Dispersive determination of the pion-kaon scattering lengths

J. Ruiz de Elvira

Institute for Theoretical Physics, University of Bern

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Outline

Why is pion-kaon scattering important?

▷ Relevance of pion-kaon scattering lengths

Chiral perturbation in SU(3)

▷ Convergence of the chiral series

Dispersive techniques in meson-meson scattering

▷ Roy-Steiner equation for pion-kaon scattering

On the dispersive determination of πK scattering lengths

in collaboration with Gilberto Colangelo and Stefano Maurizio

Motivation: Why πK scattering?

- **Low energies:** test chiral dynamics in the strange-quark sector

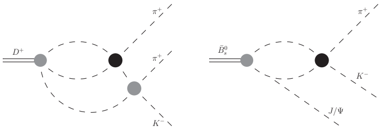
  ▷ Scattering lengths lowest energy observables

  $\hookrightarrow$ Spontaneous and explicit chiral symmetry breaking

- **Higher energies:** resonances, hadron spectrum

  $\hookrightarrow$ $\kappa(800)$ non-ordinary meson, PDG "needs confirmation"

- **Input for Heavy-meson decays**: CP-violation and New Physics searches



- **Crossed channel $\pi\pi \to \bar{K}K$**: first inelastic contribution to $\pi\pi$ scattering

  $\hookrightarrow$ $\Gamma(f_0(500) \to \bar{K}K)$ nature of the σ meson

  $\hookrightarrow$ Nucleon form factors, $g-2$ ...

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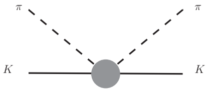

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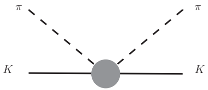
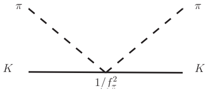


- Leading-order $\mathcal{O}(p^2) = \mathcal{O}(m_i^2)$ predictions for πK

$$a^- = \frac{m_\pi m_K}{8\pi(m_\pi + m_K) f_\pi^2} + \mathcal{O}(m_i^4) \qquad a^+ = \mathcal{O}(m_i^4)$$



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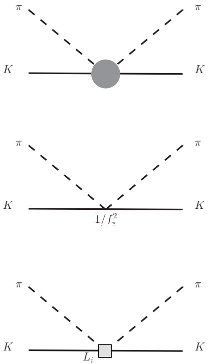
- Next-to-leading order: low-energy constants L_{1-8}

  ▷ Loop contribution suppressed at threshold



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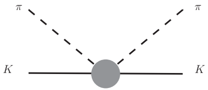

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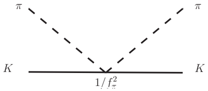

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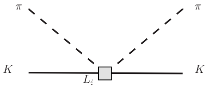

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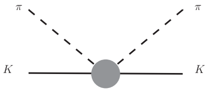

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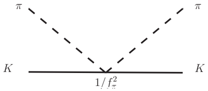

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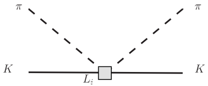

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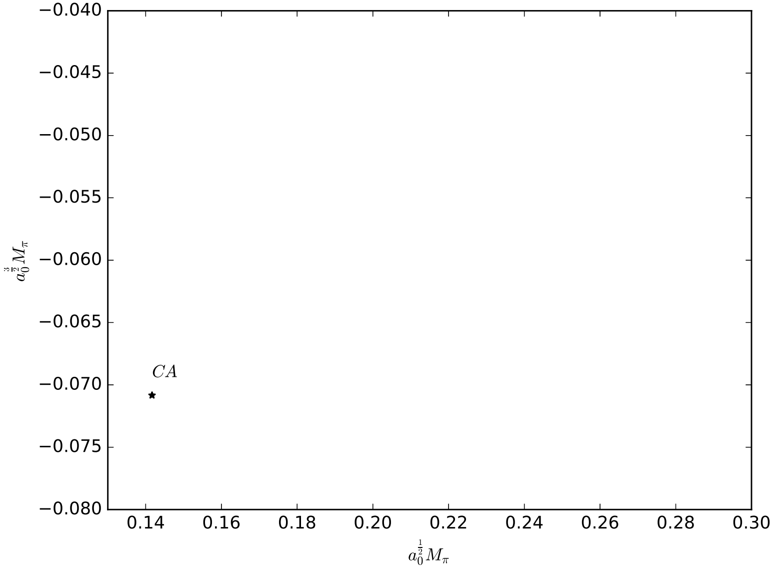
- Next-to-next to leading order: C_{1-32}, 10 for a^- and 23 for a^+

[Bijnens,Dhonte,Talavera 2004], [Bijnens, Ecker 2014]

πK scattering lengths



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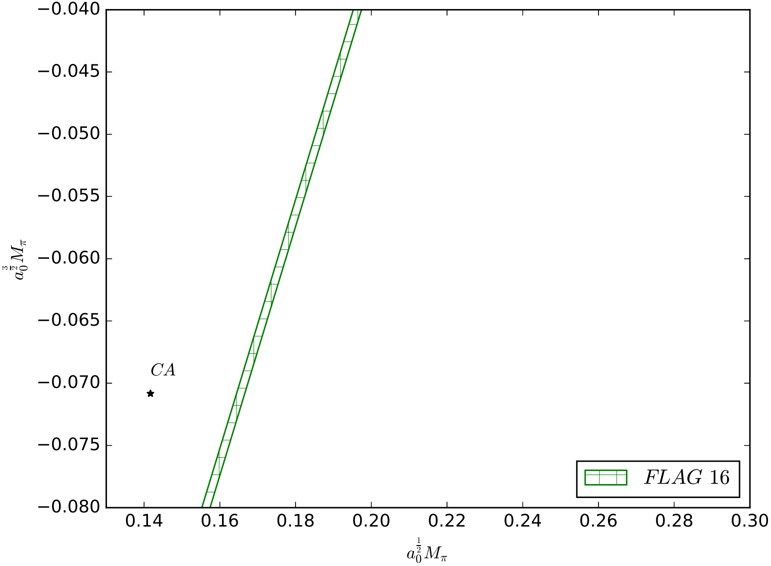

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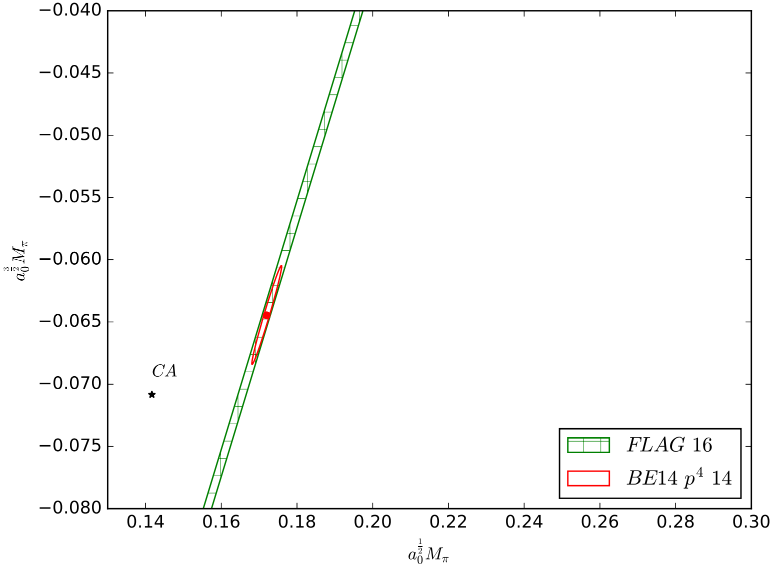

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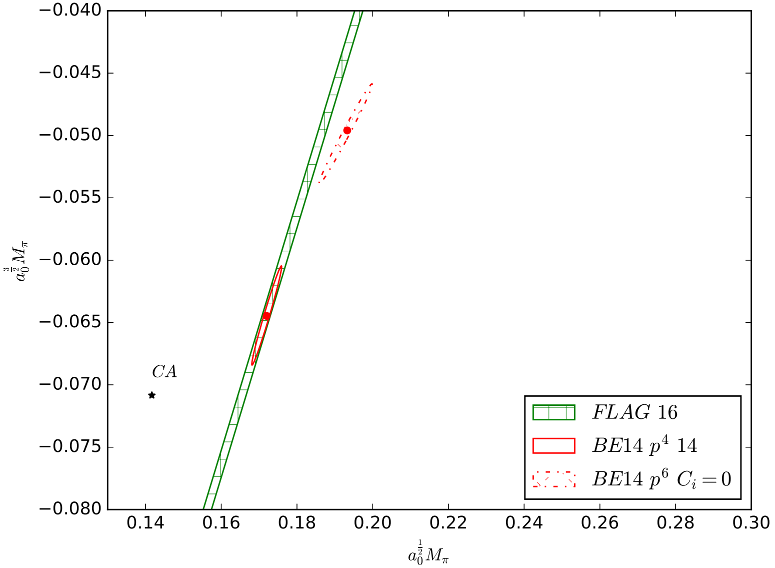

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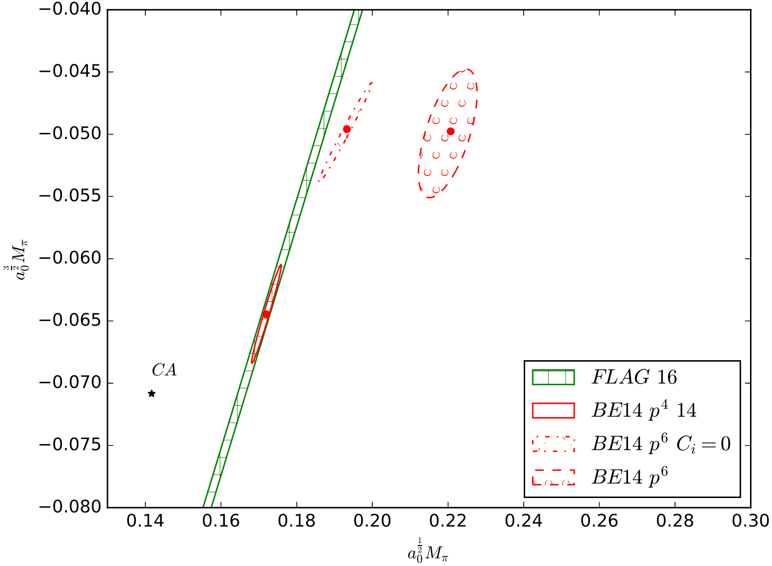

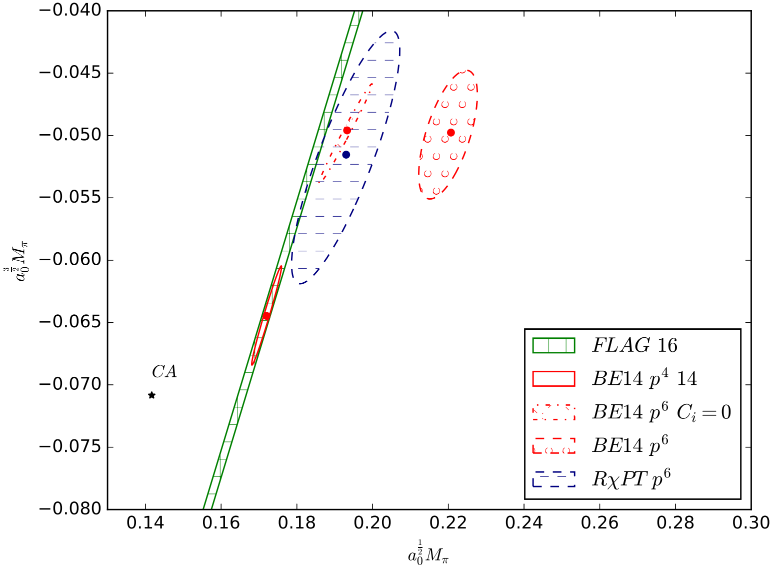

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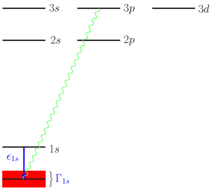
- **Experimental values**: DIRAC collaboration



▷ lifetime of πK atoms at CERN ⇒ isovector scattering length

$$\Gamma_{1S} \propto \left| T_{(\pi^+K^- \to \pi^0 K^0)} \right|^2 \propto |a^-|^2$$

[Deser, Goldberger, Baumann, Thirring 1954]

▷ Current result:

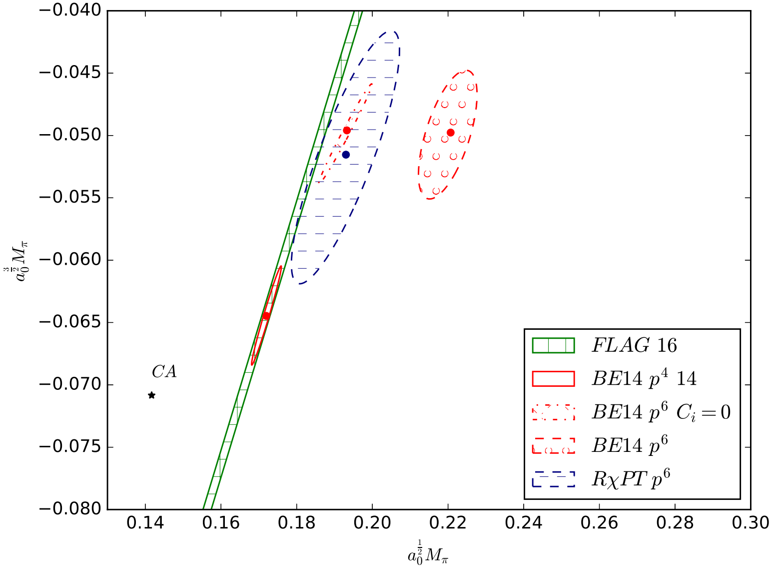
$$a^- = (-0.072^{+0.031}_{-0.020}) m_\pi^{-1}$$

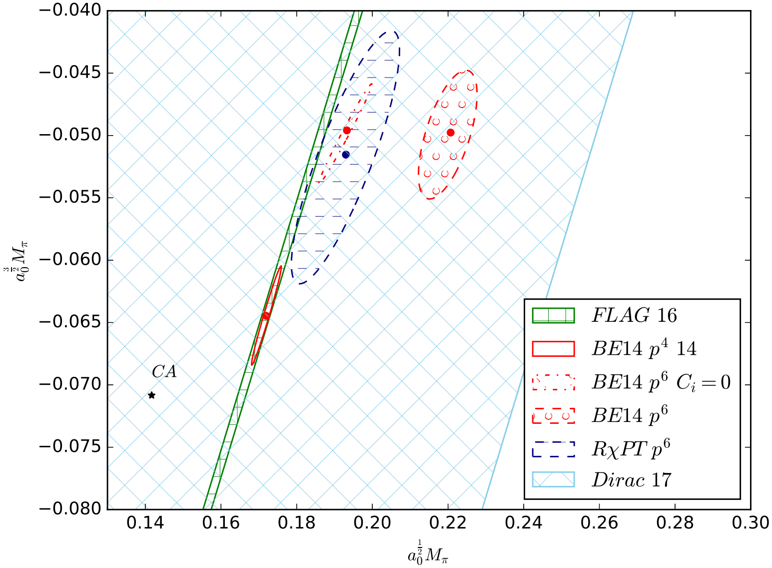
[DIRAC 2017]

↪ **huge uncertainties**

▷ Room for improvement: near future increase statistics by 10

πK scattering lengths





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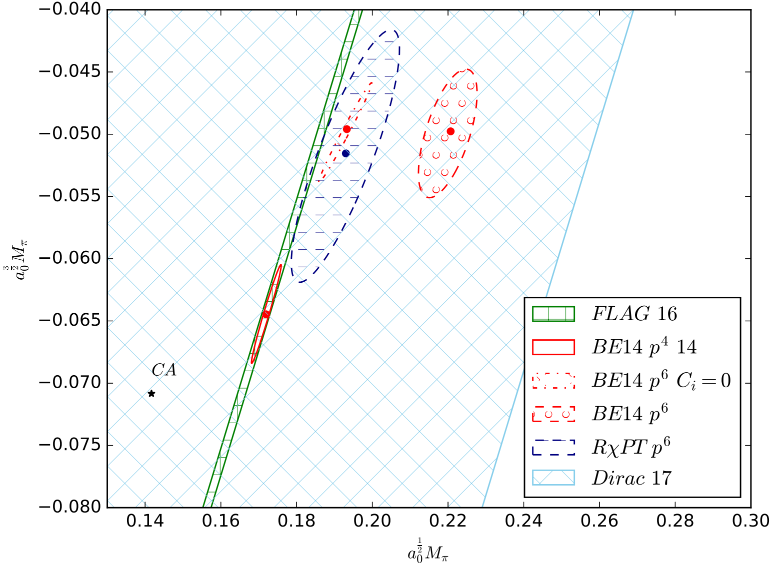

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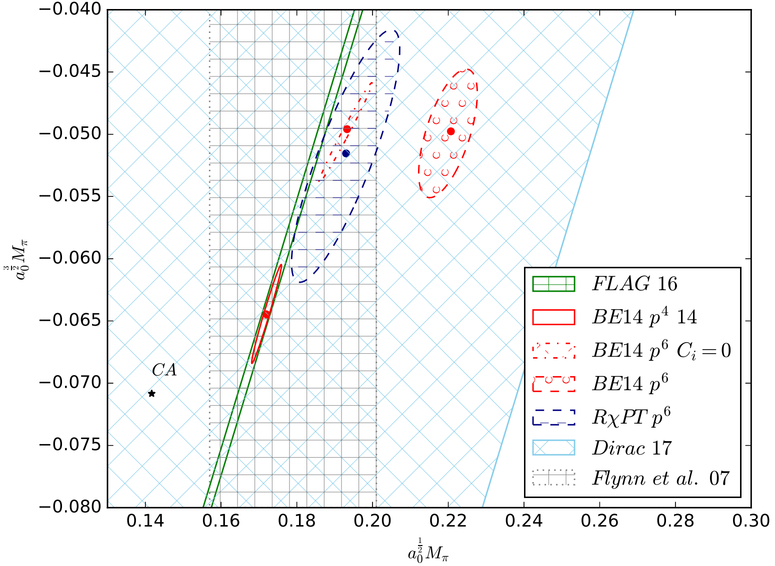

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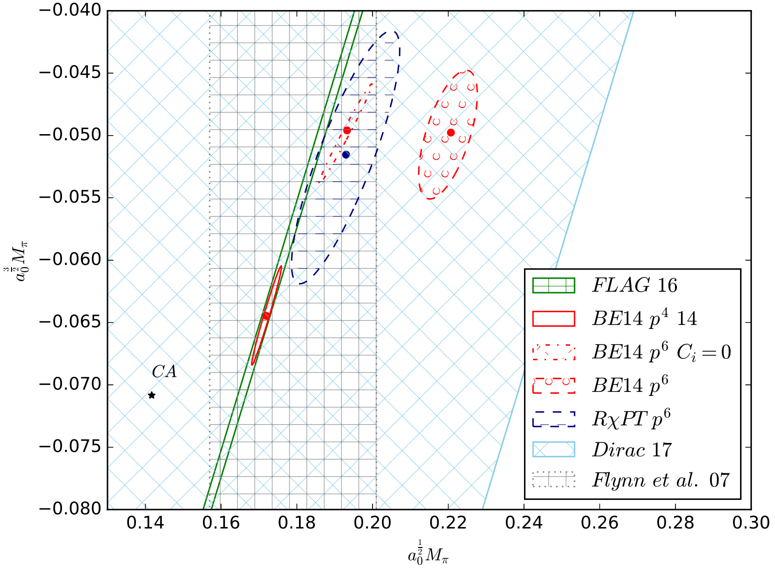

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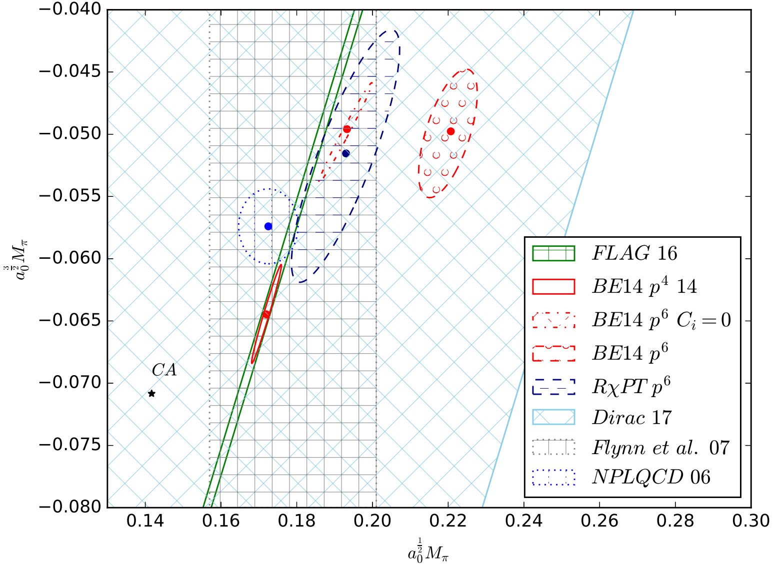

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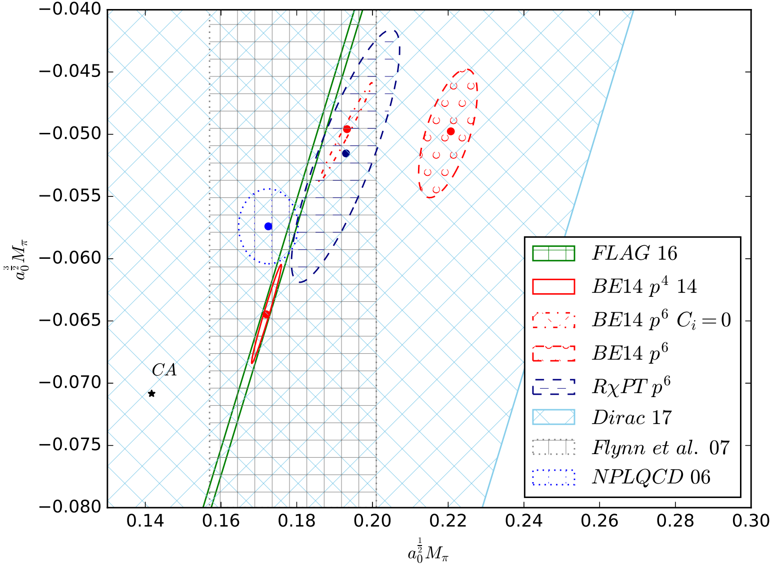

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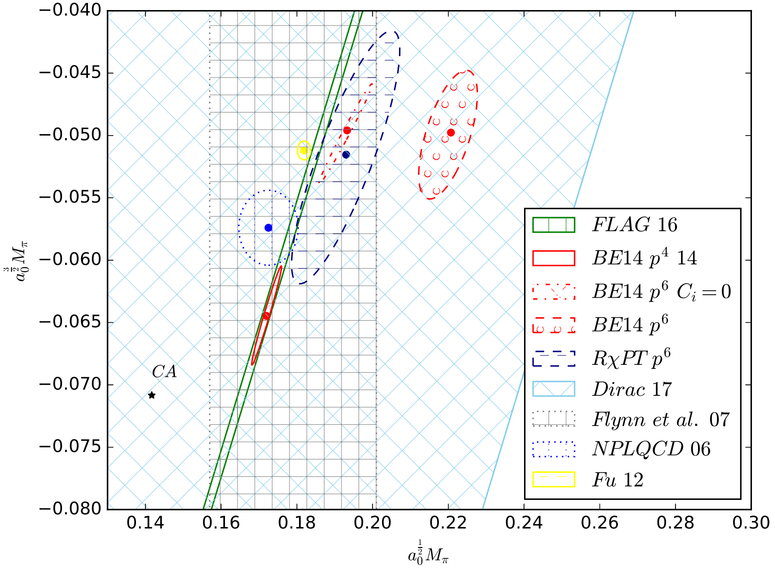

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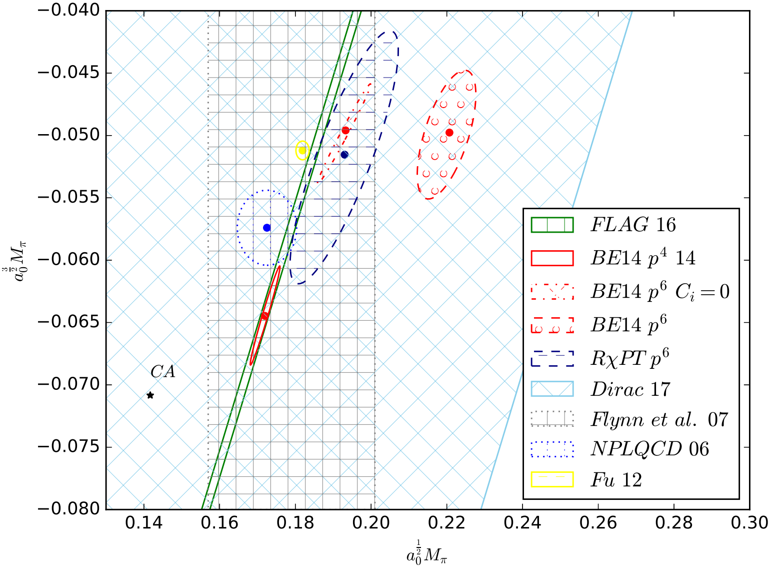
▷ PACs: dynamical $N_f = 2 + 1$ fermions, $m_\pi = 170 - 710$ MeV [PACs-Cs 2014]

$$a^{1/2} = 0.142(14)(27) m_\pi^{-1}, \quad a^{3/2} = -0.047(2)(2) m_\pi^{-1}$$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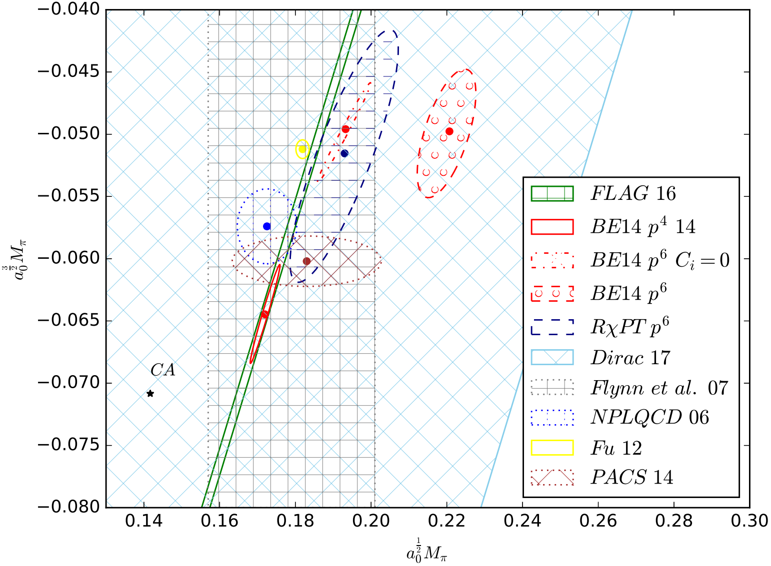

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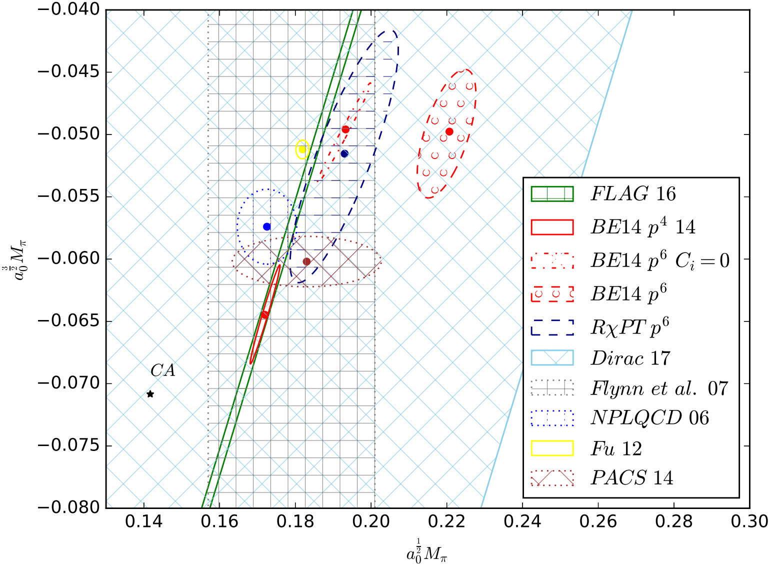

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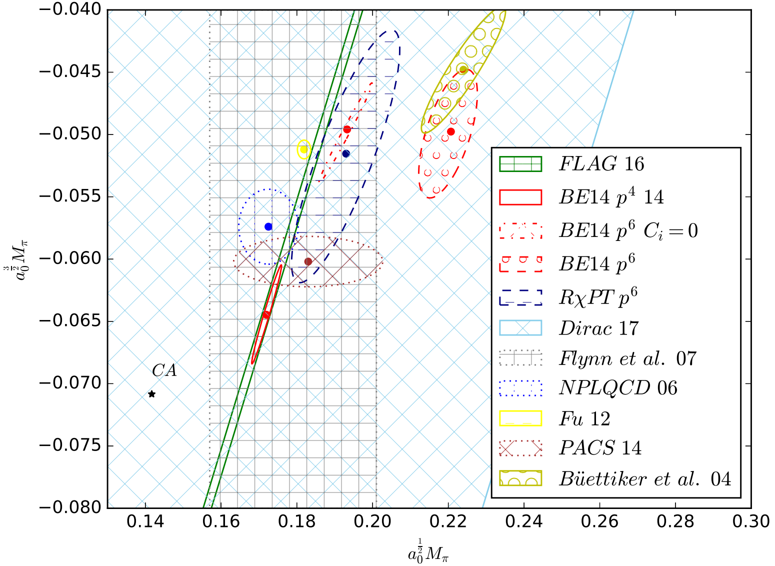

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- **This talk**: where does this discrepancy come from?

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  ↪ Overview of Roy-Steiner equation for πK scattering

Why Roy-Steiner equations?

Roy(-Steiner) eqs. = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries**: analyticity, unitarity, crossing
- **Model independent** $\Rightarrow$ the actual parametrization of the data is irrelevant once it is used in the integral.
- Framework allows for systematic improvements (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with high precision:
  - $\pi\pi$-scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - πK-scattering: [Büttiker et al. (2004)]
  - $\gamma\gamma \to \pi\pi$ scattering: [Hoferichter et al. (2011)]
  - πN scattering: [Hoferichter et al. (2015)]

Roy equations for $\pi\pi$ scattering

- $\pi\pi \to \pi\pi \Rightarrow$ fully crossing symmetric in Mandelstam variables s, t, and $u = 4M_\pi - s - t$
- Start from twice-subtracted **fixed-t** DRs

$$T^I(s,t) = c(t) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2}\left[\frac{s^2}{(s'-s)} - \frac{u^2}{(s'-u)}\right] \mathrm{Im}\, T^I(s',t)$$

- Subtraction functions c(t) are determined via crossing symmetry

  $\hookrightarrow$ functions of the I=0,2 scattering lengths: a_0^0 and a_0^2

- PW-projection and expansion yields the **Roy-equations** [Roy (1971)]

$$t_J^I(s) = ST_J^I(s) + \sum_{J'=0}^{\infty}(2J'+1)\sum_{I'=0,1,2}\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s',s)\, \mathrm{Im}\, t_{J'}^{I'}(s')$$

- $K_{JJ'}^{II'}(s',s) \equiv$ kernels $\Rightarrow$ analytically known

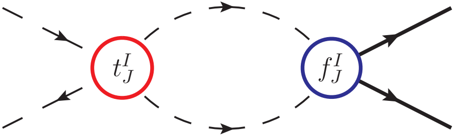
Key differences compared to $\pi\pi$ Roy equations

- Crossing: coupling between $\pi K \to \pi K$ (s-channel) and $\pi\pi \to \bar{K}K$ (t -channel)

  $\Rightarrow$ need a different kind of dispersion relations [Hite, Steiner 1973, Büttiker et al. 2004]

- Unitarity in t-channel, e.g. single-channel approximation

$$\mathrm{Im} f_{\pm}^{J}(t) = \sigma_t^{\pi} f_{\pm}^{J}(t) t_J^{I}(t)^{*}$$



$\Rightarrow$ **Watson's theorem**: phase of $f_{\pm}^{J}(t)$ equals δ_{IJ} [Watson 1954]

$\triangleright$ solution in terms of Omnès function [Muskhelishvili 1953, Omnès 1958]

$\hookrightarrow$ $\pi\pi$ scattering δ_{IJ} need as input

Roy–Steiner equations for πK: flow of information

Limited range of validity

$$\sqrt{s} \leq \sqrt{s_m} = 1.11\,\mathrm{GeV} \qquad \sqrt{t} \leq \sqrt{t_m} = 1.3\,\mathrm{GeV}$$

Input/Constraints

- S- and P-waves above matching point $s > s_m$ $(t > t_m)$
- Inelasticities
- Higher waves (D-, F-, $\cdots$)
- $\pi\pi$ phase shifts below the $\bar{K}K$ threshold

Output

- S- and P-wave phase-shifts at low energies $s < s_m$ $(t < t_m)$
- Subtraction constants

$\hookrightarrow$ πK **scattering lengths**

Solving Roy-Steiner equations for πK

- **Fixed-t** dispersion relation

  ▷ $c^I(t)$ extracted from a hyperbolic dispersion relation at threshold

$$s \cdot u = m_K^2 - m_\pi^2$$

- Twice/once subtracted version for $I = \{+, -\}$

$$c^+(t) = 8\pi(m_\pi + m_K)a^+ + b^+ t$$

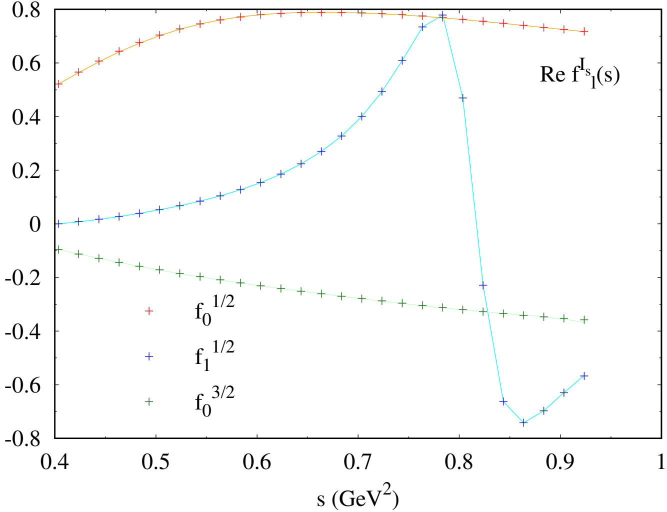
$$c^-(t) = \frac{8\pi(m_\pi + m_K)}{m_\pi m_K} a^- p_\pi(t) q_K(t)$$

- b^+ extracted from a sum rule involving a^-
- a^+ and a^- extracted from RS self-consistent solution

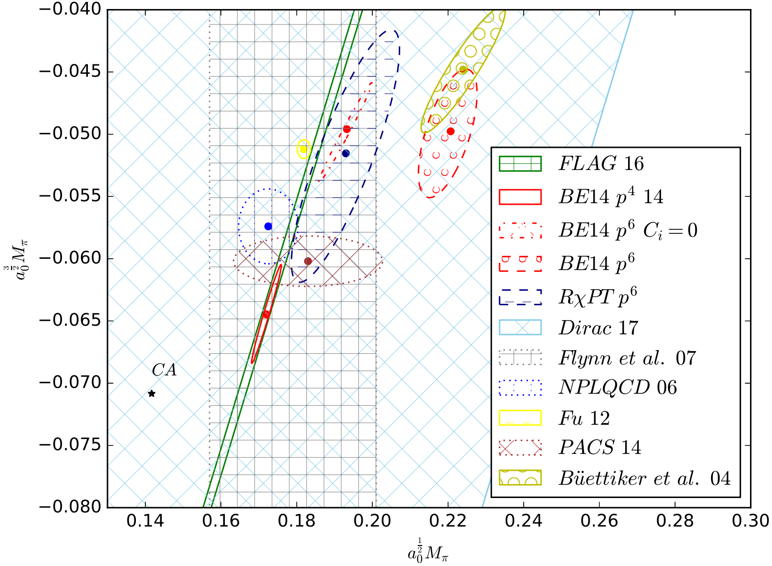
$$\chi^2_{\text{phys}} = \sum_{l, I_s} \sum_{j=1}^{N} \frac{\left(\text{Re}\, f_l^{I_s}(s_j) - F[f_l^{I_s}](s_j)\right)^2}{\text{Re}\, f_l^{I_s}(s_j)^2}$$

  ▷ $F[f_l^{I_s}](s_j) \equiv$ right hand side of RS-equations

Roy-Steiner solution for πK



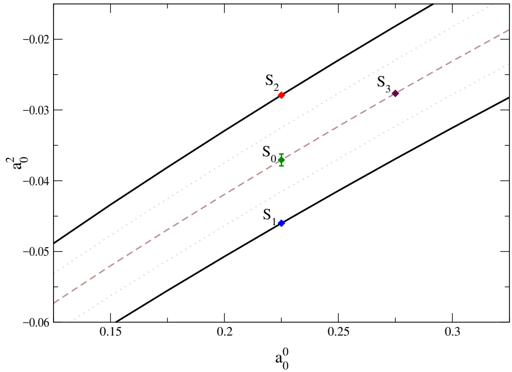
Dispersive determination of πK scattering lengths



Stability of the solution

- How **stable** is the solution?

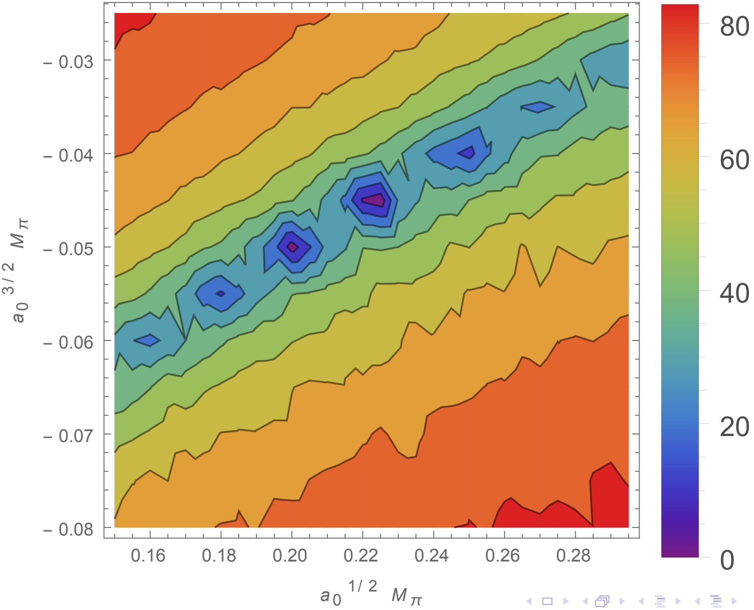
$\hookrightarrow$ $\pi\pi$ scattering case $\Rightarrow$ universal band in the a_0^0, a_0^2 plane [Ananthanarayan et al. (2001)]



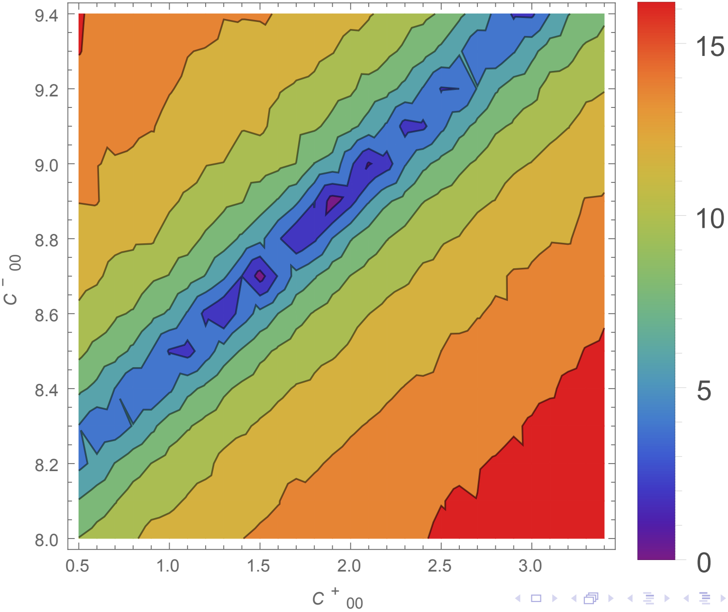
$\triangleright$ similar for πN scattering [Hoferichter et al. (2015)]

$\hookrightarrow$ look for πK RS solutions in a grid

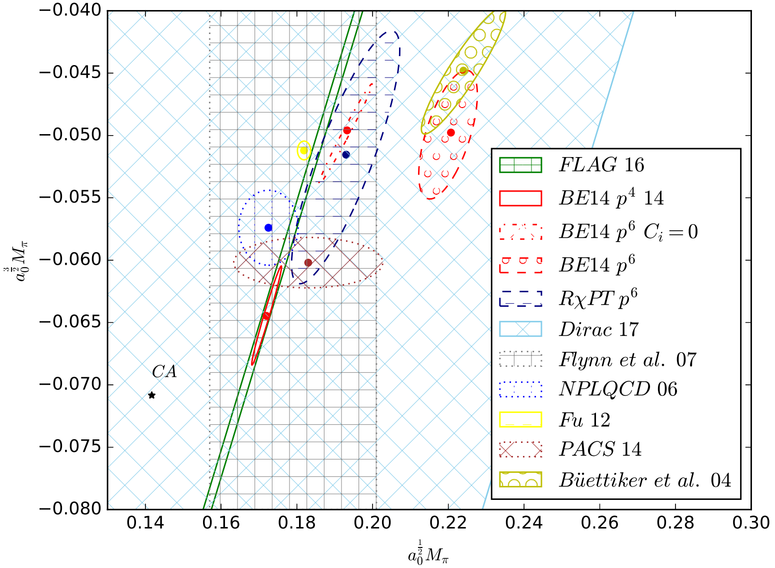
Universal band on the scattering length plane



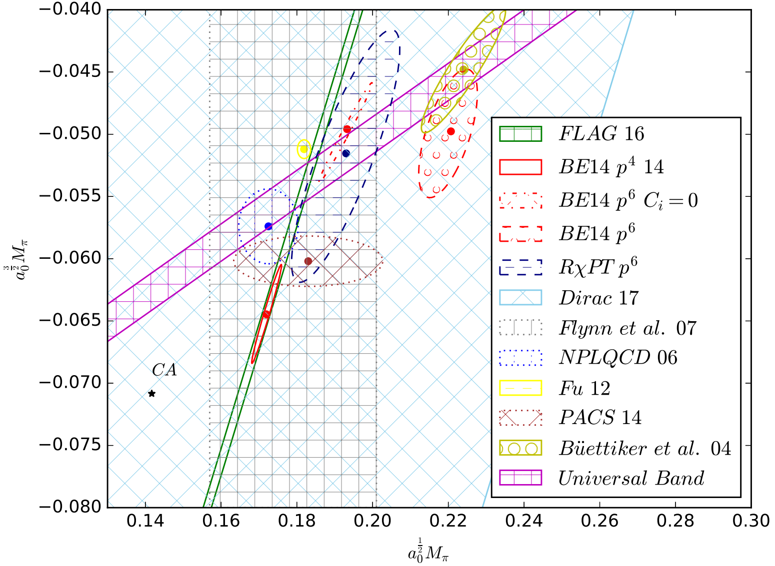
Subthreshold parameter plane



πK scattering lengths: preliminary results



πK scattering lengths: preliminary results



Looking for a unique solution

- **Unique solution**: as many subtractions as necessary to match d.o.f [Gasser, Wanders 1999]

  ↪ two non-cusp conditions ⇒ two free scattering lengths [Büttiker, Descontes-Genon, Moussallam 2003]

- Are the two scattering lengths independent?

$$\frac{8\pi(m_\pi + m_K)a^-_{SR}}{m_\pi m_K} = \frac{1}{\pi}\int_{s_{\rm th}}^{\infty}\frac{ds'}{\lambda(s')}\,\mathrm{Im}\,F^-(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty}\frac{dt'}{t'}\,\mathrm{Im}\,\frac{G^1(t',s')}{8p_\pi(t')p_K(t')}$$

	K_s	K_t	D_s	D_t	A_s	A_t	Total
$a^-_{i,SR}$ $(m_\pi 10^2)$	0.86	**6.10**	0.60	0.13	0.00	0.37	**8.05**
a^- $(m_\pi 10^2)$	–	–	–	–	–	–	8.96

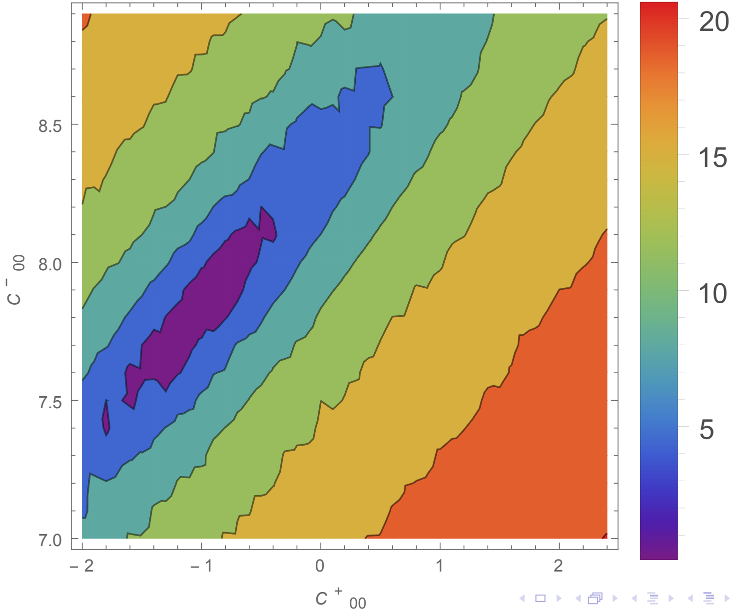
↪ once a^+ is fixed, a^- is obtained from a convergent sum rule

- New approach: **HDR** subtracted at **subthreshold**

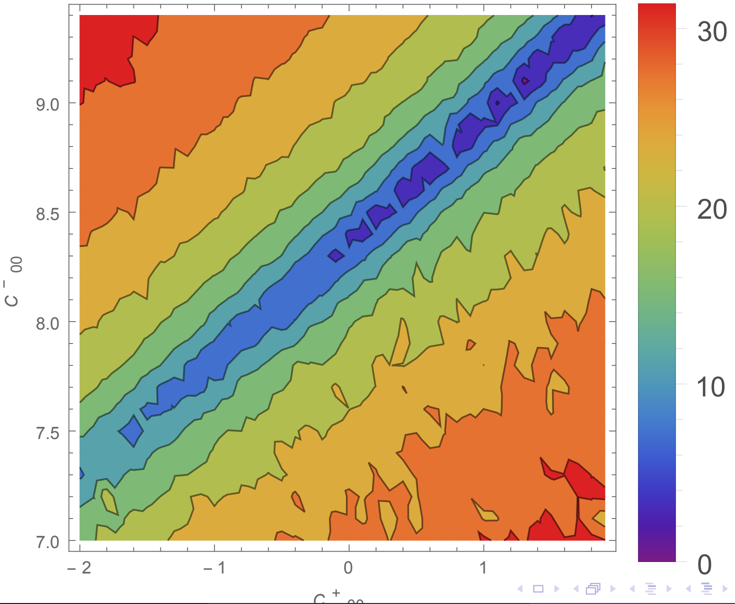
  ▷ three non-cusp conditions ⇒ three subthreshold parameters **constrained** by SR

$$\chi^2_{\rm tot} = \chi^2_{\rm phys} + \frac{\left(c^-_{00} - c^-_{00,SR}\right)^2}{\Delta {c^-_{00}}^2} + \frac{\left({c_{10}}^+ - c^+_{10,SR}\right)^2}{\Delta {c^+_{10}}^2}$$

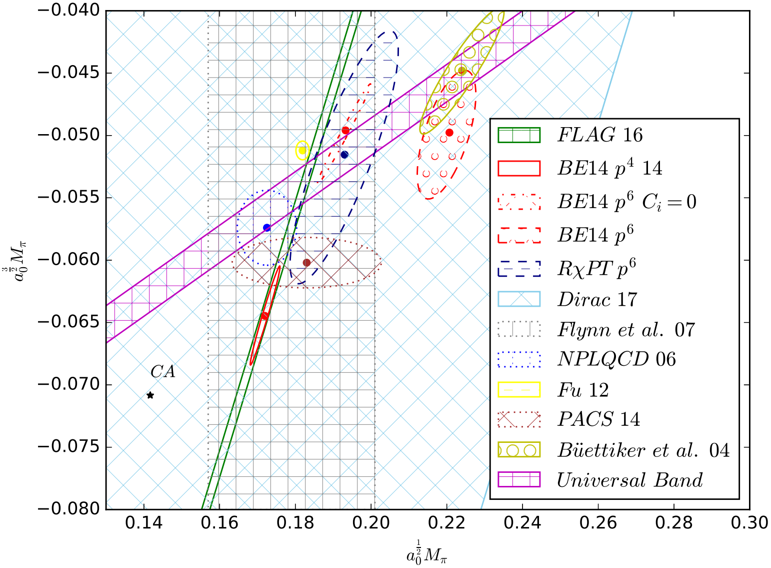
Unique solution for πK scattering



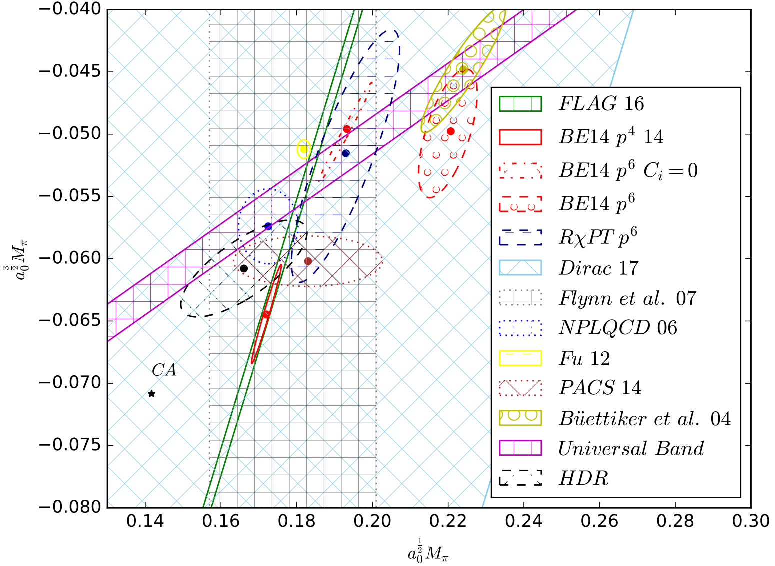
πK scattering without $\chi^2_{c^-_{00}}$



πK scattering lengths: preliminary results



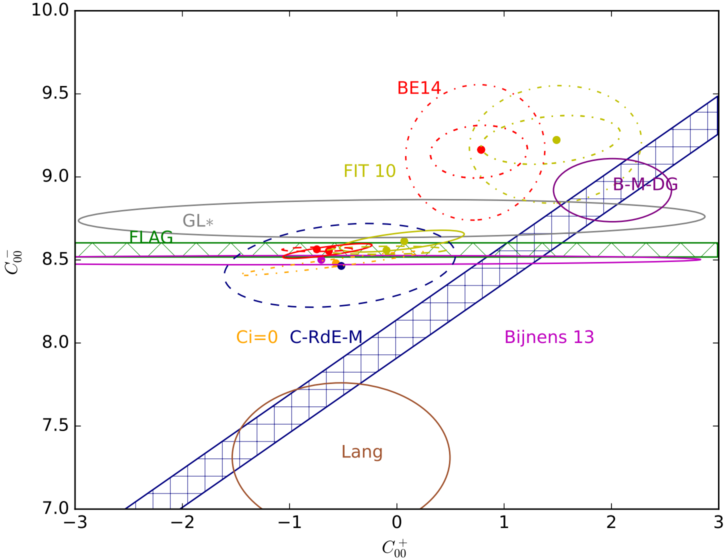
πK scattering lengths: preliminary results



Thank you

Spare slides

πK subthreshold parameter plane



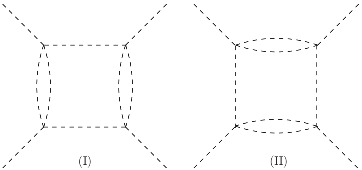
Roy equations: range of convergence

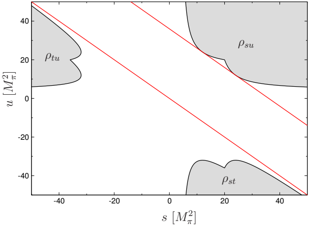
- Convergence for $T^I(s,t)$ guaranteed for $t < 4m^2$
- Where does the partial wave expansion converge?
- Assumption: Mandelstam analyticity [Mandelstam (1958,1959)]

$$T(s,t) = \frac{1}{\pi^2}\iint \mathrm{d}s'\mathrm{d}t' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

$\hookrightarrow$ integration on the support of the double spectral densities ρ

- Boundaries of ρ





- Lehmann ellipses

$\hookrightarrow$ largest ellipses, which do not enter any ρ [Lehmann (1958)]

Roy-equations: existence and uniqueness

- Solution characterized by subtraction constants and high-energy input (a, A)
- **Existence** and **uniqueness** depends on δ_i dynamically at s_m

$$m = \sum_i m_i, \qquad m_i = \begin{cases} \left\lfloor \frac{2\delta_i(s_{\rm m})}{\pi} \right\rfloor & \text{if } \delta_i(s_{\rm m}) > 0, \\ -1 & \text{if } \delta_i(s_{\rm m}) < 0, \end{cases}$$

$\lfloor x \rfloor \Rightarrow$ largest integer $\leq x$.

[Gasser, Wanders 1999, Wanders 2000]

- $m = 0$, a unique solution exists for any (a, A)
- $m > 0$, *m*-parameter family of solutions for any (a, A)
- $m < 0$, only for a specific choice of the input constrained by $|m|$ conditions
- **Physical solution** characterized by *smooth* matching

Solving Roy-equations: flow information

- **Roy-equations** rigorously valid for a finite energy range

  $\Rightarrow$ introduce a matching point s_m

- only partial waves with $J \leq J_{\max}$ are solved
- Assume **isospin limit**
- **Input**
  - High-energy region: $\mathrm{Im}\, t^I_J(s)$ for $s \geq s_m$ and for all J
  - Higher partial waves: $\mathrm{Im}\, t^I_J(s)$ for $J > J_{\max}$ and for all s
  - Inelasticities $\eta(s)$
- **Output**
  - Self-consistent solution for $\delta_{IJ}(s)$ for $J \leq J_{\max}$ and $s_{\rm th} \leq s \leq s_m$
  - Subtraction constants