Three-particle dynamics on the lattice

Akaki Rusetsky, University of Bonn

In collaboration with M. Döring, H.-W. Hammer, M. Mai, J.-Y. Pang, F. Romero-López, C. Urbach and J. Wu

JHEP 09 (2017) 109, JHEP 10 (2017) 115, arXiv:1802.03362 + ongoing work

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Plan

- Introduction
- Formalism:
  - Non-relativistic EFT and dimer picture
  - Quantization condition
  - Symmetries of the box and reduction of the quantization condition
- The finite volume spectrum: bound and scattering states
- Conclusions, outlook

Extraction of the observables on the lattice

Motivation:

- Decays into the three-particle final states:
  - $K \to 3\pi$, $\eta \to 3\pi$, $\omega \to 3\pi$
  - $a_1(1260)$, $a_1(1420)$, XYZ-states
  - Roper resonance
- Many-body physics on the lattice

Three-particle sector: continuum

- bound states, elastic scattering, rearrangement, breakup...
- 3-particle decay matrix elements (infinite volume)

Three-particle sector: finite volume

- two-particle *and* three-particle energy levels *both below and above* the pertinent thresholds
- 3-particle decay matrix elements (finite volume)

Example: $K \to 2\pi$ decays

L. Lellouch and M. Lüscher, Comm. Math. Phys. 219 (2001) 31

see also M. Hansen and S. Sharpe, PRD 86 (2012) 016007: Multiple channels

2-pion states: $O_n(x_0) = \sum_{\mathbf{k}\in\Omega_n} \int_0^L d^3\mathbf{x} d^3\mathbf{y}\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\, \phi(x_0,\mathbf{x})\phi(x_0,\mathbf{y})$

$K \to 2\pi$ decay:

$$\int_0^L d^3\mathbf{y} \langle 0|O_n(x_0)\mathcal{L}_w(0)K^\dagger(y_0)|0\rangle$$

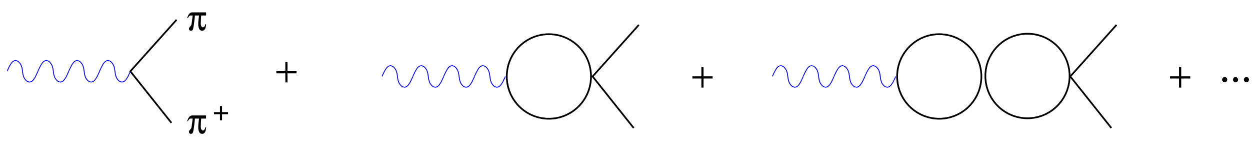
$$= \quad g\nu_n \int \frac{dP_0}{2\pi i} \frac{e^{iP_0x_0+M_Ky_0}}{(2W_\pi)^2(2M_K)(P_0-2iW_\pi)} \frac{p\cot\delta(p)}{p\cot\delta(p) - \frac{2}{\sqrt{\pi}L} Z_{00}(1;q^2)}$$

matrix element: $\langle \pi\pi l|H_w|K\rangle = \frac{g\cos\delta}{\sqrt{8\pi}} \left(\frac{p}{M_K}\right)^{3/2} \frac{1}{\sqrt{p\delta'(p)+q\phi'(q)}}$

$$\hookrightarrow \quad |\langle \pi\pi l|H_w|K\rangle|^2 = \frac{|A^\infty|^2}{8\pi} \left(\frac{p}{M_K}\right)^3 \frac{1}{p\delta'(p)+q\phi'(q)}\,, \qquad A^\infty = g e^{i\delta}\cos\delta$$

Example: form factors

Timelike e.m. form factor of the pion: H. Meyer, PRL 107 (2011) 072002



$$|F_\pi(E)|^2 = \left(q\phi'(q) + k\frac{\partial\delta_1(k)}{\partial k}\right)\frac{3\pi E^2}{2k^5}\,|A_\psi|^2$$

$$L^{3/2}\langle[\pi^b(k_\sigma)\pi^c(-k_\sigma)]_A|V_\rho^a(0)|0\rangle = \sqrt{2}\varepsilon^{abc}\delta_{\sigma\rho}\,A_\psi$$

Resonance form factors: V. Bernard, D. Hoja, U.-G. Meißner and AR, JHEP 1209 (2012) 023; A. Agadjanov, V. Bernard, U.-G. Meißner and AR, NPB 886 (2014) 1199, NPB (2016) 387; R.A. Briceno and M.T. Hansen, PRD 92 (2015) 074509; PRD 94 (2016) 013008; R.A. Briceno *et al.,* PRD 93 (2016) 114508

What is the analog in case of three-particle final states?

⟶ First: study the three-body final states in a finite volume!

Extracting three-particle observables from the lattice

K. Polejaeva and AR, EPJA 48 (2012) 67
Finite volume energy levels determined solely by the S-matrix

M. Hansen and S. Sharpe, PRD 90 (2014) 116003; PRD 92 (2015) 114509
Quantization condition

R. Briceno and Z. Davoudi, PRD 87 (2013) 094507
Dimer formalism, quantization condition

P. Guo, PRD 95 (2017) 054508
Quantization condition in the 1+1-dimensional case

S. Kreuzer and H.-W. Hammer, PLB 694 (2011) 424; EPJA 43 (2010) 229; PLB 673 (2009) 260; S. Kreuzer and H. W. Grießhammer, EPJA 48 (2012) 93
Dimer formalism, numerical solution

M. Mai and M. Döring, EPJA 53 (2017) 240
Three-body unitarity + analyticity (similar in spirit to the present approach)

$\hookrightarrow$ Is the finite-volume spectrum determined solely by the three-body S-matrix elements in the infinite volume?

The strategy

W.-W. Hammer, Jin-Yi Pang and AR, JHEP 09 (2017) 109, JHEP 10 (2017) 115

↪ Do not try to extract the amplitudes directly from data, in analogy to Lüscher's formula!

↪ Extract low-energy couplings, get amplitudes by solving scattering equations in the infinite volume!

- NREFT: relativistic kinematics will be included later
- Effective couplings: only exponentially suppressed effects at large volumes!

? Is the information about the S-matrix sufficient to uniquely determine the spectrum? Do the *off-shell couplings,* which are not fixed from matching to the S-matrix, contribute to the finite-volume energies?

NREFT: dimer picture in the two-particle sector

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi + \mathcal{L}_2$$

$$\mathcal{L}_2 = -\frac{C_0}{2}\,\psi^\dagger\psi^\dagger\psi\psi - \frac{C_2}{4}\,(\psi^\dagger\nabla^2\psi^\dagger\psi\psi + \text{h.c.}) + \cdots$$

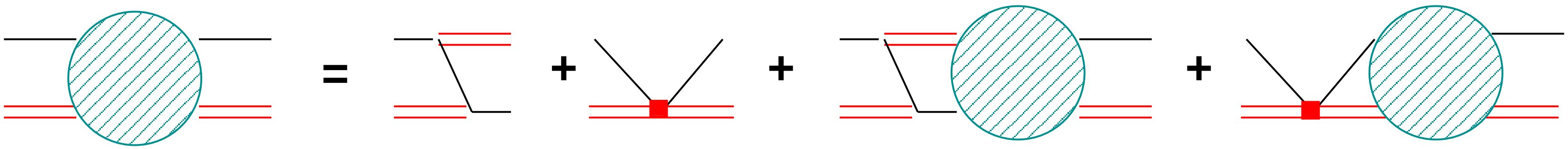
$C_0, C_2, \ldots$ matched to $p\cot\delta(p) = -\frac{1}{a} + \frac{r}{2}\,p^2 + \cdots$

dimer:

$$\mathcal{L}_2 \to \mathcal{L}_2^{\text{dimer}} = \sigma T^\dagger T + \left(T^\dagger \left[f_0\psi\psi + f_1\psi\nabla^2\psi + \cdots \right] + \text{h.c.} \right)$$

- Dimer framework algebraically equivalent to the three-particle framework
- Higher partial waves can be included: dimers with arbitrary spin
- Can be generalized to the non-rest frames

The scattering equation



$$\mathcal{M}(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + \int_{\mathbf{k}}^{\Lambda} Z(\mathbf{p},\mathbf{k};E)\tau(\mathbf{k};E)\mathcal{M}(\mathbf{k},\mathbf{q};E)$$

$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{p}\mathbf{q}-mE} + H_0 + H_2(\mathbf{p}^2+\mathbf{q}^2) + \cdots$$

$H_0, H_2, \ldots$ are related to the couplings $h_0, h_2, \ldots$

$$\tau^{-1}(\mathbf{k};E) = k^* \cot\delta(k^*) + \underbrace{\sqrt{\frac{3}{4}\,\mathbf{k}^2 - mE}}_{=k^*}$$

Finite volume

$$\mathbf{k}=\frac{2\pi}{L}\,\mathbf{n}\,,\quad \mathbf{n}\in\mathbb{Z}^3\,,\qquad\qquad \int_{\mathbf{k}}^{\Lambda}\to\frac{1}{L^3}\sum_{\mathbf{k}}^{\Lambda}$$

$$\mathcal{M}_L(\mathbf{p},\mathbf{q};E)=Z(\mathbf{p},\mathbf{q};E)+\frac{8\pi}{L^3}\sum_{\mathbf{k}}^{\Lambda}Z(\mathbf{p},\mathbf{q};E)\tau_L(\mathbf{k};E)\mathcal{M}_L(\mathbf{k},\mathbf{q};E)$$

$$\tau_L^{-1}(\mathbf{k};E)=k^*\cot\delta(k^*)-\frac{4\pi}{L^3}\sum_{\mathbf{l}}\frac{1}{\mathbf{k}^2+\mathbf{l}^2+\mathbf{k}\mathbf{l}-mE}$$

- Poles in the amplitude $\to$ finite-volume energy spectrum
- $k^*\cot\delta(k^*)$ fitted in the two-particle sector; $H_0,H_2,\ldots$ should be fitted to the three-particle energies
- S-matrix in the infinite volume $\to$ equation with $H_0,H_2,\ldots$
- No “off-shell” effects in the finite volume spectrum!

Quantization condition

The particle-dimer scattering amplitude:

$$\mathcal{M}_L = Z + Z\tau_L\mathcal{M}_L$$

The three-particle scattering amplitude:

$$T_L^{(3)} = \tau_L + \tau_L\mathcal{M}_L\tau_L = (\tau_L^{-1} - Z)^{-1}$$

The quantization condition:
the three-body energy levels coinside with the poles of $T_L^{(3)}$:

$$\det(\tau_L^{-1} - Z) = 0$$

- The spectrum is determined only by the on-shell input!
- Agrees with: Polejaeva and AR, Hansen and Sharpe, Briceno and Davoudi, Mai and Döring!

Reduction of the quantization condition: symmetries

On the lattice, rotational symmetry $\rightarrow$ octahedral symmetry O_h

Analog for a sphere $|\mathbf{k}| = \text{const}$ for a cube: *shells*

$$\mathbf{k} = g\mathbf{k}_0 \,, \quad g \in O_h$$

For an arbitrary function of the momentum $\mathbf{p}$, belonging to a shell s,

$$f(\mathbf{p}) = f(g\mathbf{p}_0) = \sum_\Gamma \sum_{ij} T_{ij}^{(\Gamma)}(g) f_{ji}^{(\Gamma)}(\mathbf{p}_0) \,, \quad \Gamma = A_1^\pm, A_2^\pm, E^\pm, T_1^\pm, T_2^\pm$$

Projecting back the components:

$$\frac{G}{s_\Gamma} f_{ji}^{(\Gamma)}(\mathbf{p}_0) = \sum_{g \in O_h} (T_{ij}^{(\Gamma)}(g))^* f(g\mathbf{p}_0) \,, \quad G = \text{dim}(O_h) = 48$$

The quantization condition in the new basis partially diagonalizes

Reduction of the equation

The equation determining the energy spectrum:

$$f(\mathbf{p}) = \frac{8\pi}{L^3} \sum_s \sum_{g\in O_h} \frac{\vartheta(s)}{G} Z(\mathbf{p}, g\mathbf{k}_0(s))\tau(s) f(g\mathbf{k}_0(s))$$

$\vartheta(s)$: the multiplicity of the shell s

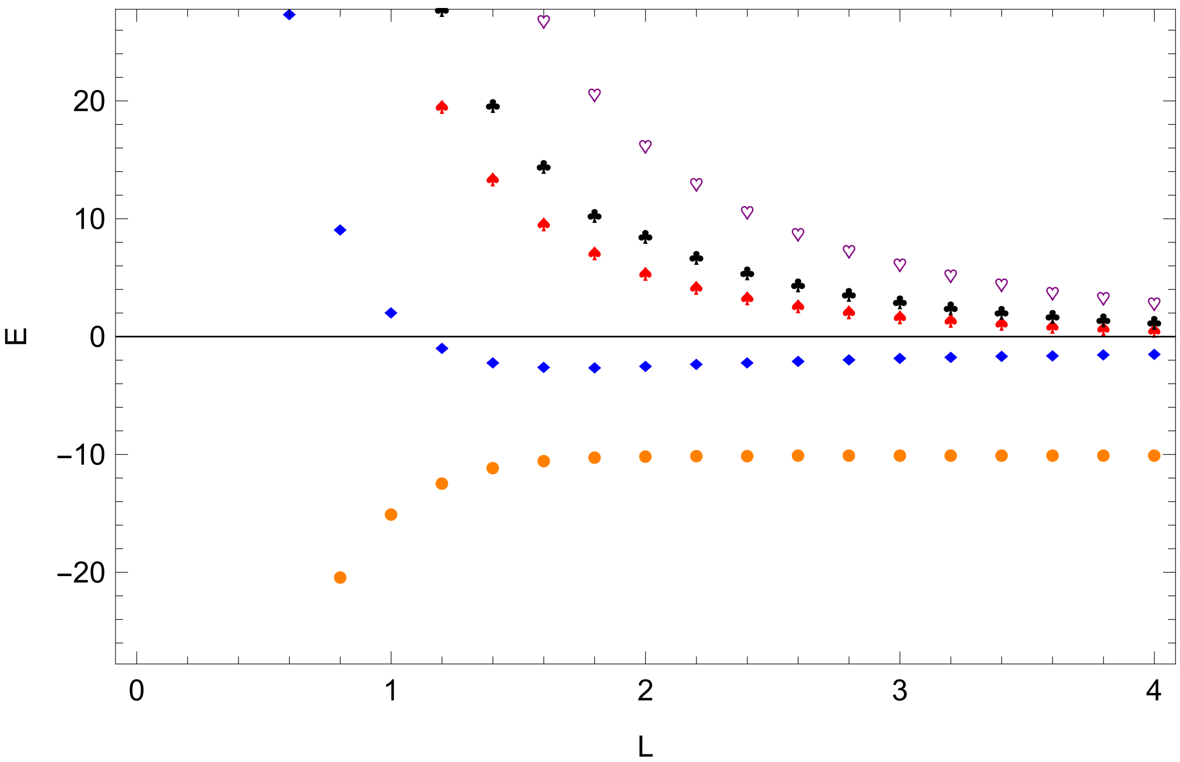
Projecting the equation on a given irrep Γ:

$$f_i^{(\Gamma)}(r) = \frac{8\pi}{L^3} \sum_s \frac{\vartheta(s)\tau(s)}{G} \sum_j Z_{ij}^{(\Gamma)}(r,s) f_j^{(\Gamma)}(s)\,.$$

The quantization condition partially diagonalizes

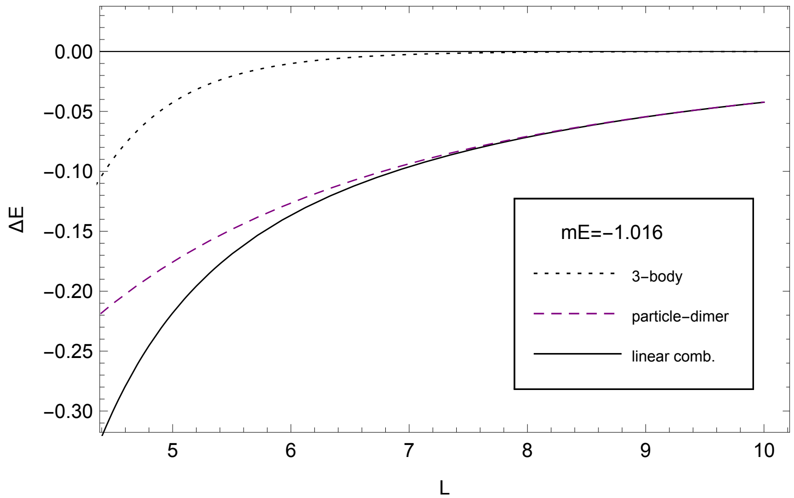
$$\det\left(\tau(s)^{-1}\vartheta(s)^{-1}\delta_{rs}\delta_{ij} - \frac{8\pi}{L^3}\,\frac{1}{G}\,Z_{ij}^{(\Gamma)}(r,s)\right) = 0\,.$$

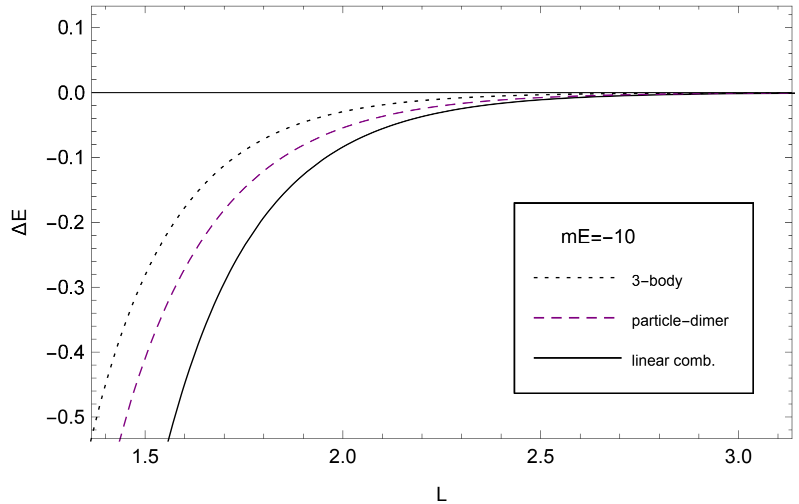
The finite-volume spectrum in the A_1 irrep, CM frame



- $m = a = 1,\ \Lambda = 225,\ H_0(\Lambda) = 0.192$
- The spectrum both below and above the three-particle threshold

Bound-state spectrum: $E = -1.016$ and $E = -10$



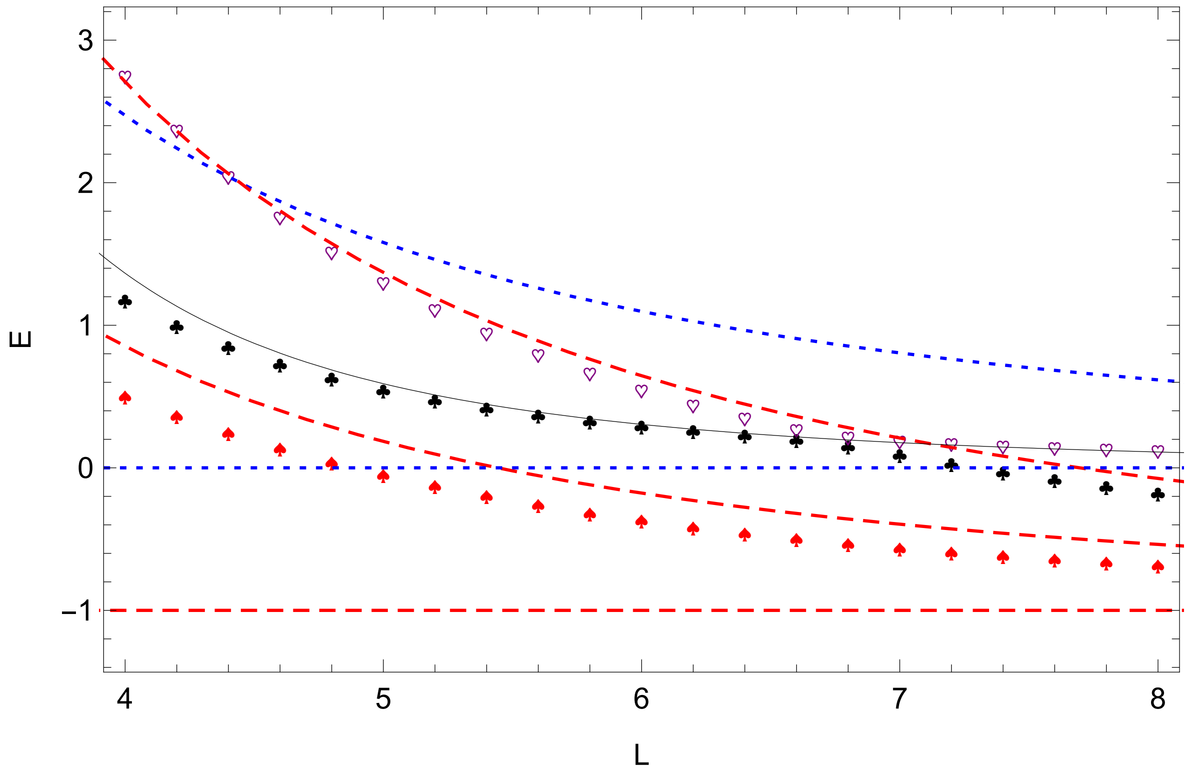


Three-particle: $\dfrac{C}{L^{3/2}}\,\exp\left(-\dfrac{2}{\sqrt{3}}\,\kappa L\right)$

Particle-dimer: $\dfrac{C'}{L}\,\exp\left(-\dfrac{2}{\sqrt{3}}\,\sqrt{\kappa^2 - a^{-2}}L\right)$

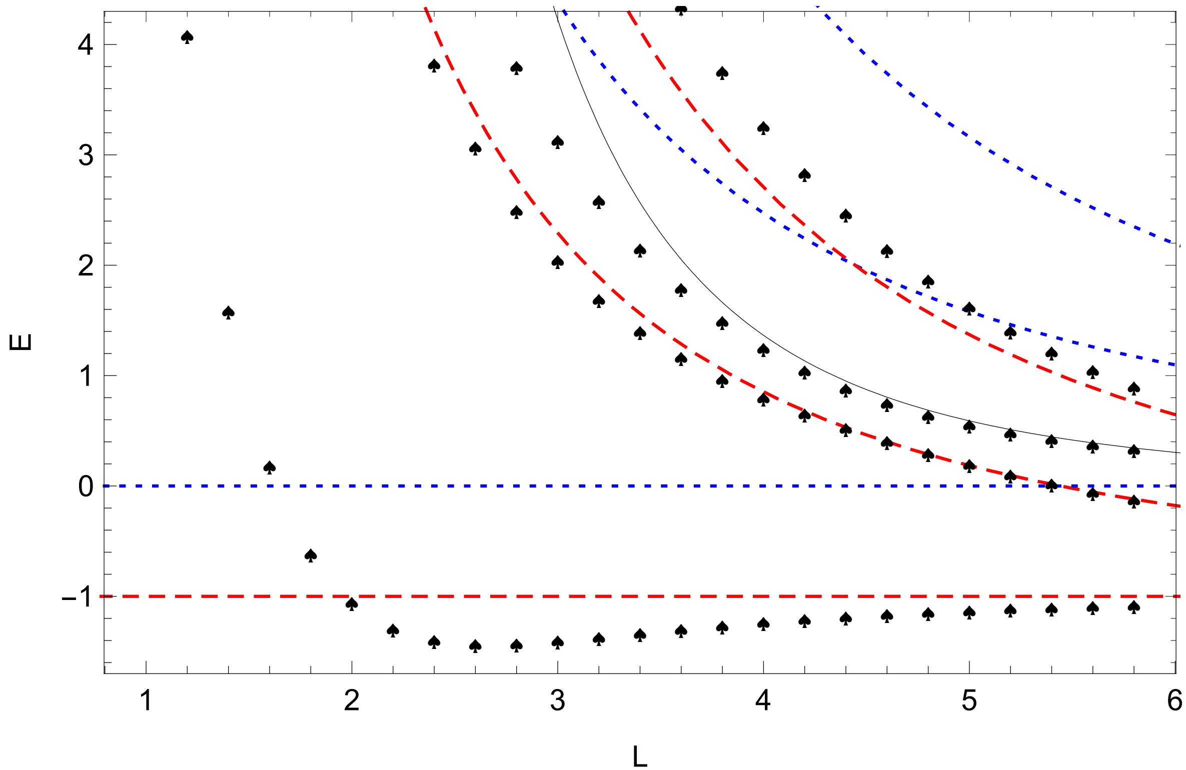
…or, a linear combination thereof

Scattering states: particle-dimer and 3-particle



$$\Delta E = \frac{12\pi a}{L^3} - \frac{12a^2}{L^4}\mathcal{I} + \frac{12a^3}{\pi L^5}(\mathcal{I}^2 + \mathcal{J}) + O(L^{-6})\,, \quad \mathcal{I} \simeq -8.914,\ \mathcal{J} \simeq 16.532$$

Pushing up the energy level by a shallow bound state



- **?** Where is the energy level, corresponding to the displaced particle-dimer threshold?
- **!** The shallow bound state is pushing it up
- Displaced threshold can be easily identified!

Conclusions

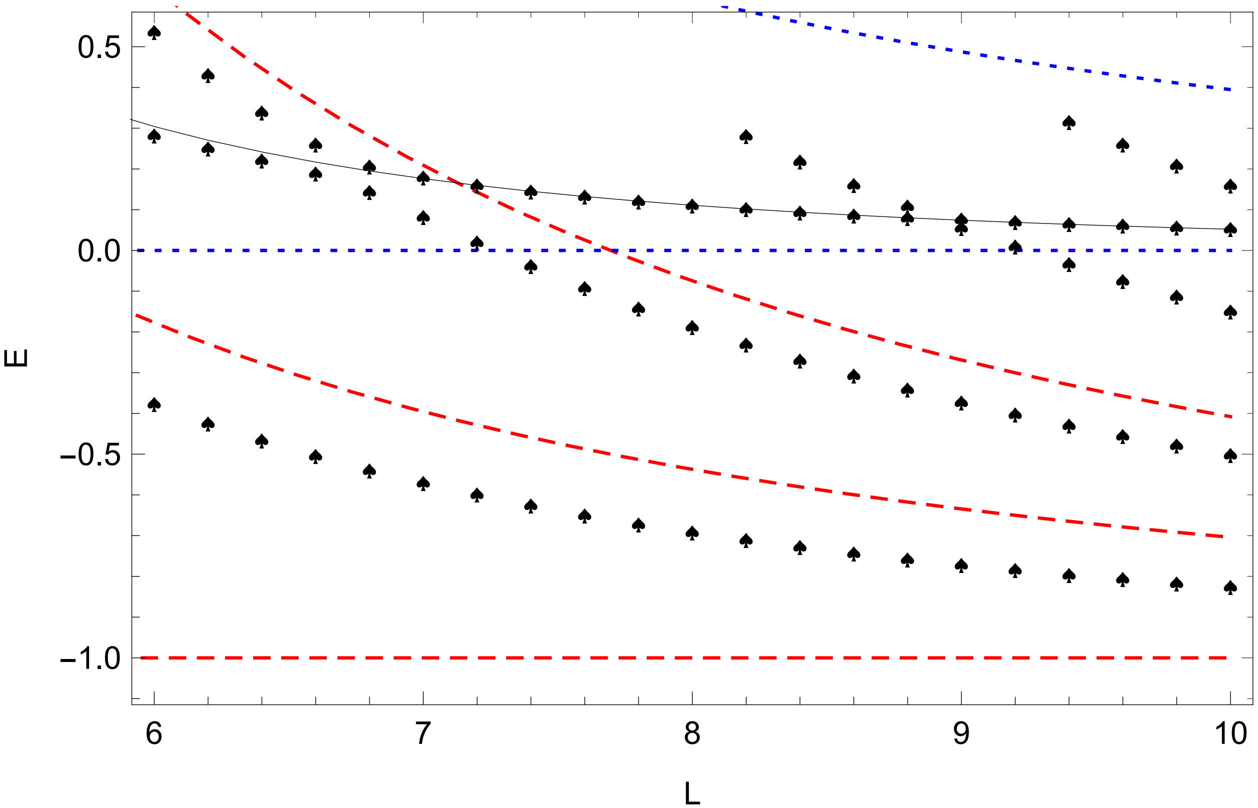
- An EFT formalism in a finite volume is proposed to analyze the data in the three-particle sector
- The low-energy three-body couplings are fitted to the spectrum; S-matrix is obtained through the solution of equations
- A systematic approach: allows the inclusion of higher partial waves, derivative couplings, two $\rightarrow$ three transitions, relativistic kinematics,$\ldots$
- Equivalent to other known approaches, much easier to use!
- Reduction of the quantization condition is possible, according to the octahedral symmetry
- Outlook: Three-particle Lellouch-Lüscher formula + $\ldots$

Spare: The expansion of the kernel

The kernel is invariant under O_h: $Z(g\mathbf{p}, g\mathbf{q}) = Z(\mathbf{p}, \mathbf{q})$

$$
\begin{aligned}
Z_{nm}^{(\Gamma\Gamma',ij)}(r,s) &= \sum_{g,g'\in O_h} (T_{in}^{(\Gamma)}(g'))^* Z(g'\mathbf{p}_0(r), g\mathbf{k}_0(s)) T_{jm}^{(\Gamma')}(g) \\
&= \sum_{g,g'\in O_h} (T_{in}^{(\Gamma)}(g'))^* Z(\underbrace{g^{-1}g'}_{=g''}\mathbf{p}_0(r), \mathbf{k}_0(s)) T_{jm}^{(\Gamma')}(g) \\
&= \sum_{g,g''\in O_h} \sum_k (T_{ik}^{(\Gamma)}(g))^* (T_{kn}^{(\Gamma)}(g''))^* Z(g''\mathbf{p}_0(r), \mathbf{k}_0(s)) T_{jm}^{(\Gamma')}(g) \\
&= \sum_{g''\in O_h} \sum_k \frac{G}{s_\Gamma} \delta_{\Gamma\Gamma'}\delta_{ij}\delta_{km} (T_{kn}^{(\Gamma)}(g''))^* Z(g''\mathbf{p}_0(r), \mathbf{k}_0(s)) \\
&= \frac{G}{s_\Gamma}\delta_{\Gamma\Gamma'}\delta_{ij} \sum_{g\in O_h} (T_{mn}^{(\Gamma)}(g))^* Z(g\mathbf{p}_0(r), \mathbf{k}_0(s)) \\
&= \frac{G}{s_\Gamma}\delta_{\Gamma\Gamma'}\delta_{ij} Z_{nm}^{(\Gamma)}(r,s)
\end{aligned}
$$

Spare: Avoided level crossing



- Avoided level crossing between 3-particle and particle-dimer states
- Where is the (displaced) particle-dimer threshold?