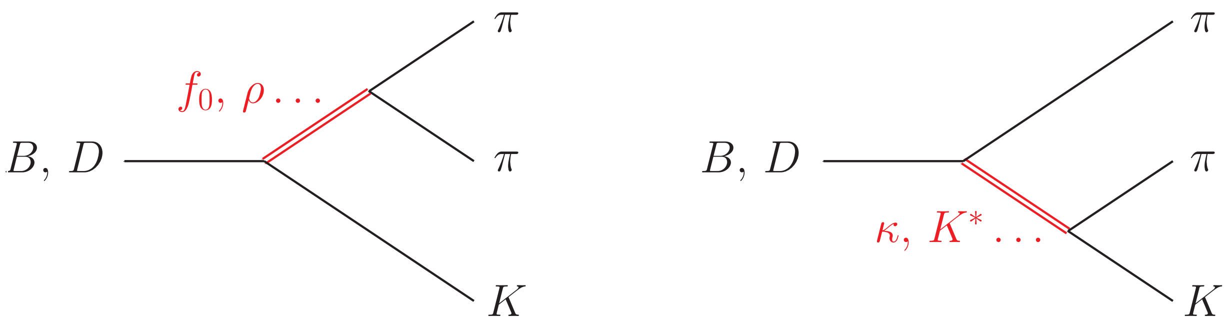
- Dalitz plot $\hat{=}$ density distribution in two kinematical variables
- resonances $\longrightarrow$ rapid phase variation enhances CP-violation in parts of the decay region
- how well do we control strong phase motion with information from hadron physics?

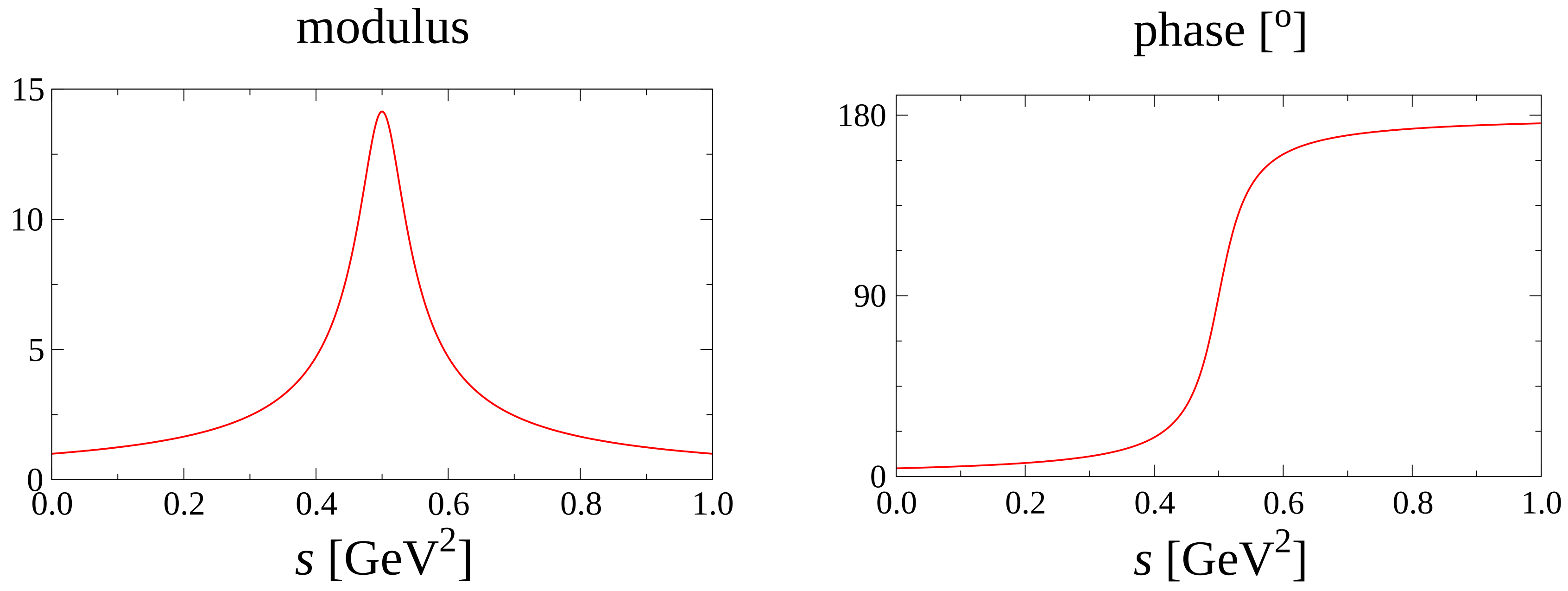



Crystal Barrel

# Amplitude analyses in Dalitz plots

**The traditional picture: isobar model / Breit–Wigner resonances**



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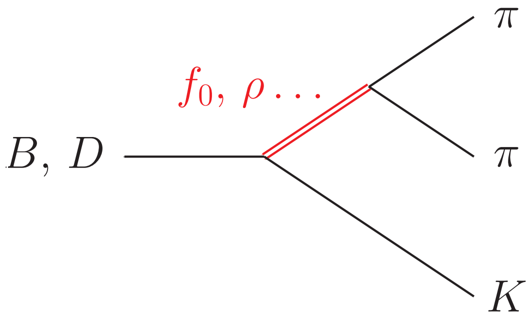


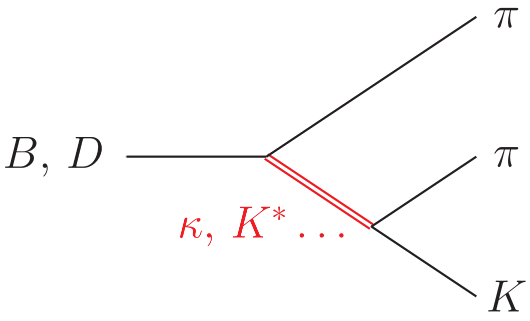


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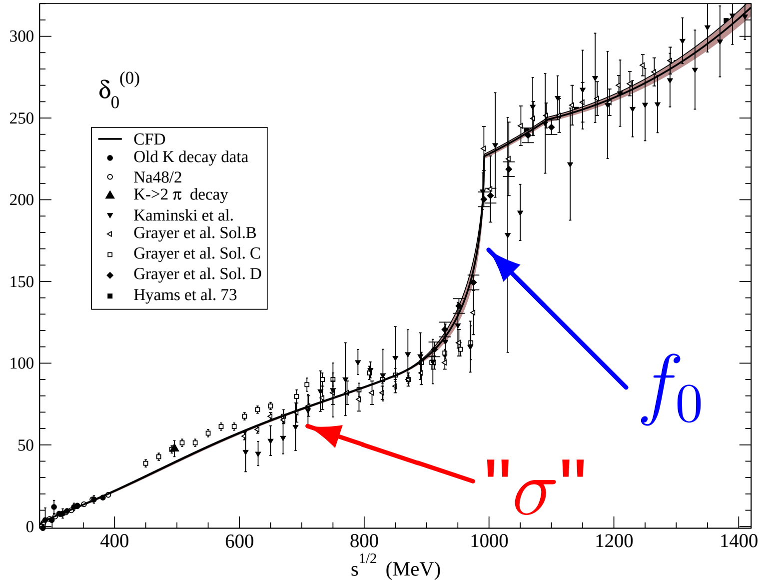
- some resonances don't look like Breit–Wigners at all!

  $\longrightarrow$ use exact scattering phase shifts instead



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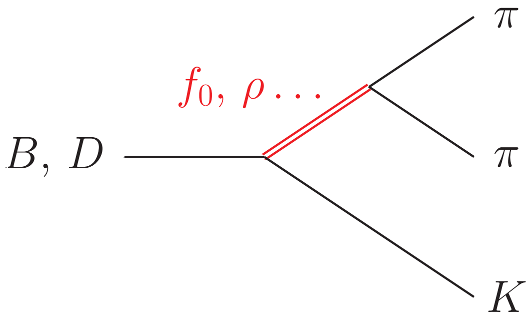


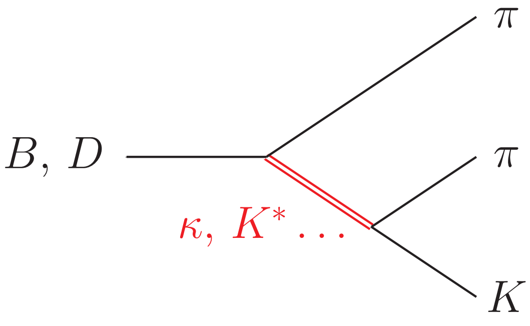


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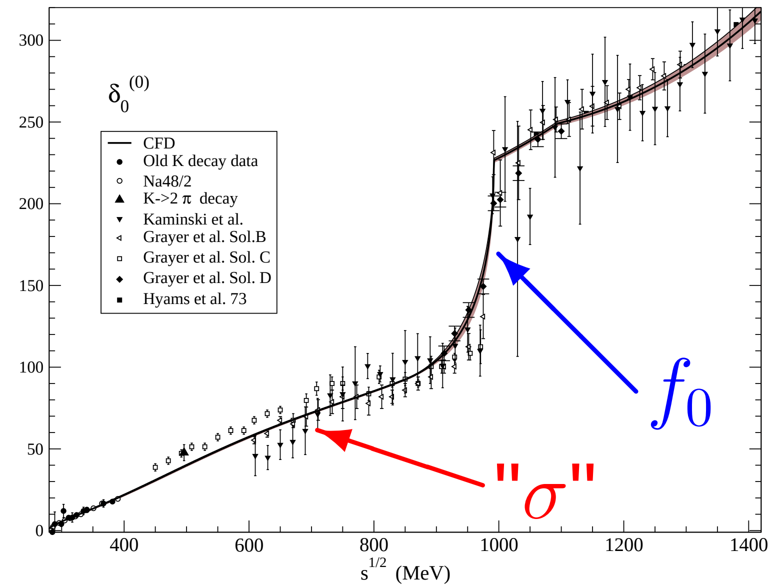
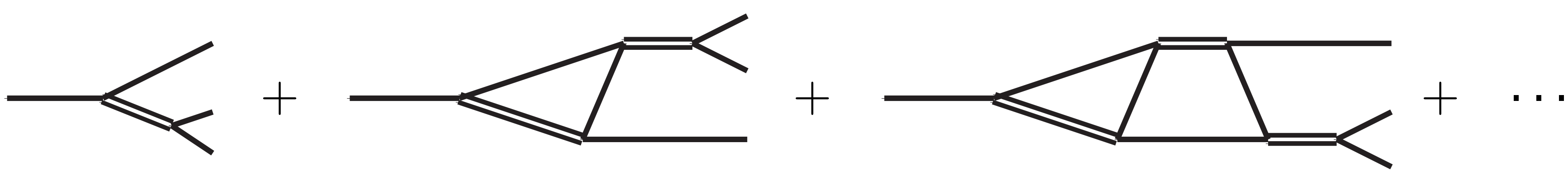
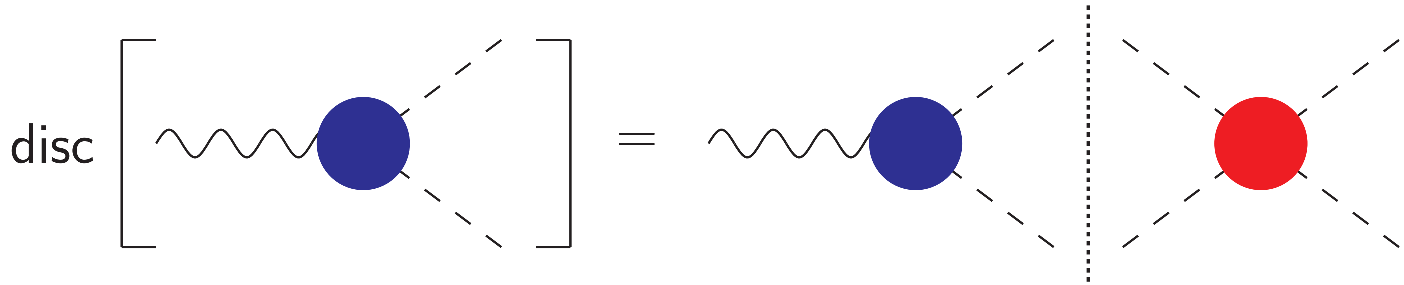



- 3-particle rescattering



# Two-body decays: form factors

- just two particles in final state: form factor; from unitarity:




$$\frac{1}{2i}\,\text{disc}\,F_I(s) \;=\; \text{Im}\,F_I(s) \;=\; F_I(s)\times\theta(s-4\,M_\pi^2)\times\sin\delta_I(s)\,e^{-i\delta_I(s)}$$

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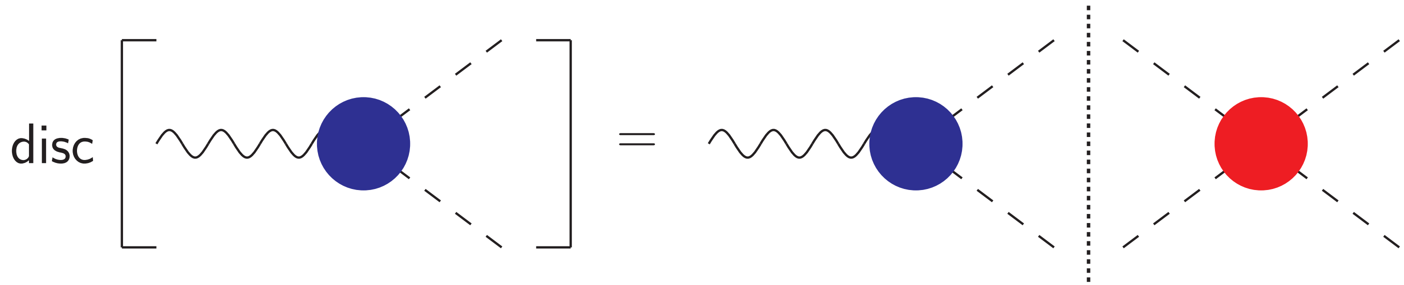


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$\longrightarrow$ final-state theorem: phase of $F_I(s)$ is just $\delta_I(s)$ — Watson 1954

- solution to this homogeneous integral equation known:

$$F_I(s) = P_I(s)\Omega_I(s)\;, \quad \Omega_I(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\,\frac{\delta_I(s')}{s'(s'-s)}\right\}$$

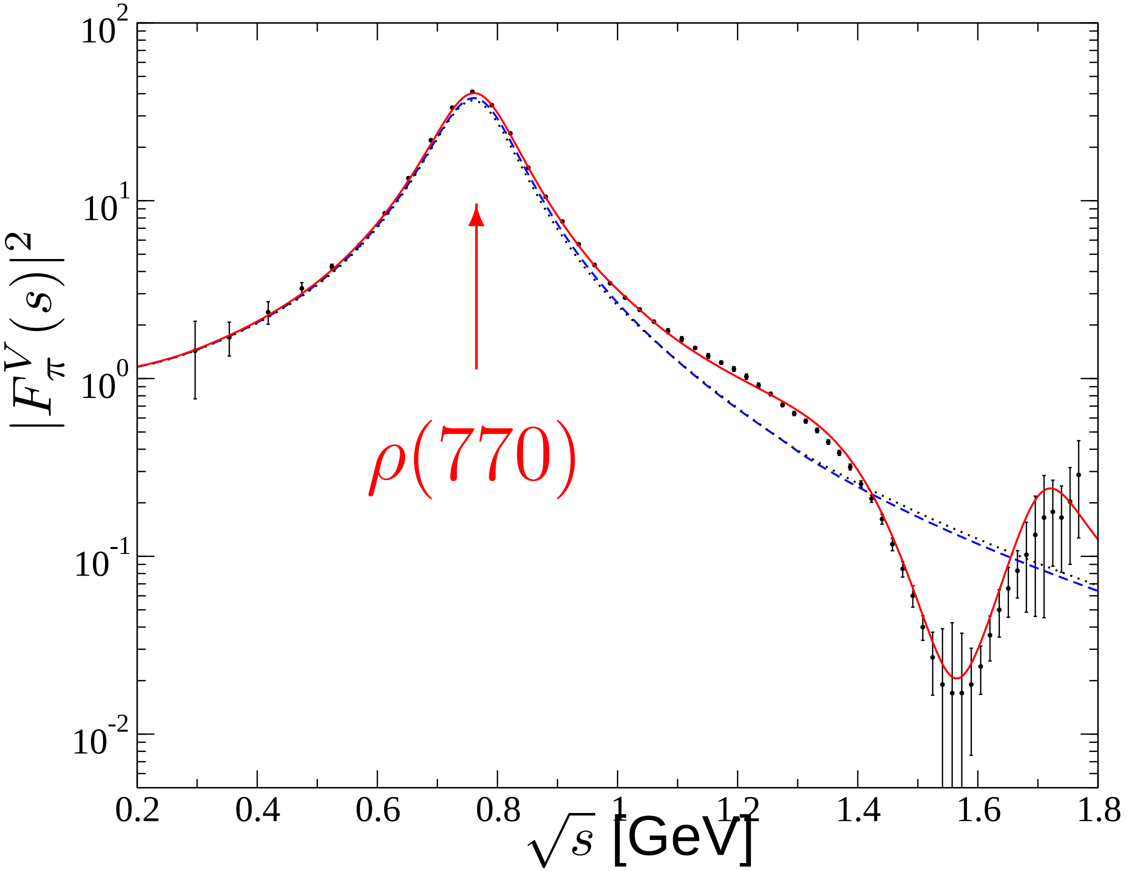
$P_I(s)$ polynomial, $\Omega_I(s)$ Omnès function — Omnès 1958

- today: high-accuracy $\pi\pi$, $\pi K$ phase shifts available

Ananthanarayan et al. 2001, García-Martín et al. 2011 / Büttiker et al. 2004

# Pion vector form factor vs. Omnès representation

**Data on pion form factor in $\tau^- \to \pi^-\pi^0\nu_\tau$** Belle 2008

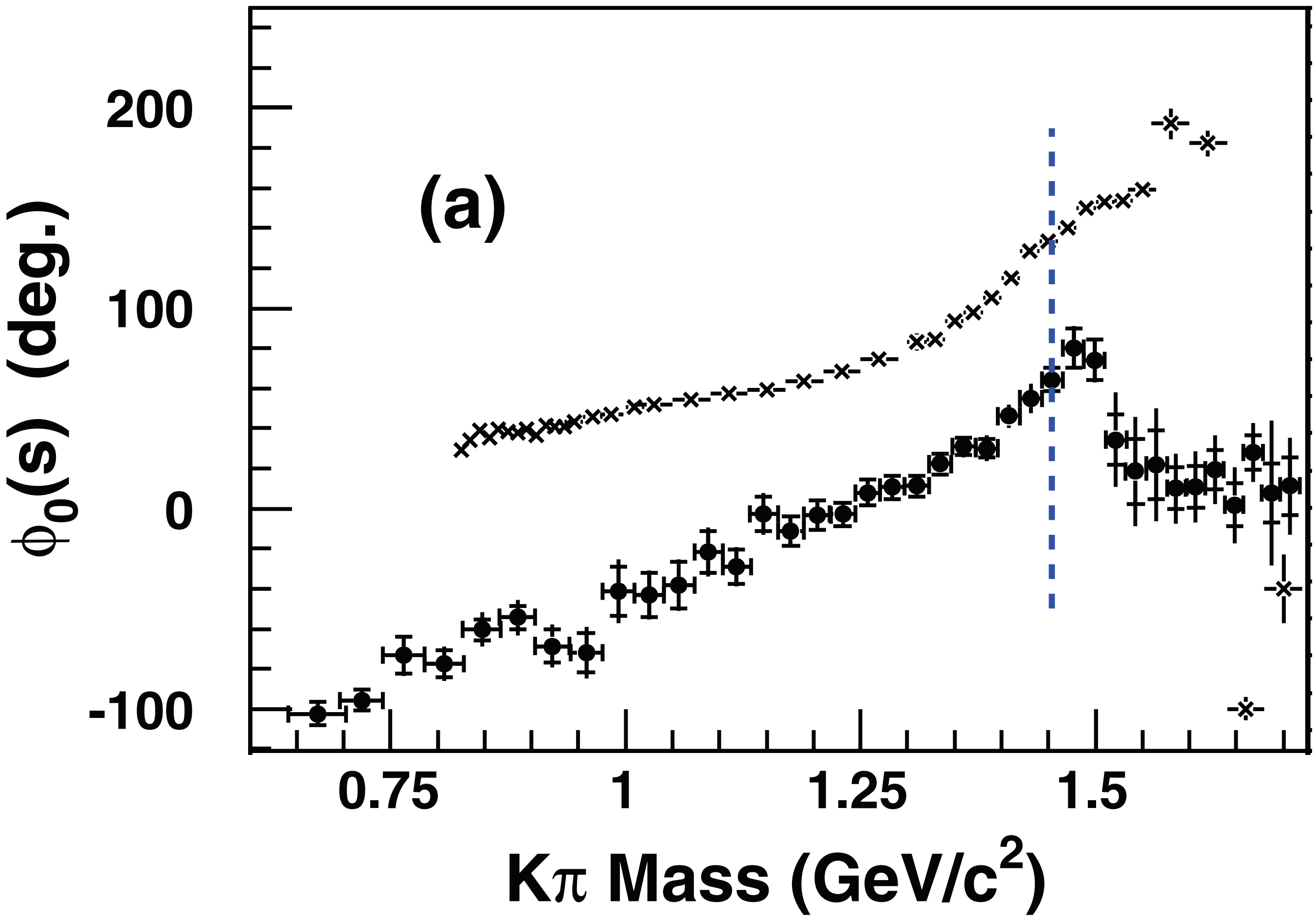



Schneider et al. 2012

- linear polynomial below 1 GeV: $F_\pi^V(s) \approx (1 + 0.1\,\text{GeV}^{-2} s)\Omega(s)$
- above: inelastic resonances $\rho'$, $\rho''$...

# Three-body decays: phase universality??

- extraction of $\pi K$ $S$-wave from $D^+ \to K^-\pi^+\pi^+$ seen to have different phase from elastic scattering: E791 2006

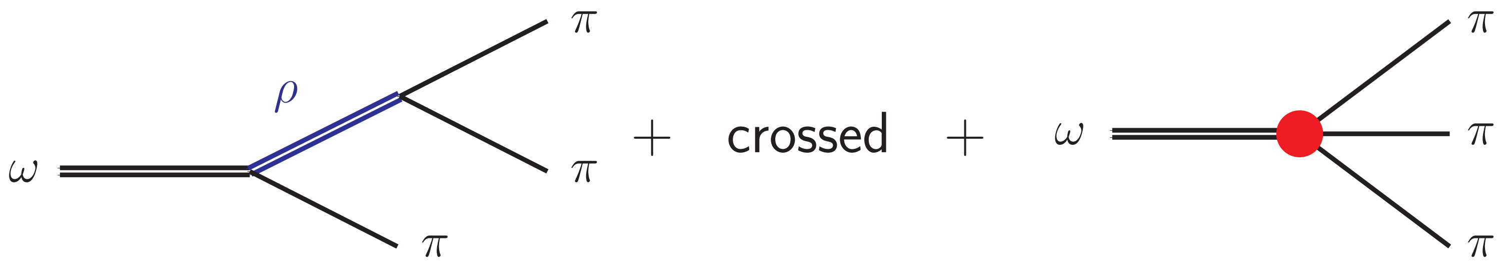



$\longrightarrow$ third-particle effect?? compared to Aston et al. 1988

# Simpler three-body decays: $\omega/\phi \to 3\pi$

**Naive:** sum of 3 Breit–Wigners ($\rho^+$, $\rho^-$, $\rho^0$)

+ constant background term



## Simpler three-body decays: $\omega/\phi \to 3\pi$

**Decay amplitude** can be decomposed into single-variable functions

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu} p^{\nu}_{\pi^+} p^{\alpha}_{\pi^-} p^{\beta}_{\pi^0} \left\{\mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)\right\}$$

$\longrightarrow$ only odd partial waves; neglect F-waves and higher

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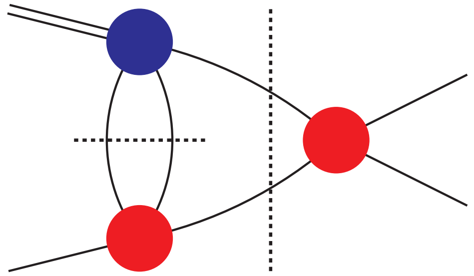

- inhomogeneities $\hat{\mathcal{F}}(s)$: partial-wave projections of $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s) = a\,\Omega(s)\left\{1+\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right\}$$

$$\hat{\mathcal{F}}(s) = \frac{3}{2}\int_{-1}^{1}dz\,(1-z^2)\mathcal{F}\big(t(s,z)\big)$$

Anisovich, Leutwyler 1998

$\longrightarrow$ left- and right-hand cuts iterated self-consistently

# Three-body decays: analytic structure

$$\mathcal{F}(s) = a\,\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\,\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s-i\epsilon)}\right\}$$

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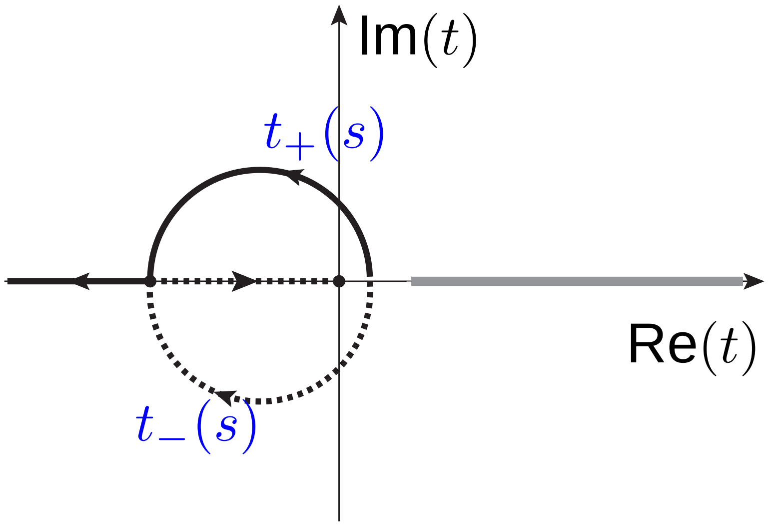
- complication:
  analytic continuation in
  decay mass $M_V$ required
- $M_V < 3M_\pi$:
  okay



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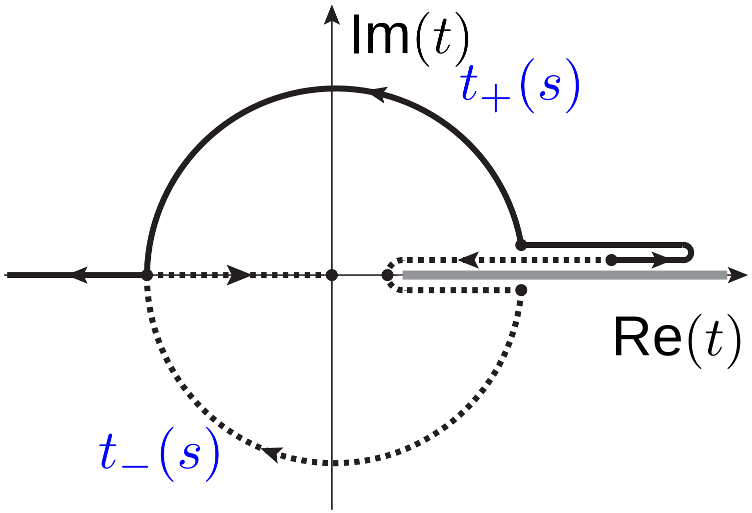


Khuri, Treiman 1960

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- $M_V > 3M_\pi$:
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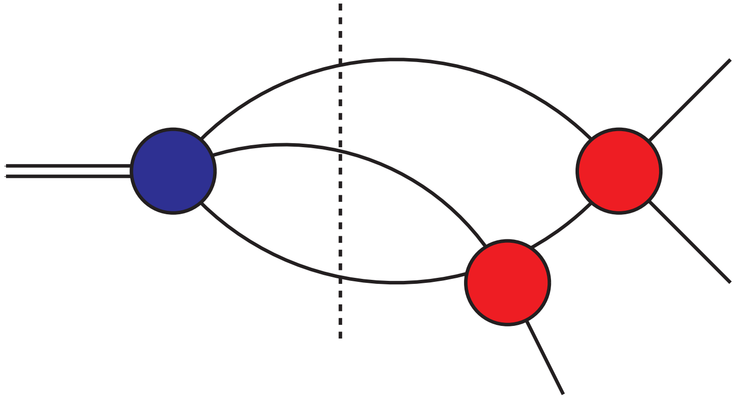
  $\longrightarrow$ generates 3-particle cuts
  $\longrightarrow$ no simple phase relation



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# Crossed-channel rescattering

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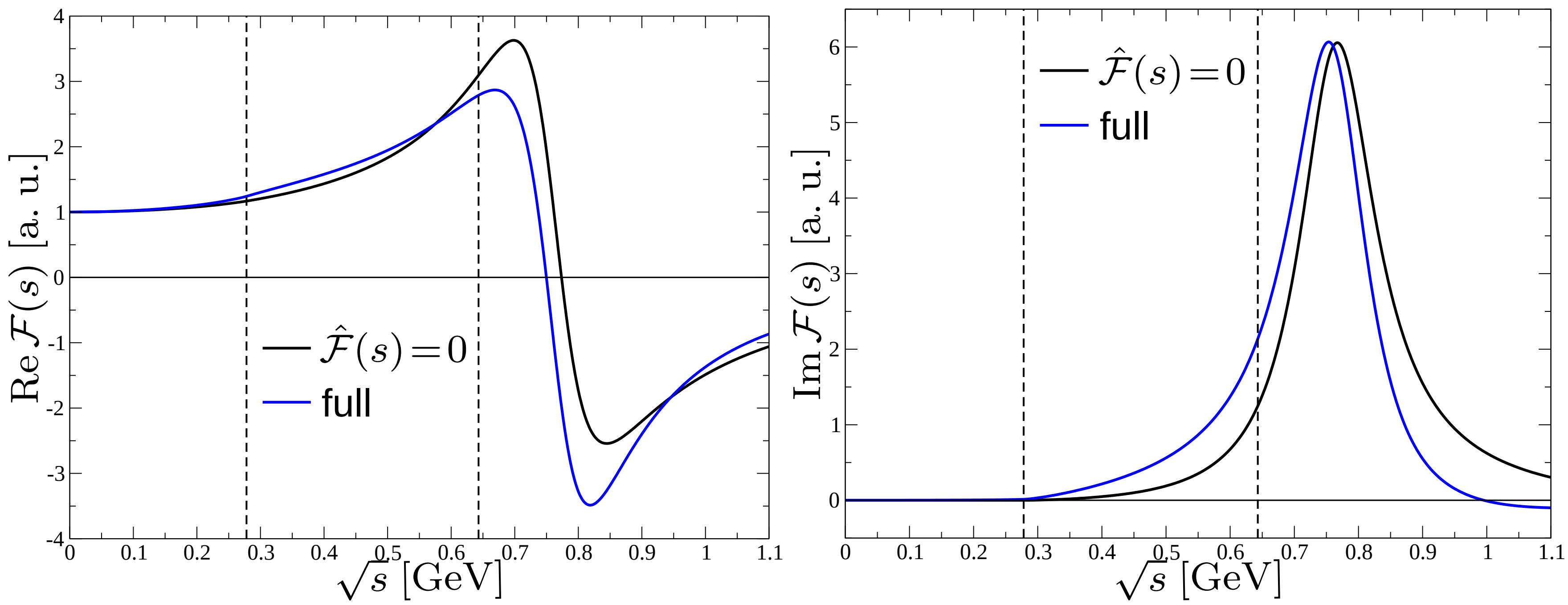
- $\omega \to 3\pi$ vs. $\phi \to 3\pi$: crossed-channel effects depend on decay mass!




Niecknig, BK, Schneider 2012

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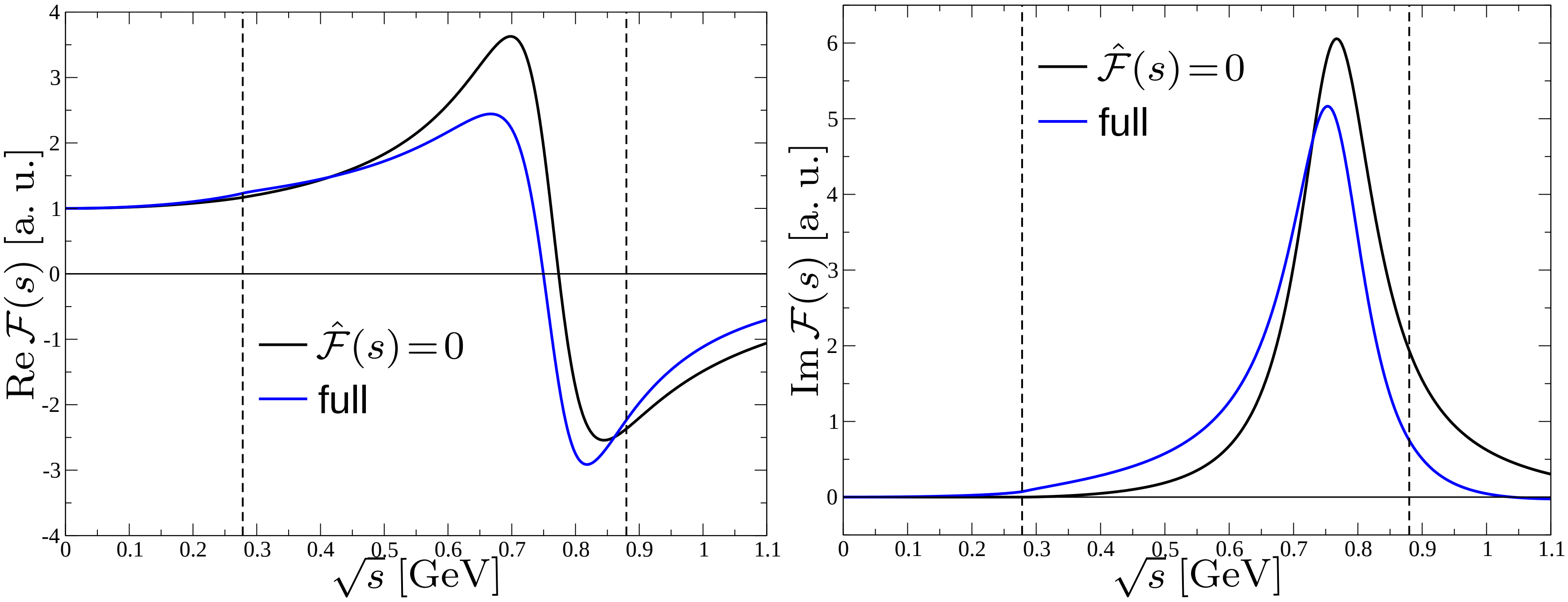


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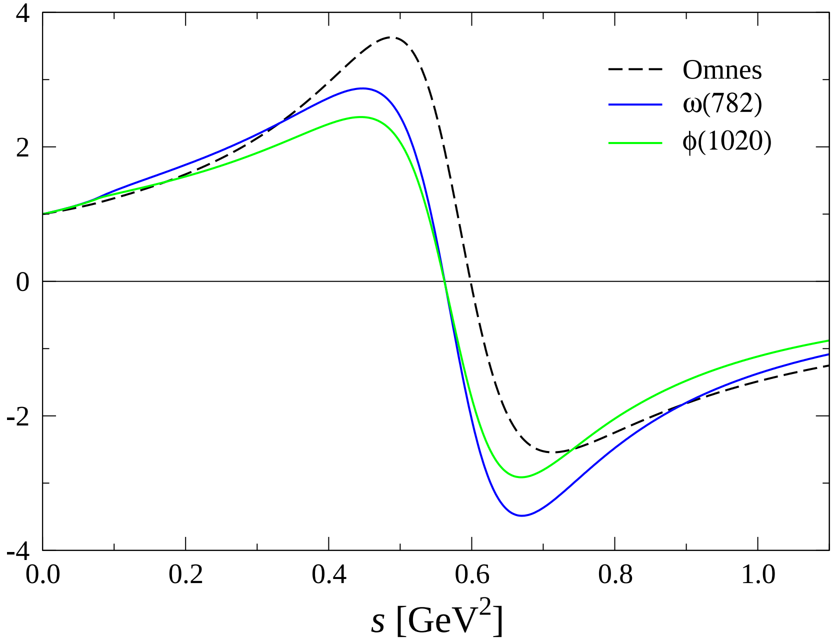
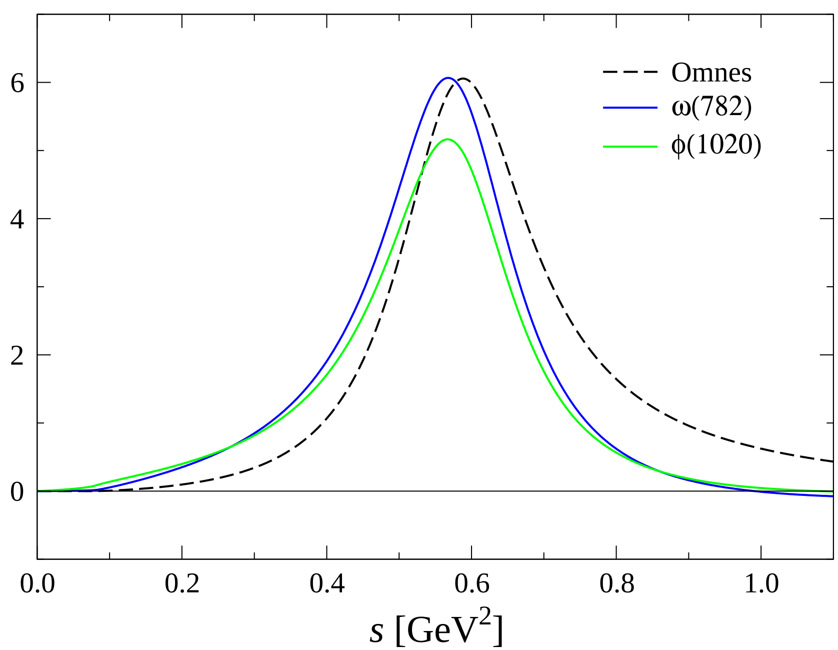
**Extension to higher energies: $D,\ B \to 3\pi$ etc.?**

- well-defined high-energy limit of Khuri–Treiman amplitudes:

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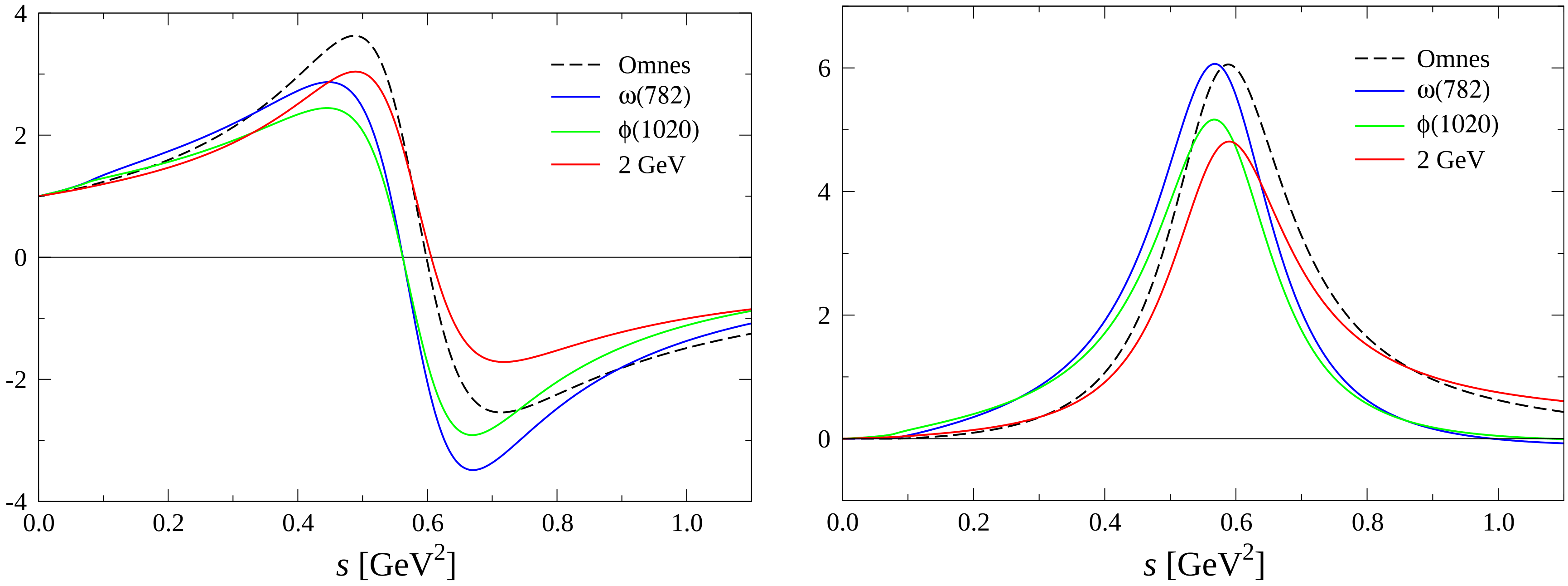


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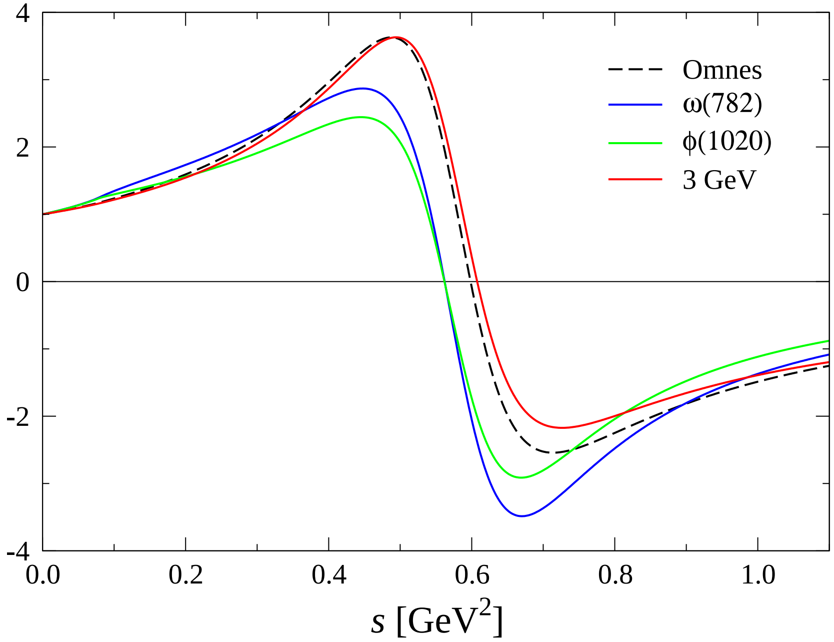


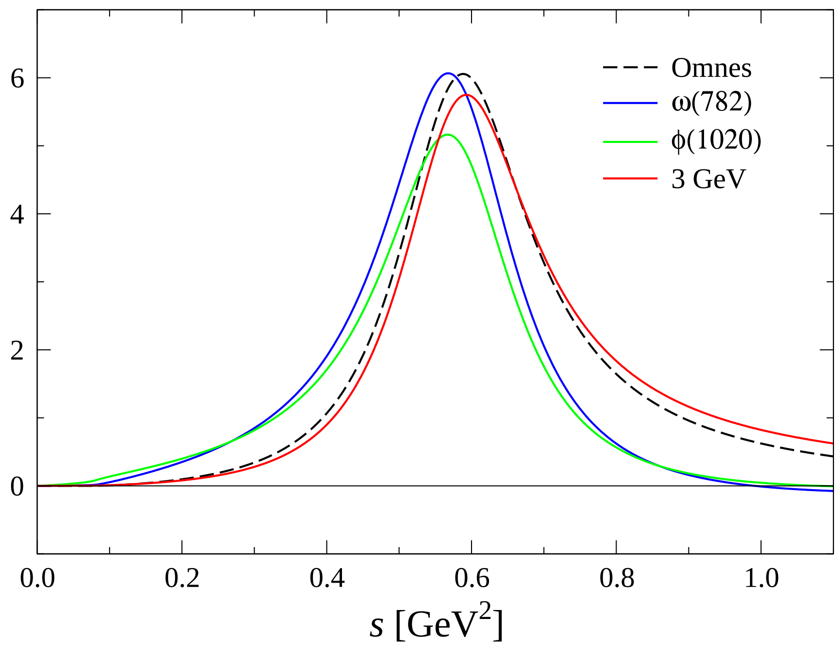


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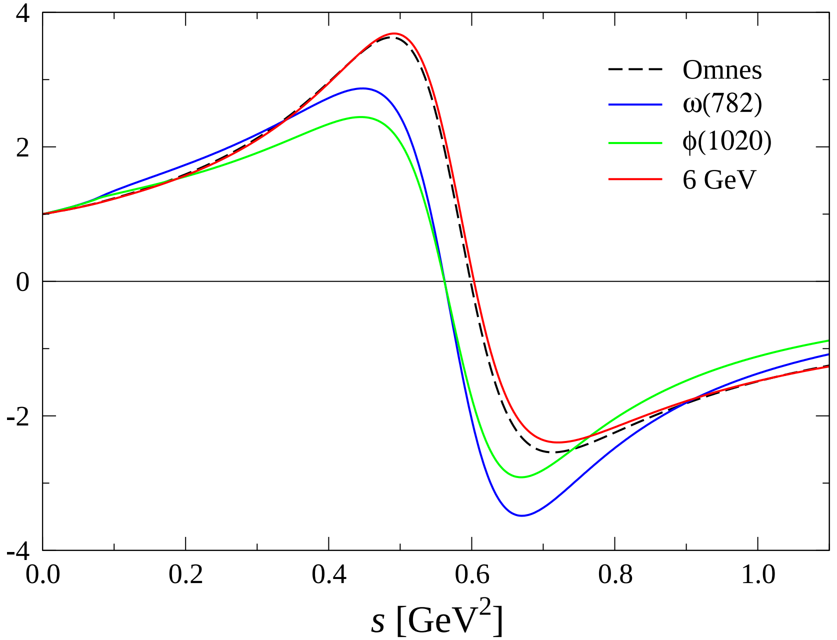


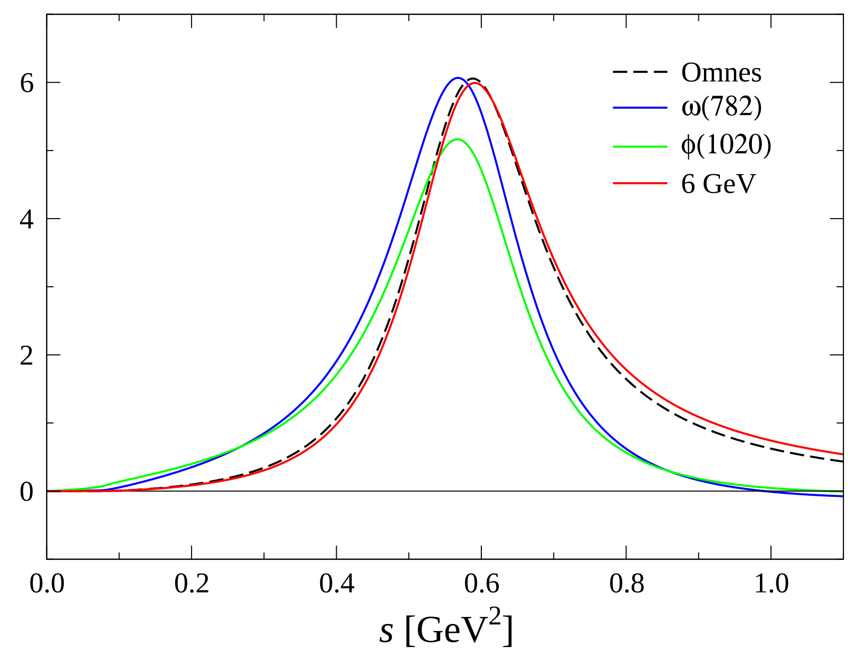


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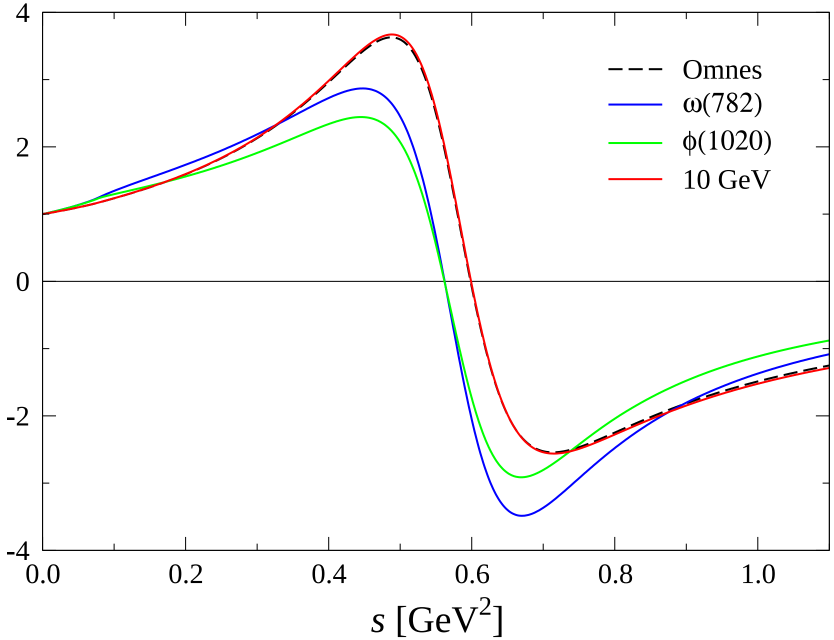


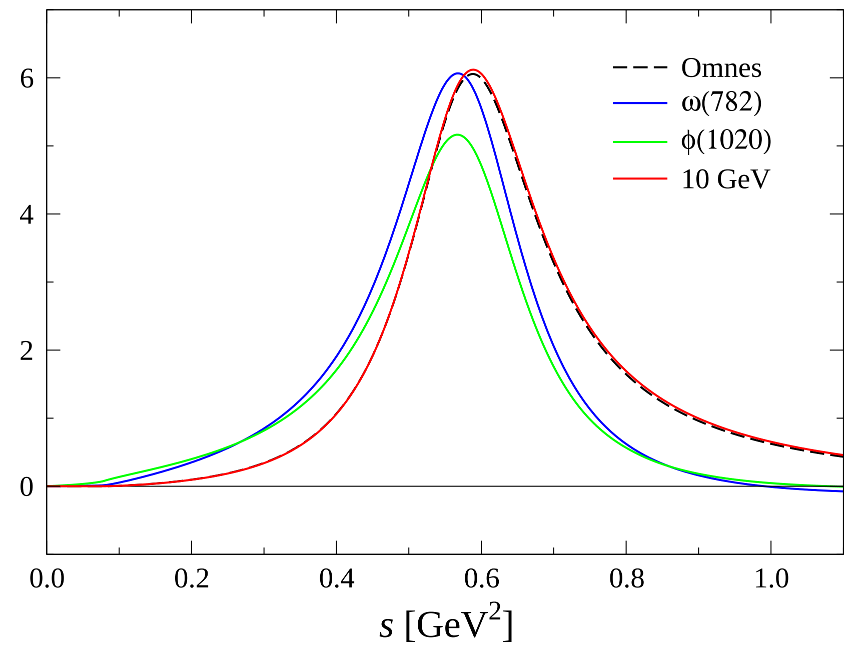


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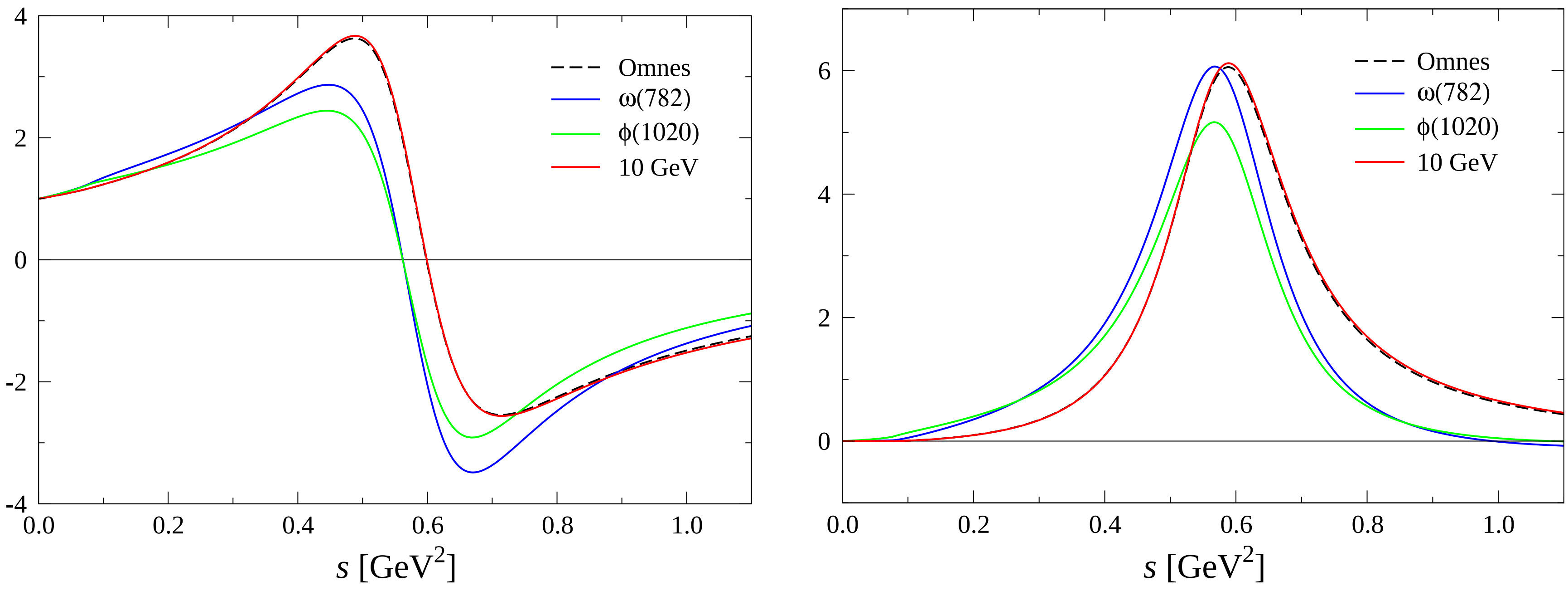
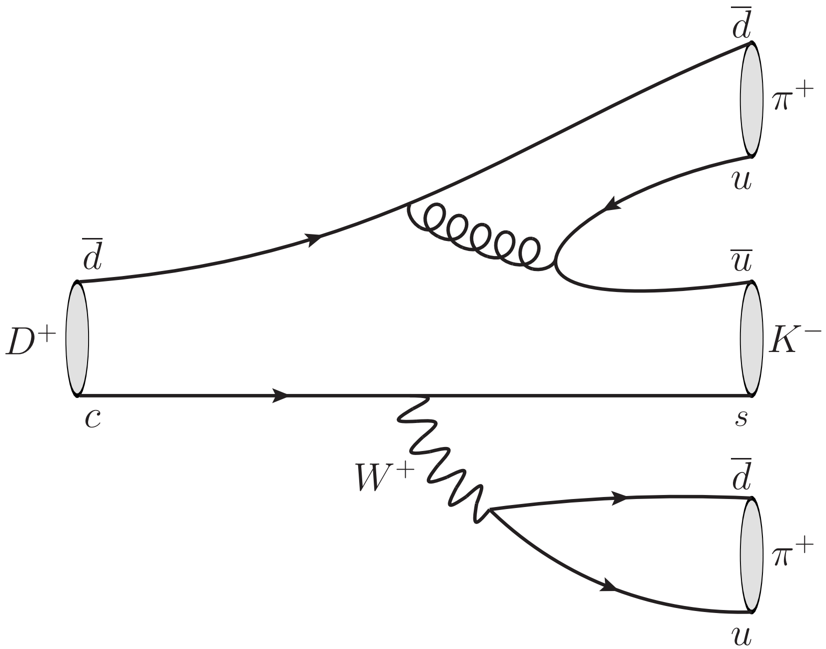



Niecknig, BK

$\longrightarrow$ third-particle interaction vanishes at high decay masses

$\longrightarrow$ warning: naive continuation, inelastic effects neglected. . .

# Heavier decays: $D^+ \to \bar{K}\pi\pi^+$




- Cabibbo-favoured decays, several experiments with good statistics for $D^+ \to K^-\pi^+\pi^+$ E791 2006, CLEO 2008, FOCUS 2009
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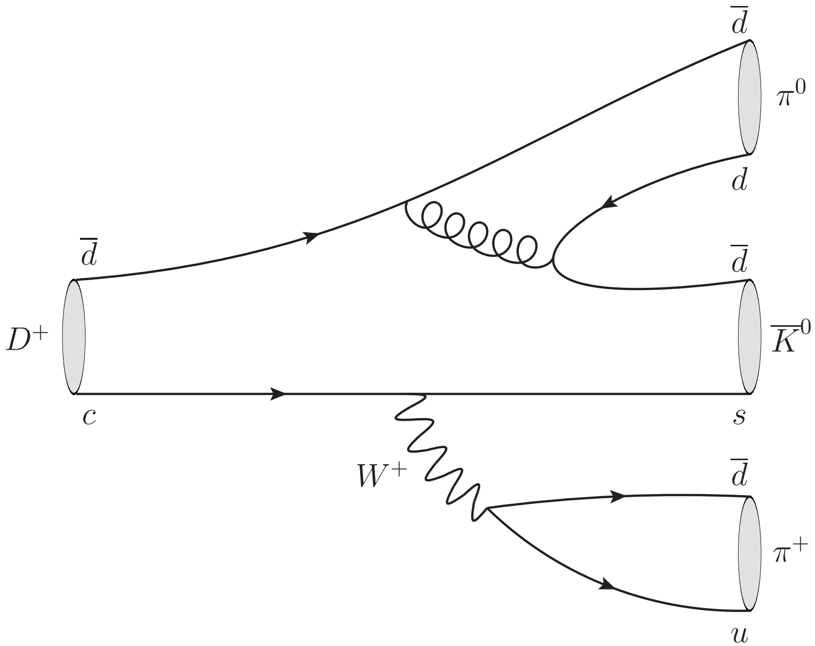


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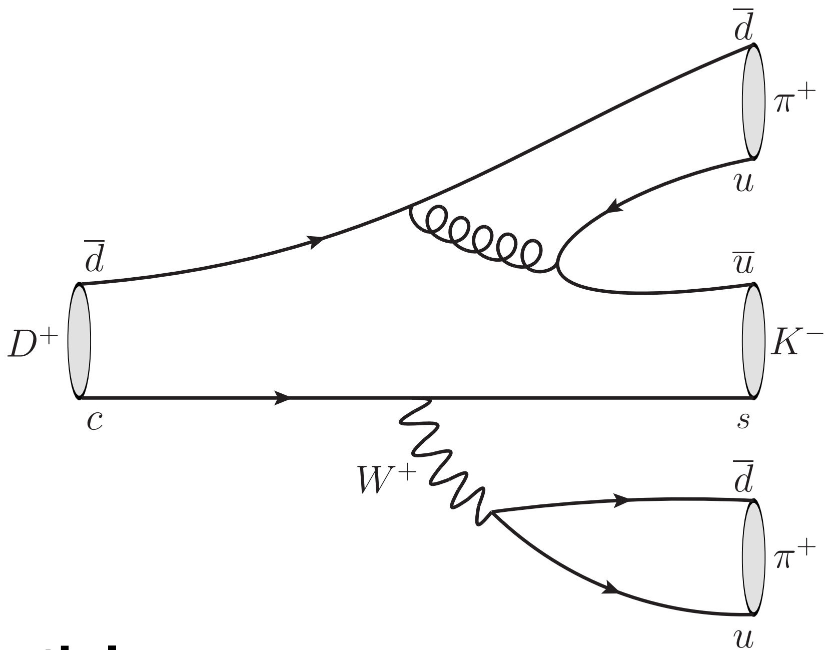


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**Partial waves:**

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|---|---|---|
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| pion–kaon | $S^{1/2}_{\pi K}$ | $S^{3/2}_{\pi K}$ |
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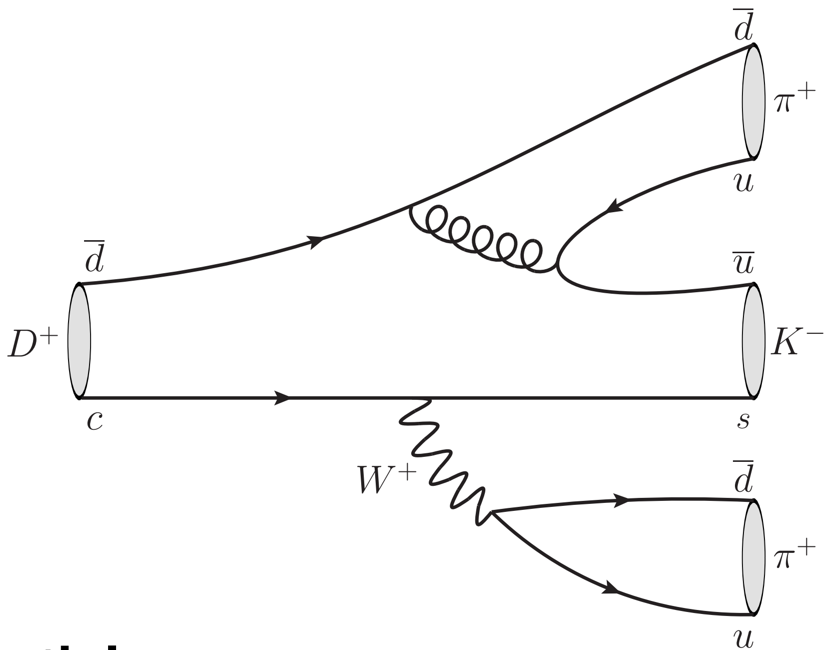


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$\longrightarrow$ exotic partial waves, weak, repulsive

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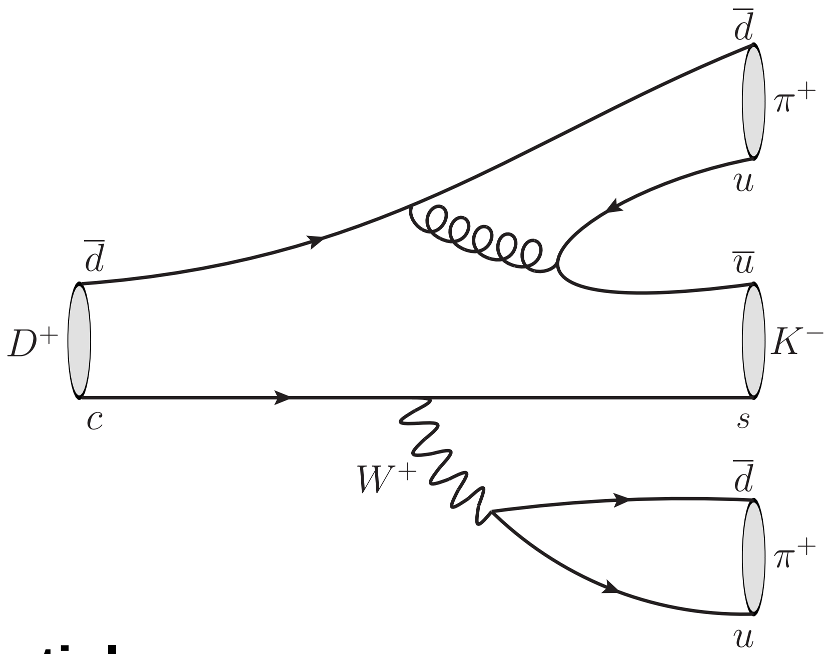


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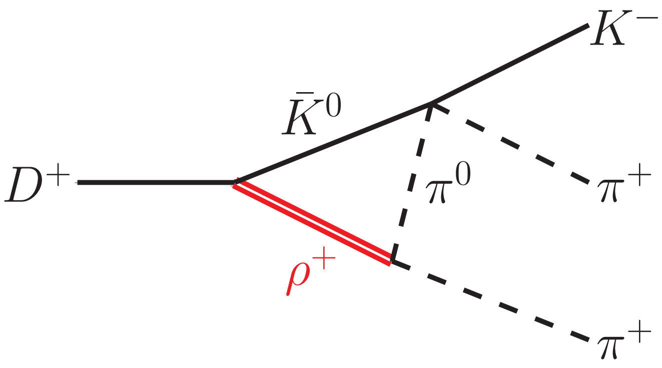


$\longrightarrow$ $\pi\pi$ P-wave only couples indirectly via $D^+ \to \bar{K}^0\pi^0\pi^+$

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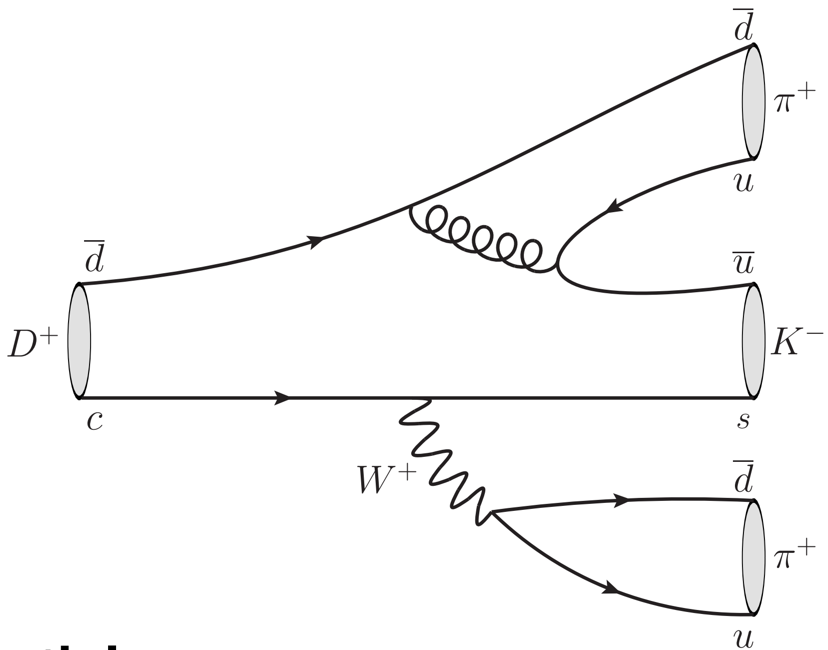


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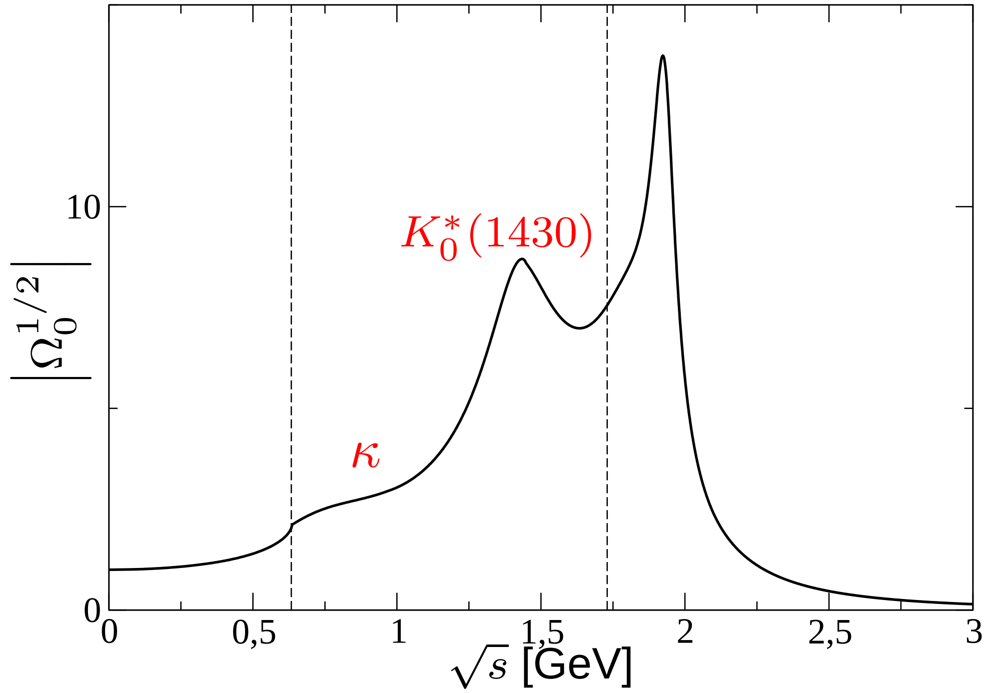

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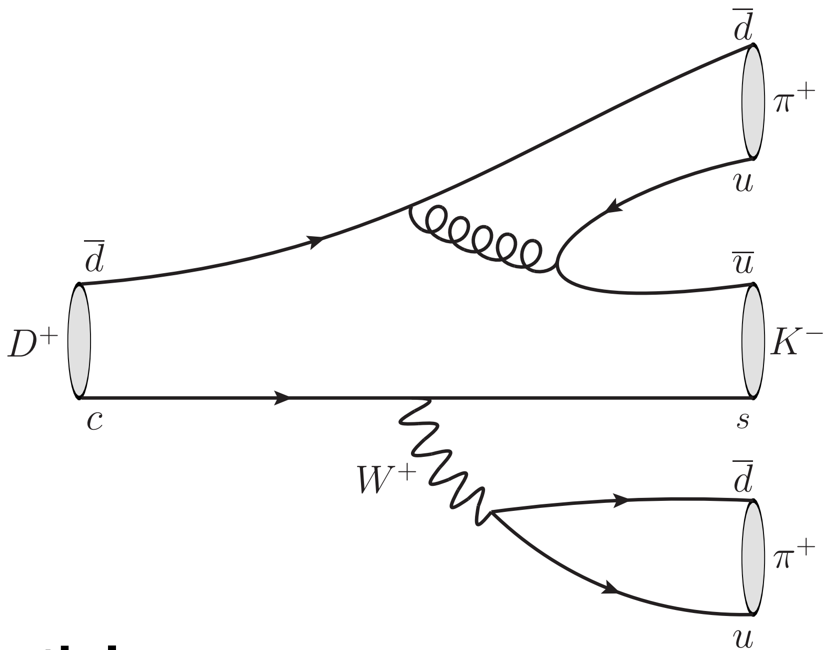


- Cabibbo-favoured decays, several experiments with good statistics for $D^+ \to K^-\pi^+\pi^+$ E791 2006, CLEO 2008, FOCUS 2009
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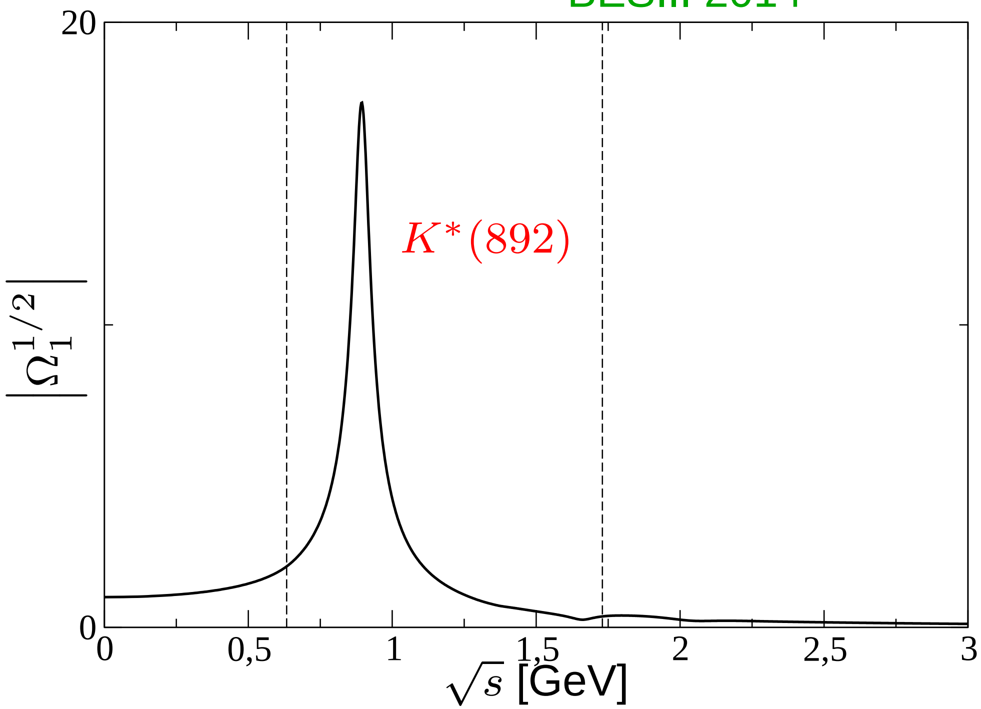


**Partial waves:**

| | | |
|---|---|---|
| pion–pion | $P^1_{\pi\pi}$ | $S^2_{\pi\pi}$ |
| pion–kaon | $S^{1/2}_{\pi K}$ | $S^{3/2}_{\pi K}$ |
| | $P^{1/2}_{\pi K}$ | $P^{3/2}_{\pi K}$ |
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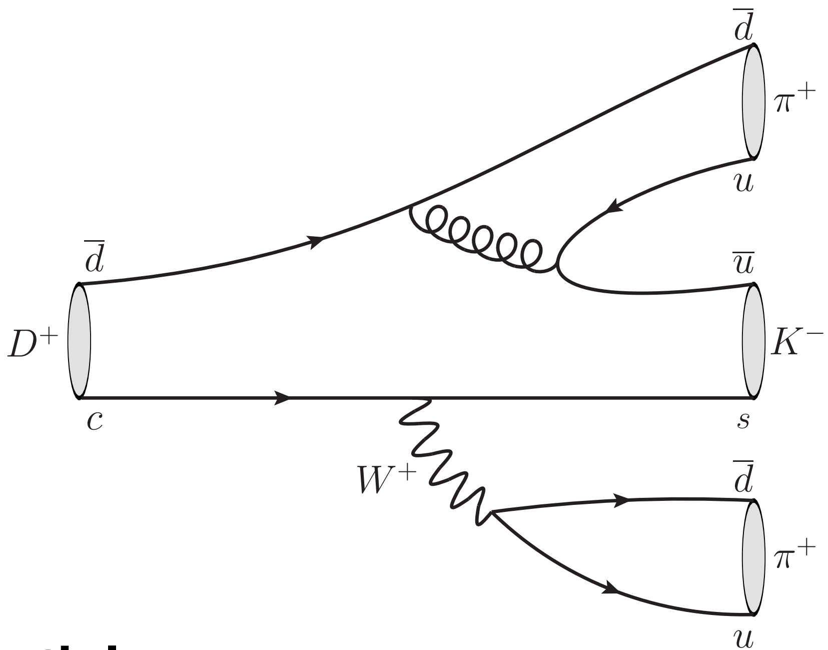


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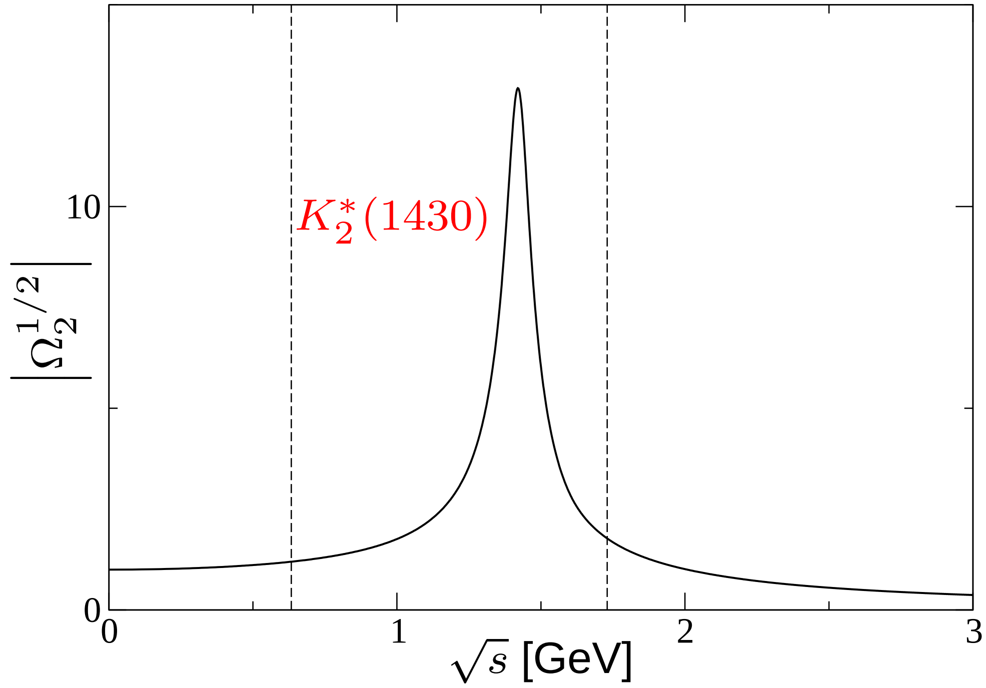


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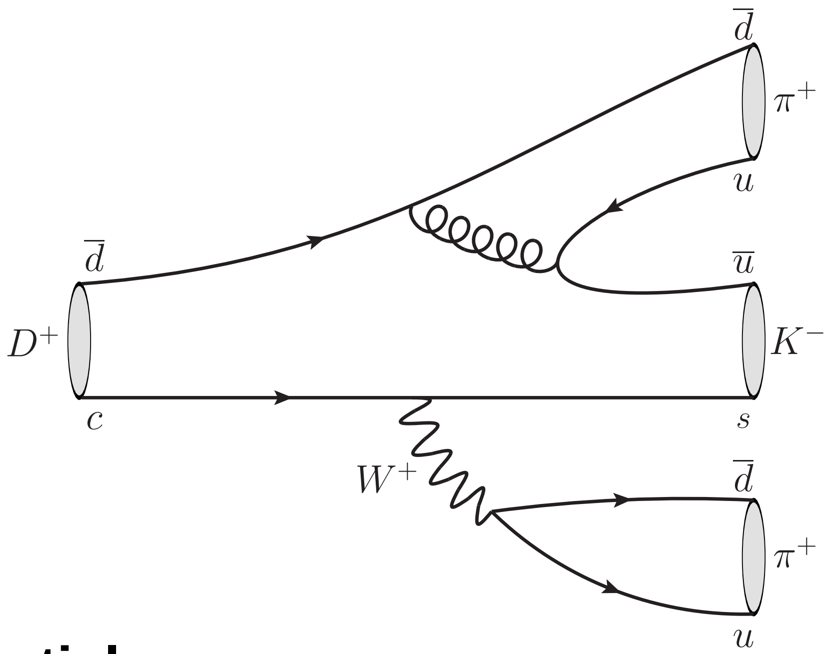


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**Partial waves:**

| | | | | |
|---|---|---|---|---|
| pion–pion | $P^1_{\pi\pi}$ | 2 | $S^2_{\pi\pi}$ | 2 $\to$ 0 |
| pion–kaon | $S^{1/2}_{\pi K}$ | 4 | $S^{3/2}_{\pi K}$ | 2 $\to$ 0 |
| | $P^{1/2}_{\pi K}$ | 1 | $P^{3/2}_{\pi K}$ | 0 |
| | ($D^{1/2}_{\pi K}$) | 0 | | |

# subtraction constants $\longrightarrow$ using $s + t + u =$ const. $\longrightarrow$ 7 altogether

# Full system of dispersion relations

$$\mathcal{F}_0^2(u) = \Omega_0^2(u)\frac{u^2}{\pi}\int_{u_{\rm th}}^\infty \frac{{\rm d}u'}{u'^2}\frac{\hat{\mathcal{F}}_0^2(u')\sin\delta_0^2(u')}{|\Omega_0^2(u')|\,(u'-u)}$$

$$\mathcal{F}_1^1(u) = \Omega_1^1(u)\bigg\{c_0 + c_1 u + \frac{u^2}{\pi}\int_{u_{\rm th}}^\infty \frac{{\rm d}u'}{u'^2}\frac{\hat{\mathcal{F}}_1^1(u')\sin\delta_1^1(u')}{\big|\Omega_1^1(u')\big|(u'-u)}\bigg\}$$

$$\mathcal{F}_0^{1/2}(s) = \Omega_0^{1/2}(s)\bigg\{c_2 + c_3 s + c_4 s^2 + c_5 s^3 + \frac{s^4}{\pi}\int_{s_{\rm th}}^\infty \frac{{\rm d}s'}{s'^4}\frac{\hat{\mathcal{F}}_0^{1/2}(s')\sin\delta_0^{1/2}(s')}{\big|\Omega_0^{1/2}(s')\big|(s'-s)}\bigg\}$$

$$\mathcal{F}_0^{3/2}(s) = \Omega_0^{3/2}(s)\bigg\{\frac{s^2}{\pi}\int_{s_{\rm th}}^\infty \frac{{\rm d}s'}{s'^2}\frac{\hat{\mathcal{F}}_0^{3/2}(s')\sin\delta_0^{3/2}(s')}{\big|\Omega_0^{3/2}(s')\big|(s'-s)}\bigg\}$$

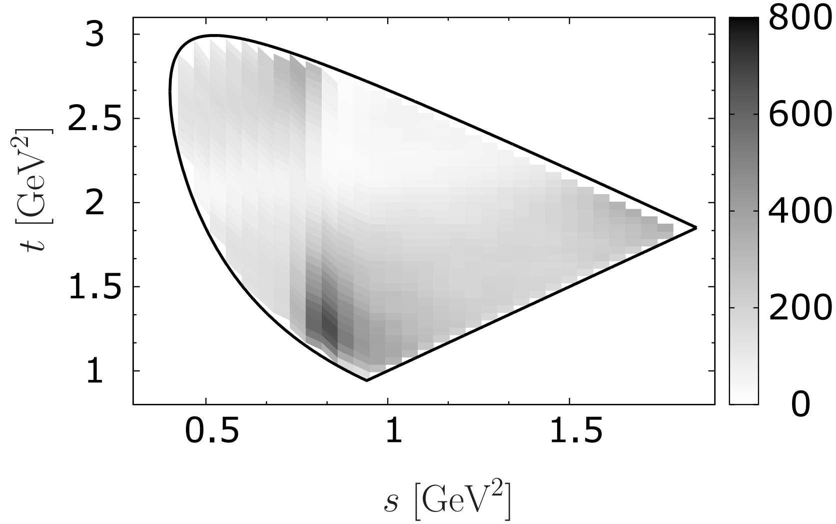
$$\mathcal{F}_1^{1/2}(s) = \Omega_1^{1/2}(s)\bigg\{c_6 + \frac{s}{\pi}\int_{s_{\rm th}}^\infty \frac{{\rm d}s'}{s'}\frac{\hat{\mathcal{F}}_1^{1/2}(s')\sin\delta_1^{1/2}(s')}{\big|\Omega_1^{1/2}(s')\big|(s'-s)}\bigg\}$$

$$\mathcal{F}_2^{1/2}(s) = \Omega_2^{1/2}(s)\frac{1}{\pi}\int_{s_{\rm th}}^\infty {\rm d}s'\frac{\hat{\mathcal{F}}_2^{1/2}(s')\sin\delta_2^{1/2}(s')}{\big|\Omega_2^{1/2}(s')\big|(s'-s)}$$

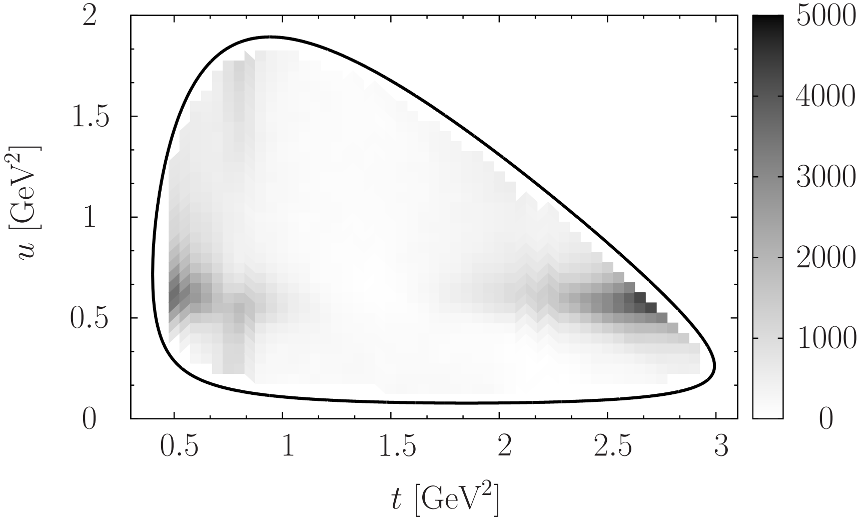
- *D*-wave contributions to the inhomogeneities neglected
- system linear in subtractions: $\mathcal{M}(s,t,u) = \sum_{i=0}^{6} c_i \underbrace{\mathcal{M}_i(s,t,u)}_{\text{basis functions}}$

# Dalitz plots $D^+ \to K^-\pi^+\pi^+$ / $D^+ \to \bar{K}^0\pi^0\pi^+$

$D^+ \to K^-\pi^+\pi^+$ CLEO 2008




$D^+ \to \bar{K}_S\pi^0\pi^+$ BESIII 2014



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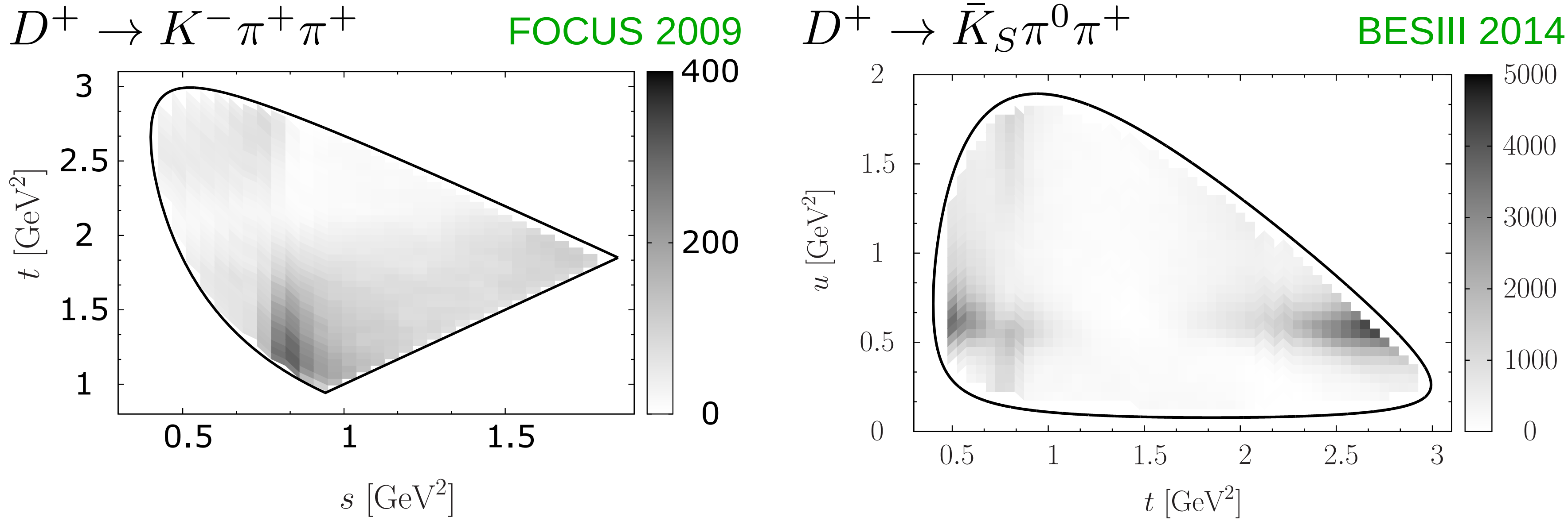


- prominent $K^*(892)$ and $\rho(770)$ (in $K_S\pi^0\pi^+$) signals
- phase space limit: $\sqrt{s},\ \sqrt{t} \leq M_D - M_\pi \approx 1.73\,\mathrm{GeV}$

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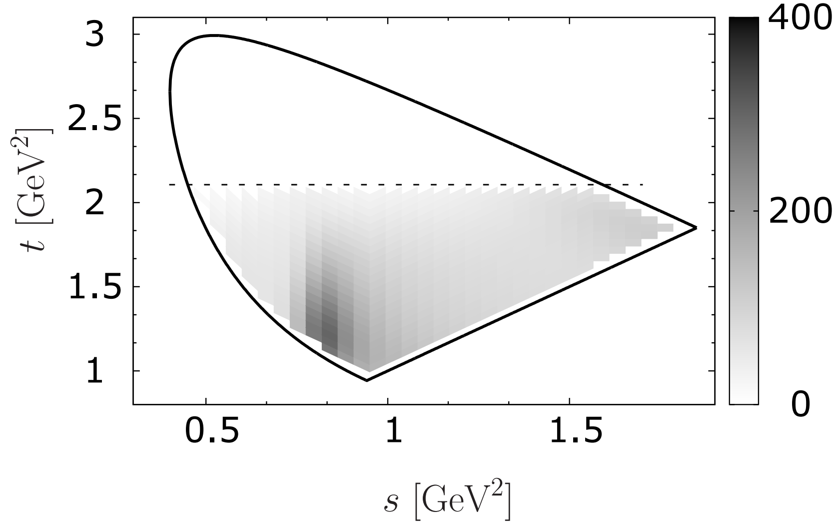


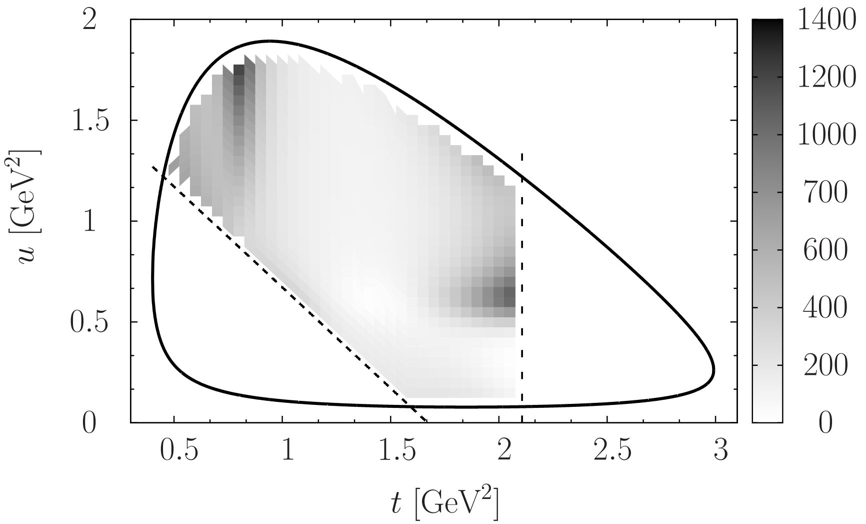


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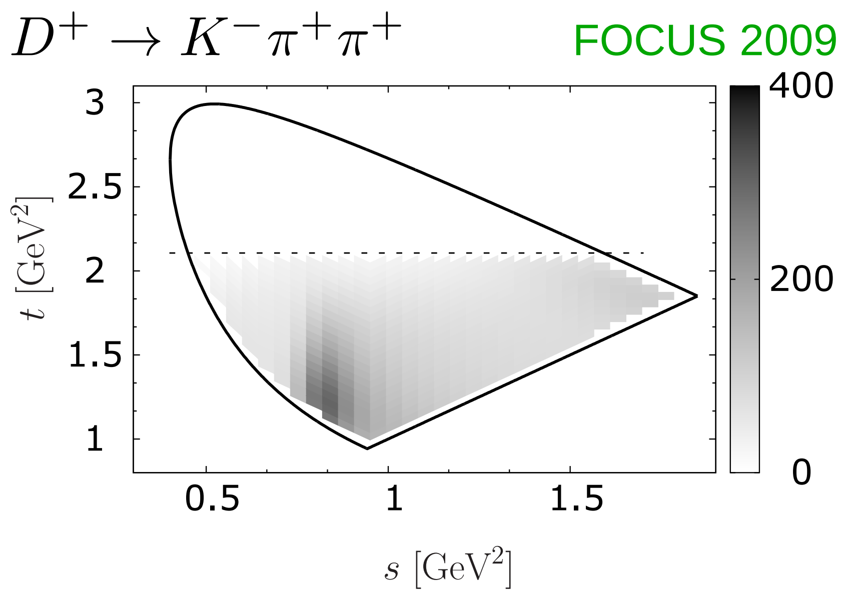


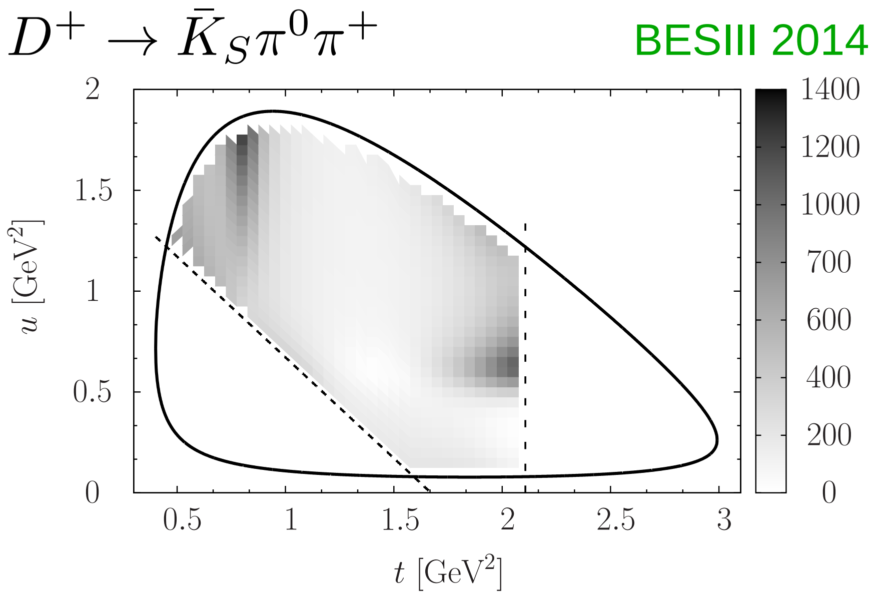


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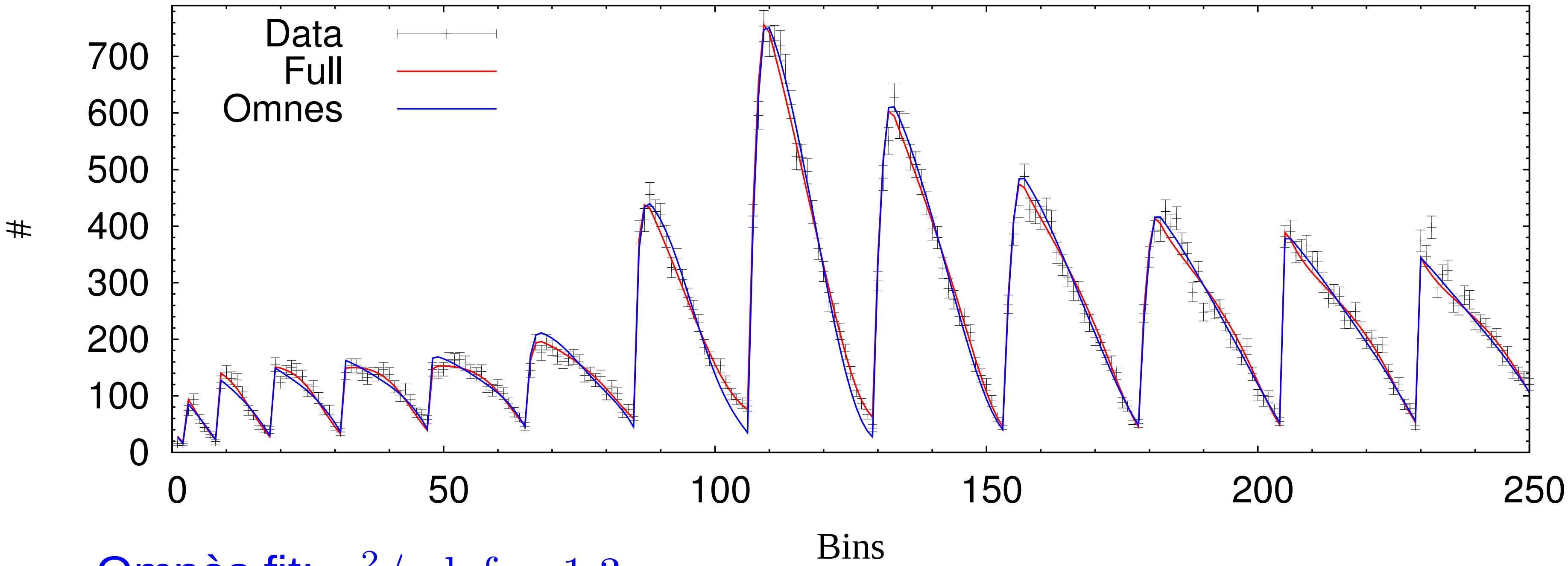
- fits to CLEO and FOCUS $D^+ \to K^-\pi^+\pi^+$ — Niecknig, BK 2015
- BESIII $D^+ \to \bar{K}_S\pi^0\pi^+$ and combined fits — Niecknig, BK 2017

# (Slices through) Dalitz plot $D^+ \to K^-\pi^+\pi^+$




- Omnès fit: $\chi^2/\mathrm{ndof} \approx 1.3$

  ("isobar model" + non-resonant background waves)

  $\longrightarrow$ but: very implausible fit fractions (huge cancellations)

- full dispersive solution: $\chi^2/\mathrm{ndof} \approx 1.2$

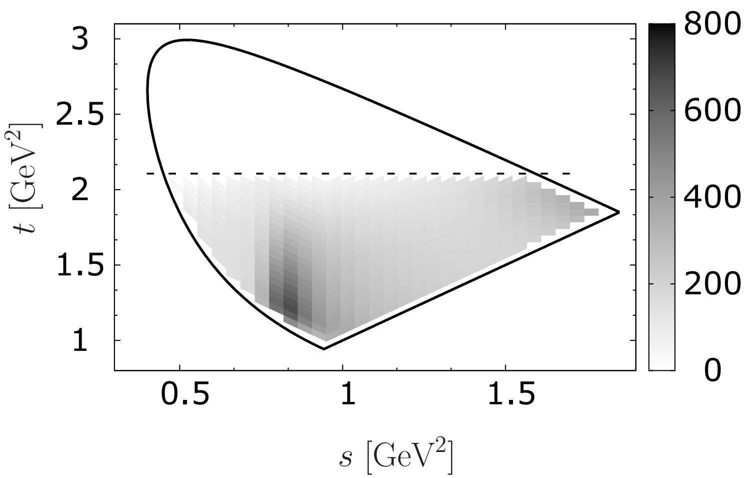
- including $D^{1/2}_{\pi K}$ wave (no add. parameter) $\longrightarrow \chi^2/\mathrm{ndof} \approx 1.1$

- 7 complex subtraction constants –1 phase –1 normalisation

Niecknig, BK 2015

# $D^+ \to K^-\pi^+\pi^+$ fits

theoretical Dalitz plot



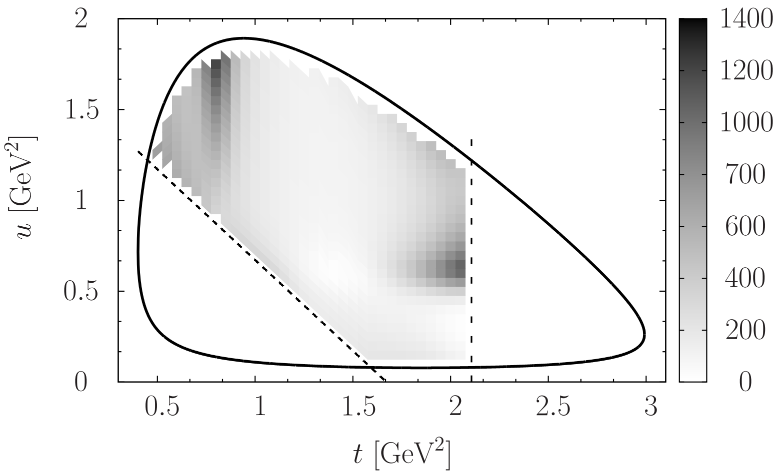

fit fractions

| CLEO | without *D*-wave | with *D*-wave |
|---|---|---|
| $\mathcal{F}_0^2$ | $(37 \pm 23)\%$ | $(8 \pm 3)\%$ |
| $\mathcal{F}_0^{1/2}$ | $(190 \pm 60)\%$ | $(72 \pm 12)\%$ |
| $\mathcal{F}_1^{1/2}$ | $(11 \pm 3)\%$ | $(10 \pm 2)\%$ |
| $\mathcal{F}_0^{3/2}$ | $(65 \pm 35)\%$ | $(16 \pm 3)\%$ |
| $\mathcal{F}_2^{1/2}$ | – | $(0.1 \pm 0.05)\%$ |
| $\chi^2$/dof | $1.2$ | $1.1$ |

- without *D*-wave:
  large destructive interference between $\pi K$ *S*-waves
- with *D*-wave: no additional parameter;
  small, but large effects on other fit fractions
- very similar FOCUS fit results
- uncertainties include input phase shift error estimates

## $D^+ \to K_S \pi^0 \pi^+$ fits

theoretical Dalitz plot




fit fractions

| BESIII | without $D$-wave | with $D$-wave |
|---|---|---|
| $\mathcal{F}_0^2$ | $(5 \pm 2)\%$ | $(5 \pm 0.3)\%$ |
| $\mathcal{F}_1^1$ | $(21 \pm 5)\%$ | $(16 \pm 3)\%$ |
| $\mathcal{F}_0^{1/2}$ | $(39 \pm 5)\%$ | $(43 \pm 4)\%$ |
| $\mathcal{F}_1^{1/2}$ | $(9 \pm 0.5)\%$ | $(7 \pm 2)\%$ |
| $\mathcal{F}_0^{3/2}$ | $(6 \pm 2)\%$ | $(9 \pm 3)\%$ |
| $\mathcal{F}_2^{1/2}$ | – | $(1.5 \pm 0.05)\%$ |
| $\chi^2$/dof | $1.27$ | $1.35$ |

- similar fit fractions with and without *D*-wave
- BESIII fit constrains subtraction constants well
- *but:* fitted constants not well compatible with $D^+ \to K^- \pi^+ \pi^+$
  combined fits with bad $\chi^2$/dof

# Combined fits

## Combined CLEO/FOCUS/BESIII data fit

- without *D*-wave: $\chi^2/\text{dof} = 1.7 \pm 0.1$
- with (parameter-free) *D*-wave: $\chi^2/\text{dof} = 2.5 \pm 0.1$ ???
- oversubtract *D*-wave:

$$\mathcal{F}_2^{1/2}(s) \to \mathcal{F}_2^{1/2}(s) + c_7\,\Omega_2^{1/2}(s)$$

- *D*-wave more flexible independent coupling
- combined $\chi^2/\text{dof} \approx 1.2$ comparable to individual experimental fits

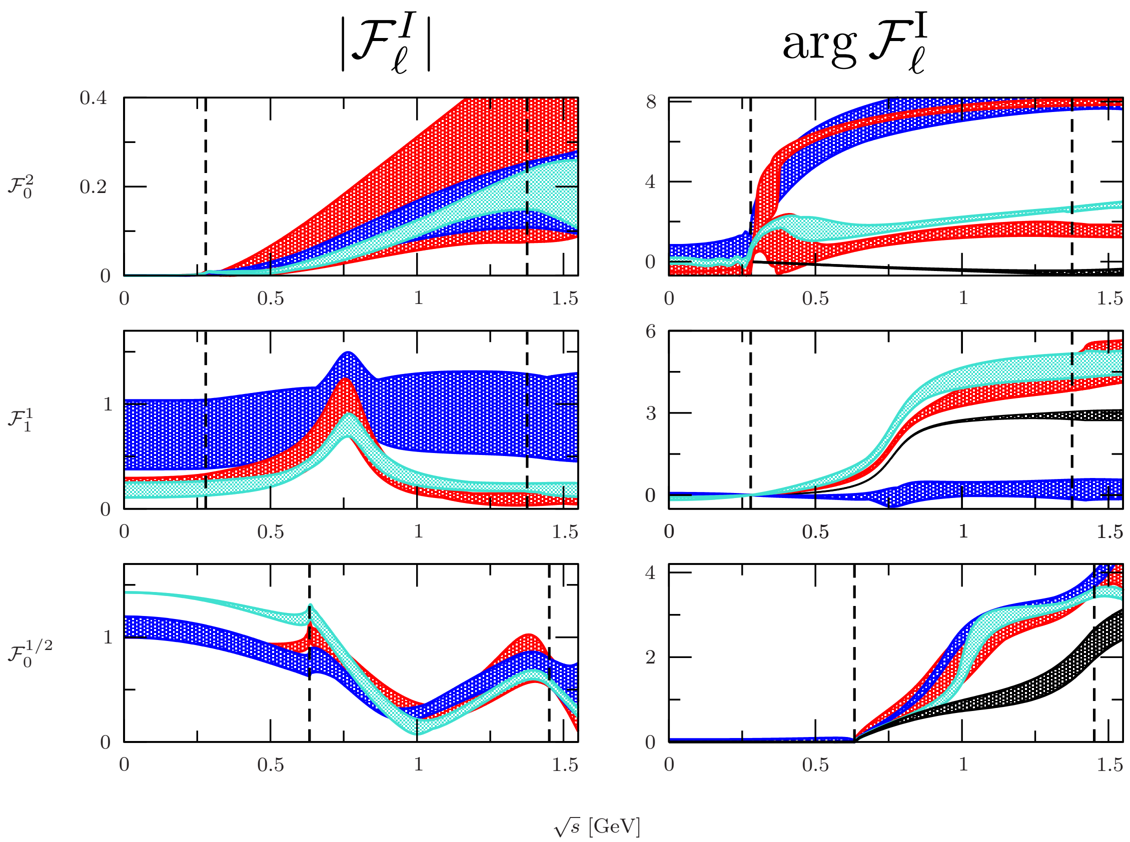
FOCUS 2009, BESIII 2014

fit fractions

| | $K^-\pi^+\pi^+$ | $\bar{K}^0\pi^0\pi^+$ |
|---|---|---|
| $\mathcal{F}_0^2$ | $(3 \pm 2)\%$ | $(1 \pm 0.3)\%$ |
| $\mathcal{F}_1^1$ | — | $(21 \pm 3)\%$ |
| $\mathcal{F}_0^{1/2}$ | $(53 \pm 3)\%$ | $(41 \pm 3)\%$ |
| $\mathcal{F}_1^{1/2}$ | $(11 \pm 1)\%$ | $(8 \pm 1)\%$ |
| $\mathcal{F}_0^{3/2}$ | $(0.6 \pm 0.1)\%$ | $(7 \pm 1)\%$ |
| $\mathcal{F}_2^{1/2}$ | $(0.3 \pm 0.1)\%$ | $(0.3 \pm 0.1)\%$ |
| $\chi^2/\text{dof}$ | $1.2$ | $1.2$ |

Niecknig, BK 2017

# Amplitudes and phases




black: scatt. phase shifts

red: fit to $D^+ \to K_S \pi^0 \pi^+$

BESIII 2014

blue: fit to $D^+ \to K^- \pi^+ \pi^+$

CLEO 2008, FOCUS 2009

turquoise: combined fit

- two channels with different sensitivity to different amplitudes
- significant differences in phase motion due to three-body rescattering!

Niecknig, BK 2015, 2017

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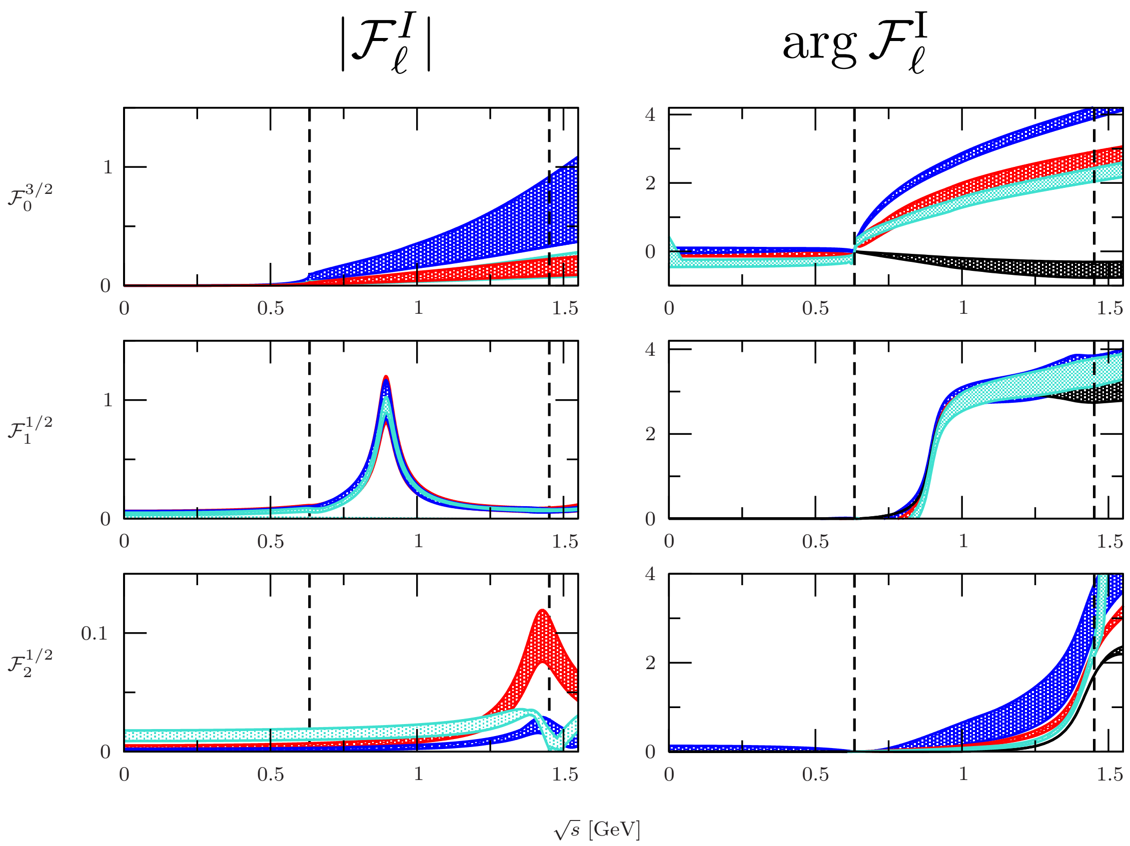


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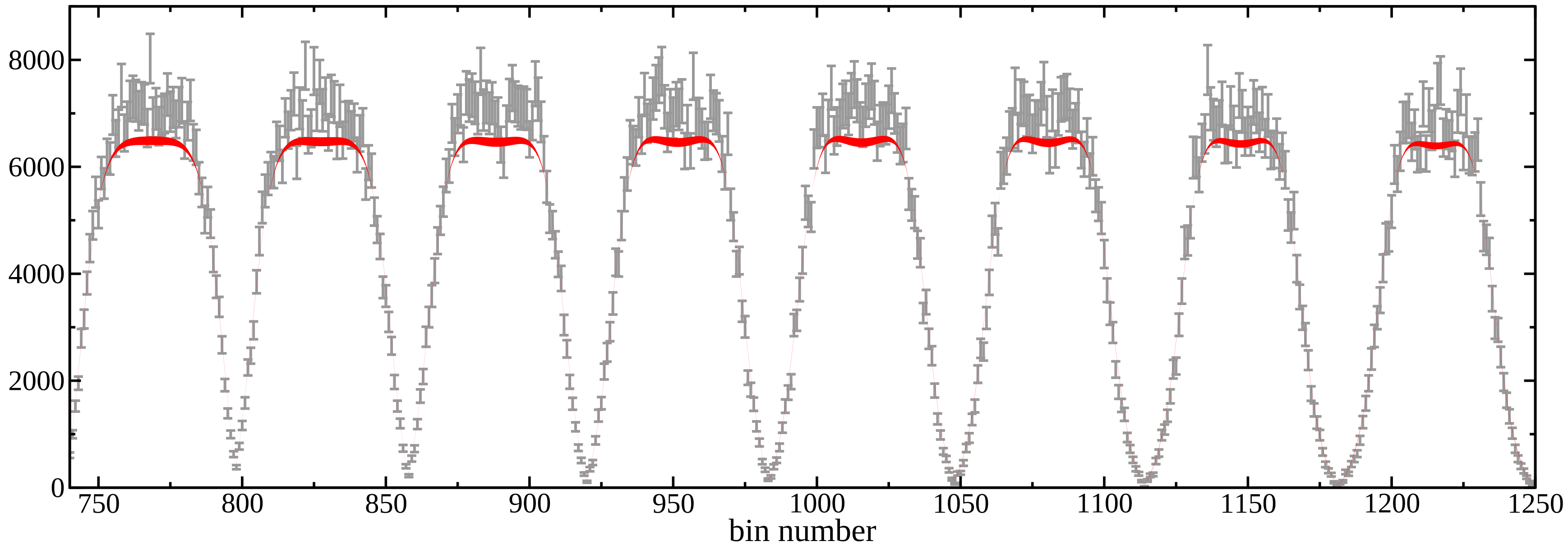
Thank you!

# Spares

# Experimental comparison to $\phi \to 3\pi$

- successive slices through Dalitz plot: Niecknig, BK, Schneider 2012




| | |
|---|---|
| $\chi^2/\mathrm{ndof}$ | $1.7 \ldots 2.1$ |

$\longrightarrow$ pairwise interaction only (with correct $\pi\pi$ scattering phase)

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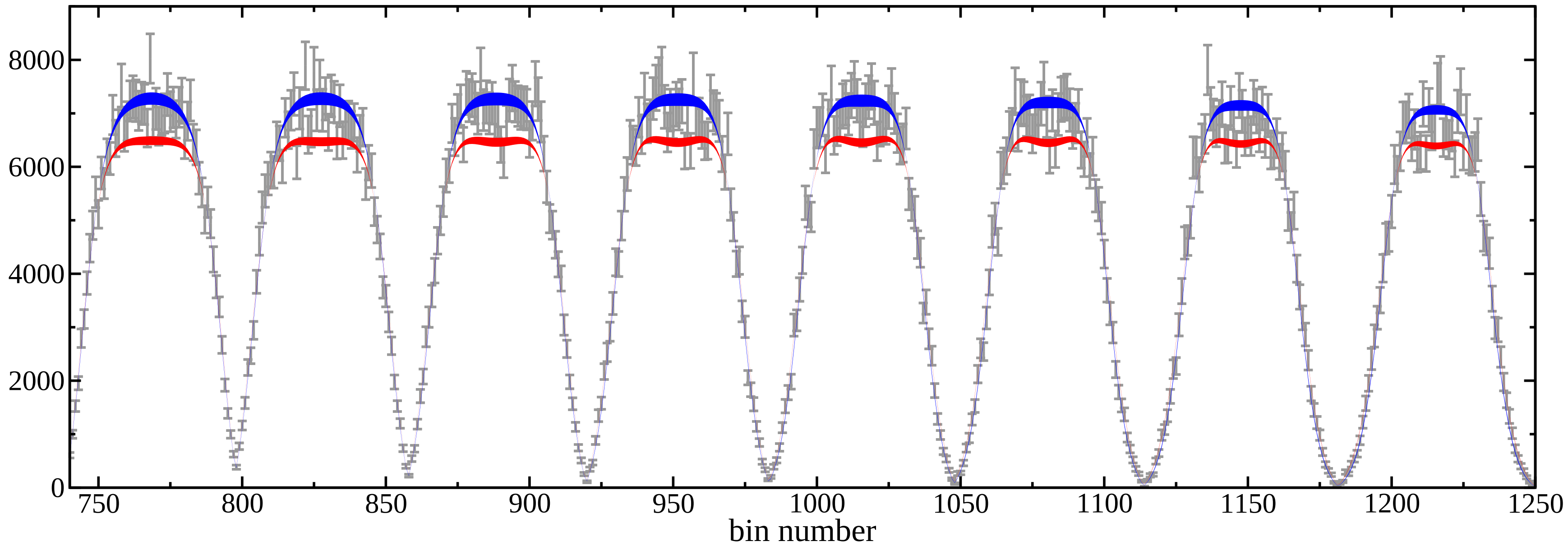


| | | |
|---|---|---|
| $\chi^2/\text{ndof}$ | $1.7 \ldots 2.1$ | $1.2 \ldots 1.5$ |

$\longrightarrow$ full 3-particle rescattering, only overall normalisation adjustable

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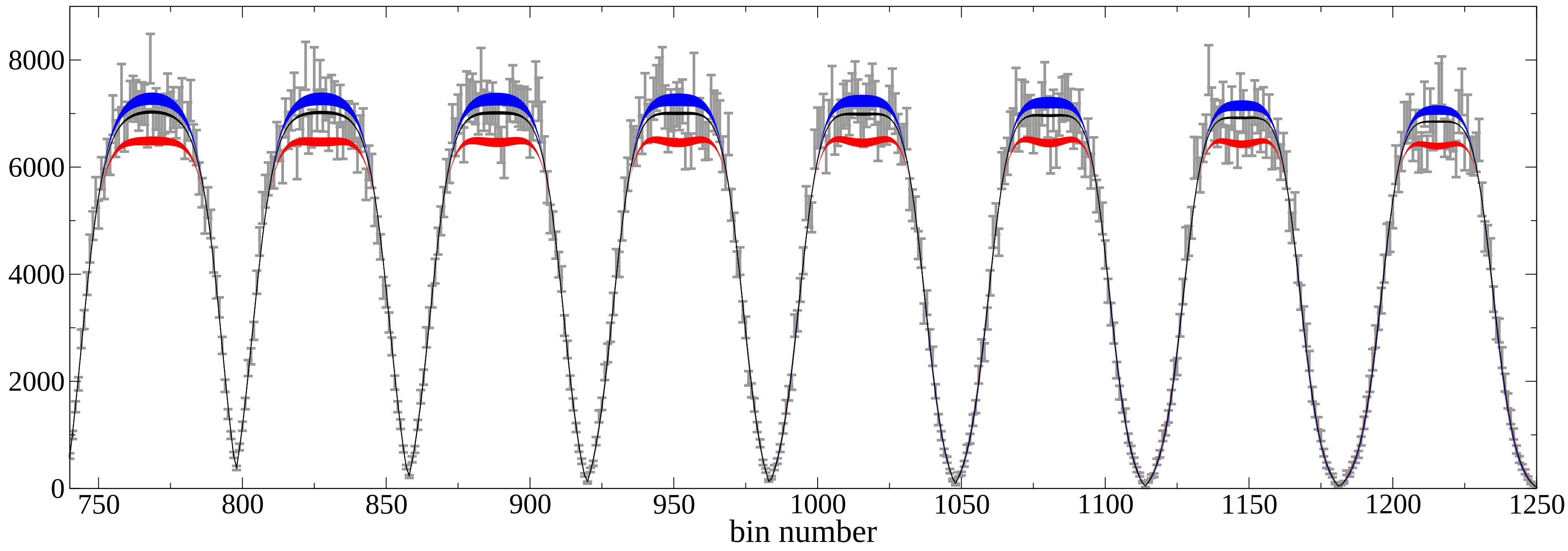


| $\chi^2/\text{ndof}$ | $1.7 \ldots 2.1$ | $1.2 \ldots 1.5$ | $1.0$ |
|---|---|---|---|

$\longrightarrow$ full 3-particle rescattering, 2 adjustable parameters
(additional "subtraction constant" to suppress inelastic effects)

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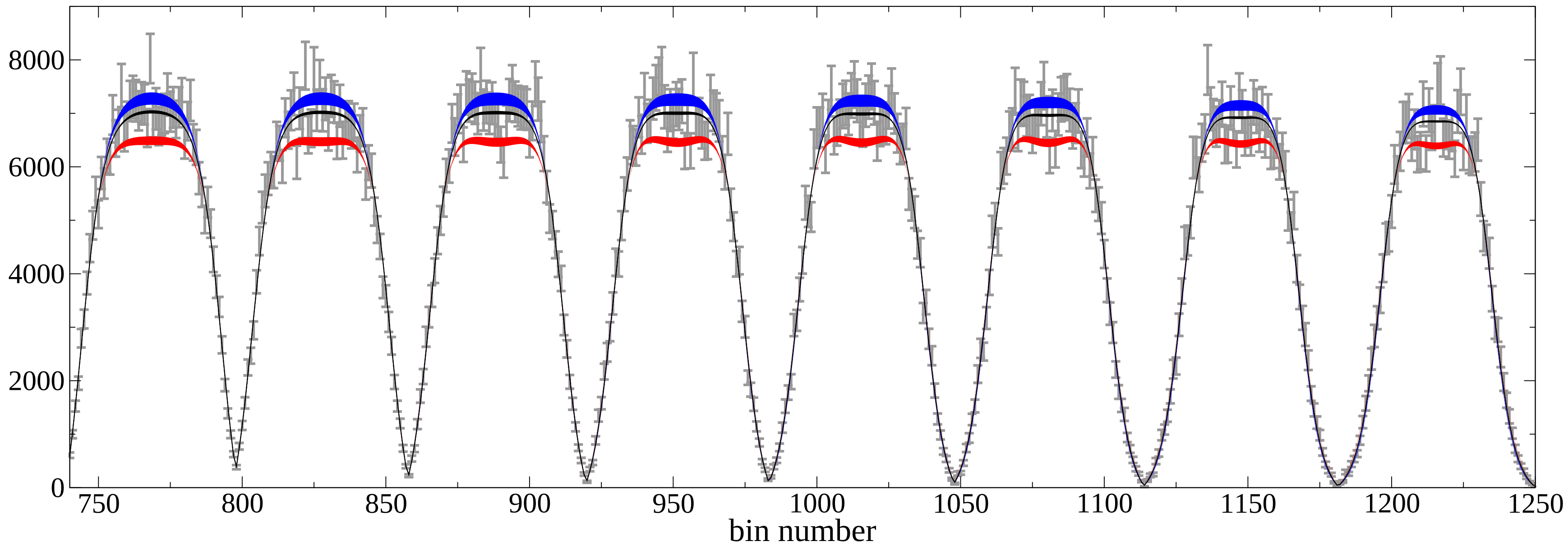
- successive slices through Dalitz plot: Niecknig, BK, Schneider 2012




| | | | |
|---|---|---|---|
| $\chi^2/\text{ndof}$ | $1.7\ldots2.1$ | $1.2\ldots1.5$ | $1.0$ |

- perfect fit respecting analyticity and unitarity possible
- contact term emulates neglected rescattering effects
- no need for "background" — inseparable from "resonance"