Transverse momentum dependent fragmentation function: matching coefficient at NNLO

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Outlook

I will report on the ondoing calculation of small-b_T matching coefficient for transverse momentum dependent (TMD) fragmentation function (FF) within (slightly-modified) [Echevarria,Idilbi,Scememi] approach at next-to-next-to-leading order ($\sim a_s^2$).

Disclaimer

The final result is not ready.

Introduction & Motivation

There are many formulations of TMD factorization.

$$W = C_{f,f'}(z_A, z_B; Q, \mu) \otimes \int \frac{db_T}{(2\pi)^2} e^{-ik_T b_T} D_{A/f}(z_A, b_T; \mu) F_{f'/B}(z_B, b_T; \mu)$$

Differences between various approaches

(see detailed comparison [Collins, 1409.5408])

- Different interpretation of non-perturbative contributions
- Different IR regularizations
- Principal possibility to define individual TMD parton densities.

With individual TMDs

"new" CSS [Collins,textbook]
"EIS" [Echevarria,Idilbi,Scimemi]

individually
$D_{A/f}(z_A, b_T; \mu, \zeta_A)$ and $F_{f'/B}(z_B, b_T; \zeta_B)$
are well-defined

Without individual TMDs

"BN" [Becher,Neubert]

only the product
$F(x_A, b_T; \mu) D(x_B, b_T; \mu)$
is well-defined

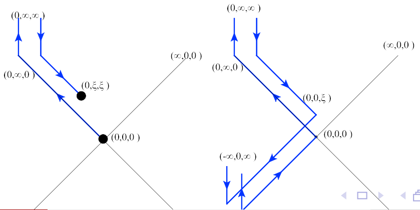
EIS formulation of TMD factorization

Within EIS formulation the TMD fractionation is given by

$$
\begin{aligned}
W \quad = \quad & H_{f,f'}(z_A, z_B; Q, \mu) \otimes \\
& \int \frac{db_T}{(2\pi)^2} e^{-ik_T b_T} \underbrace{D^{zb.}_{A/f}(z_A, b_T; \mu, \delta^+)}_{\text{TMD FF}} \underbrace{F^{zb.}_{f'/B}(z_B, b_T; \mu, \delta^-)}_{\text{TMD PDF}} \underbrace{S(b_T; \delta^+, \delta^-)}_{\text{soft factor}}
\end{aligned}
$$

- The superscript $zb.$ stands for matrix element with subtracted soft-singularities $q \sim (\lambda, \lambda, \lambda)$ (zero-bin subtraction)
- $\delta^{\pm}$ regularizes rapidity divergences.
- Zero-bin–subtraction is equal to the soft-factor [1111.4996],

$$
D^{zb.}_{A/f}(z_A, b_T; \mu, \delta^+) = \frac{D^{naive}_{A/f}(z_A, b_T; \mu, \delta^+, \delta^-)}{S(b_T, \delta^+, \delta^-)}
$$



EIS formulation of TMD factorization

$$S(b_T) = \frac{1}{N_c}\langle 0|[-\infty_n, b_T, \infty_{\bar{n}}][\infty_{\bar{n}}, 0, -\infty_n]|0\rangle, \quad [\gamma] \sim P\exp\left(-ig\int_\gamma A_\gamma\right)$$

- The collinear (and rapidity) divergence within soft-factor can appear as

$$S(b_T,\delta^+,\delta^-) = \exp\left[A(b_T)\ln(\delta^+\delta^-) + B(b_T)\right] = \sqrt{S(b_T,\delta^+,\alpha\delta^+)}\sqrt{S(b_T,\frac{\delta^-}{\alpha},\delta^-)}$$

- Re-adjusting soft-factors one defines an individual TMD parton density

$$D(z,b_T;\mu,\zeta) = \frac{\frac{\mathrm{tr}_{\mathrm{Dir,col}}}{4zN_c}\int\frac{d\xi^-}{(2\pi)^3}e^{i\xi^- k^+}\sum_X\langle 0|\gamma^+[\xi,\infty_n]q_i(\xi)|X,P_h\rangle\langle X,P_h|\bar{q}_j(0)[0,\infty_n]^\dagger|0\rangle}{\sqrt{S(b_T,\delta^+,\delta^-)}}$$

- The parameter ζ appears as a ratio of splitting $\zeta = \frac{Q^2}{\alpha} = (p^+)^2\frac{\delta^+\delta^-}{\delta^2}$
- EIS approach is equivalent to "new CSS" [Collins,Rogers,1210.2100],[EIS,1211.1947]

EIS formulation of TMD factorization

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- Re-adjusting soft-factors one defines an individual TMD parton density

$$D(z, b_T; \mu, \zeta) =$$
$$\frac{\mathrm{tr}_{\mathrm{Dir,col}}}{\;}\int \frac{d\xi^-}{\;}e^{i\xi^- k^+}\sum \langle 0|\gamma^+[\xi, \infty_n]q_i(\xi)\,|X, P_h\rangle\langle X, P_h|\bar{q}_i(0)\,[0, \infty_n]^\dagger|0\rangle$$

NOTE:

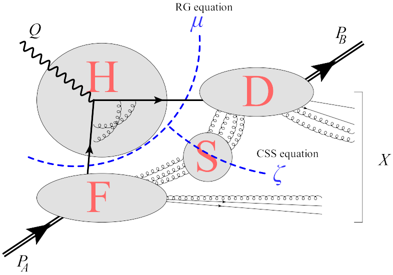
Soft factor must contain only rapidity and collinear divergences. No soft divergences.

Watch over your regulators.

Structure of TMD factorization

TMD factorization for processes with two hadrons $Q^2 \gg k_T^2$ gives

$$W = H_{f,f'}(z_A, z_B; Q, \mu) \otimes \int \frac{db_T}{(2\pi)^2} e^{-ik_T b_T} \overbrace{D_{A/f}(z_A, b_T; \mu, \zeta_A)}^{D^{naive}/\sqrt{S}} \overbrace{F_{f'/B}(z_B, b_T; \mu, \zeta_B)}^{F^{naive}/\sqrt{S}}$$



- Dependence on factorization scales is fixed by equations

$$\mu^2 \frac{d}{d\mu^2} D(z, b_T; \zeta, \mu) = 2\gamma_D(\zeta, \mu) D(z, b_T; \zeta, \mu)$$

$$\zeta \frac{d}{d\zeta} D(z, b_T; \zeta, \mu) = \frac{1}{2} \tilde{K}(b_T, \mu) D(z, b_T; \zeta, \mu)$$

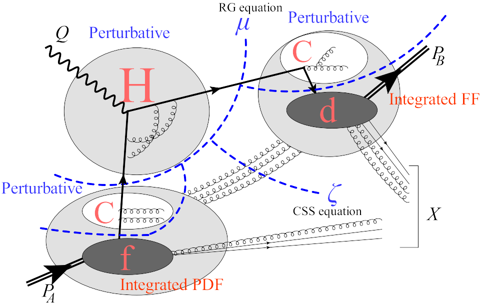
- The Cauchy-Riemman condition for existence of solution

$$\mu^2 \frac{d\tilde{K}(b_T, \mu)}{d\mu^2} = 4\zeta \frac{d\gamma_D(\zeta, \mu)}{d\zeta} = -2\gamma_K(\mu).$$

Small b_T behavior

Naively, at zero-b_T TMD-operator turns to integrated operator: i.e. at small-b_T one can perform OPE onto integrated FF
"factorization with maximum perturbative content" [Collins]

$$D(z,b_T;\zeta,\mu)=\int\frac{dy}{y}\underbrace{C\left(\frac{z}{y},b_T;\mu,\zeta,\kappa\right)}_{\text{matching coeff.}}d(z,\kappa)+\mathcal{O}(b_T)$$



- The κ-dependence is then given by DGLAP equation:

$$\kappa^2\frac{d}{d\kappa^2}d(z,\kappa)=\int_z^1\frac{dy}{y}P\left(\frac{z}{y},\kappa\right)d(y,\kappa)$$

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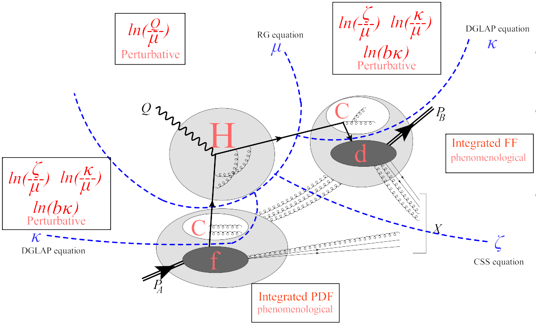
"factorization with maximum perturbative content" [Collins]

$$D(z,b_T;\zeta,\mu)=\int\frac{dy}{y}C\left(\frac{z}{y},b_T;\mu,\zeta\right)d(z,\mu_b)+\mathcal{O}(b_T)$$



- To make expression finite at $b_T \to 0$, one sets $\kappa=\mu_b\sim b_T^{-1}$.
- Works only in vicinity of $b_T=0$, further "non-perturbative" modification is needed for $b_T>0$.
- Another operator basis might work in wider b_T region, e.g. Laguerre based OPE [AV,1402.3182]

TMD PDF at NNLO [Gehrmann,Lübbert,Yang,1209.0682,1403.6451]

[Gehrmann,Lübbert,Yang,1209.0682,1403.6451] performed the NNLO calculation of coefficient function for TMD PDF in "NB" scheme.

- $$W = H(Q,\mu) \otimes \underbrace{\left[S(b_T^2)F(z_A,b_T^2)F(z_B,b_T^2)\right]}_{\text{no individual TMDs}}$$
- The analytical regulator is used:
$$\int \frac{d^dk}{(2\pi)^{d-1}}\delta_+(k^2) \;\to\; \int \frac{d^dk}{(2\pi)^{d-1}}\delta_+(k^2)\left(\frac{\nu}{k^+}\right)^\alpha$$
$$\alpha(<0?) \sim 0, \qquad \nu\; - \;\text{unphysical mass scale}$$
- The regulator is the same for both TMDs, i.e. symmetry $n \leftrightarrow \bar{n}$ is broken
- Due to "no-scale-in-dimension-regularization" argument
$$S(b_T) \equiv 1 \qquad \text{(in analytical regularization only)}$$
- The collinear divergences arrears as poles in α, and cancel in the product of TMDs and $(n \leftrightarrow \bar{n})$-symmetry is restored (the finite term is "collinear anomaly")
$$\underbrace{F_n(z_A,b_T^2)F_{\bar{n}}(z_B,b_T^2)}_{\text{finite at } \alpha\to 0}$$

TMD PDF at NNLO [Gehrmann,Lübbert,Yang,1209.0682,1403.6451]

At two-loop

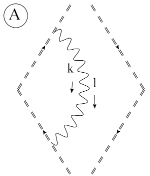
$$F_n(z_A,b_T^2)F_{\bar{n}}(z_B,b_T^2)$$
$$=\left[\mathbf{1}_A\mathbf{1}_B+a_s\underbrace{\left(\mathbf{1}_AC_B^{[1]}+C_A^{[1]}\mathbf{1}_B\right)}_{\text{finite at }\alpha\to 0}+a_s^2\underbrace{\left(\mathbf{1}_AC_B^{[2]}+C_A^{[1]}C_B^{[1]}+C_A^{[2]}\mathbf{1}_B\right)}_{\text{finite at }\alpha\to 0}+...\right]\otimes q_Aq_B$$

- Using together with RG equation one can extract individual matching coefficient functions.

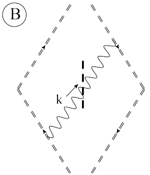
- Luck of the matching coefficient for TMD FF (needed analytical continuation)
- IR scheme dependance (??)
- Inability of analytical regulator to regularize soft-factor

Indefinite soft factor

In [Gehrmann,et al] the soft factor is set to unity. Due to "absence-of-scale-in-dimension-regularization". However, to my understanding it is just not defined.



- Analytical regulator is defined as $\left(\frac{\nu}{k^+}\right)^\alpha$ for every real gluon.
- $$\text{Diag}_A \simeq \int d^dk \frac{1}{(k^+ + i0)(k^- + i0)} \frac{1}{k^2 + i0} \sim \int \frac{dk^+}{(k^+)^{1+\epsilon}} \frac{dk^-}{(k^-)^{1+\epsilon}}$$
  =0 due to dim.reg. axiomatic.



- $$\text{Diag}_B \simeq \int d^dk \frac{e^{i(kb)_T}\delta(k^2)\theta(k^+)}{(k^+ + i0)(k^- + i0)} \left(\frac{\nu}{k^+}\right)^\alpha \sim \int_0^\infty \frac{dk^+}{(k^+)^{1+\epsilon+\alpha}} \left(\frac{b_T^2}{4}\right)^\epsilon$$
- Analytical regulator unable to regularize divergences of soft-factor, leaving it indefinite.
- Does the cancelation correct? (finite parts?)

Motivations and goals

- We would like to check the cancelation of divergences individually for every TMD
  - We need expression for soft factor
  - We need expression for naive collinear TMD
- The expression for TMD FF is under interest
  - It is novel part of information, which cannot be get from [Gehrmann,et al] (since they restrict they-self to space-like separators only)
  - It is needed for N^3LO analysis of TMD FF. So TMD FF and TMD PDF would be considered on equal footing.

$$D_{q/f}(z_A, b_T; \zeta, \mu) = \sum_j \int_{z_A}^1 \frac{dz}{z^{3-2\epsilon}} d_{q/j}(z,\mu) C_{j/f}\left(\frac{z_A}{z}, b_T; \zeta, \mu\right) + \mathcal{O}(b_T),$$

$$C_{j/f}(z, b_T; \zeta, \mu) = \underbrace{C^{[0]}_{j/f}(z, b_T; \zeta, \mu)}_{\delta_{jf}\delta(z-1)} + \overbrace{a_s}^{\frac{g^2}{(4\pi)^2}} \underbrace{C^{[1]}_{j/f}(z, b_T; \zeta, \mu)}_{\substack{\text{[Collins,textbook]} \\ \text{[EIS,1402.0869]}}} + a_s^2 \underbrace{C^{[2]}_{j/f}(z, b_T; \zeta, \mu)}_{\text{desired}} + \ldots$$

Small-b_T factorization

$$D(z, b_T) = C(z, b_T) \otimes \frac{d(z)}{z^{3-2\epsilon}},$$

$$C^{[0]} = \delta(1-z), \quad d^{[0]}(z) = \delta(1-z), \quad D^{[1]}(z, b_T) = \delta(1-z)$$

The order-by-order perturbative definition of matching coefficient:

$$C^{[1]}_{j/f} = D^{[1]}_{j/f} - \frac{d^{[1]}_{j/f}}{z^{3-2\epsilon}}$$

$$C^{[2]}_{j/f} = D^{[2]}_{j/f} - \sum_x D^{[1]}_{j/x} \otimes \frac{d^{[1]}_{x/f}}{z^{3-2\epsilon}} - \frac{d^{[2]}_{j/f}}{z^{3-2\epsilon}}$$

- The main difficulty is to calculate $D^{[2]}_{j/f}$

Perturbative structure of D

Renormalized expression for parton-parton (here parton=quark) reads

$$D \;=\; \underbrace{Z_2^{-1}}_{\substack{\text{quark}\\ \text{ren.}\\ \text{const.}}}\;\underbrace{Z_D}_{\substack{\text{TMD FF}\\ \text{ren.}\\ \text{const.}}}\;\overbrace{\Big(\underbrace{\Delta}_{\substack{\text{"naive"}\\ \text{collinear}}}\;\underbrace{S^{-\frac{1}{2}}}_{\text{SF}}\Big)}^{\text{free of IR singularities}},$$

- We perform the operator renormalization after the multiplication of Δ and $S^{-1/2}$, i.e. soft factor is a part of operator
  - There are no overlapping UV$\times$IR divergences to renormalize
  - No reason to evaluate UV part of diagrams, which significantly simplify the calculation
  - Within our regularization the renormalization and multiplication do not commute. Therefore, other way (multiplication of *renormalized* soft factor and Δ) would lead to different result (?), and may be IR-singular.
- The renormalization constants are "C-numbers" (no distributions), therefore, renormalization is multiplicative (in contrast to convolutions)

$$\Delta = \underbrace{\Big\langle\; \overbrace{qW^{\dagger}(x,b_T)}^{\substack{\text{renormalization}\\ \text{local}}}\;\; |q,X\rangle\langle q,X| \;\; \overbrace{qW^{\dagger}(0)}^{\substack{\text{renormalization}\\ \text{local}}}\;\Big\rangle}_{\text{exchange interactions are UV finite}}$$

Perturbative structure of D

$$D = \Delta^{[0]} + \boxed{\left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2} + \left(Z_D^{[1]} - Z_2^{[1]}\right)\Delta^{[0]}\right) \quad \text{1-loop}}$$

$$\boxed{\begin{aligned} &+\Bigg[\Delta^{[2]} - \frac{S^{[1]}\Delta^{[1]}}{2} - \frac{S^{[2]}\Delta^{[0]}}{2} + \frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8} + \left(Z_D^{[1]} - Z_2^{[1]}\right)\left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2}\right) \\ &\qquad + \left(Z_D^{[2]} - Z_2^{[2]} - Z_2^{[1]}Z_D^{[1]} + Z_2^{[1]}Z_2^{[1]}\right)\Delta^{[0]}\Bigg] + a_s^3 \ldots \\ &\text{2-loop} \end{aligned}}$$

Perturbative structure of D

$$D = \Delta^{[0]} + \left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2} + \underbrace{\left(Z_D^{[1]} - Z_2^{[1]}\right)\Delta^{[0]}}_{\text{pure UV}}\right) \quad \text{1-loop}$$

$$+ \left[\Delta^{[2]} - \frac{S^{[1]}\Delta^{[1]}}{2} - \frac{S^{[2]}\Delta^{[0]}}{2} + \frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8} + \left(Z_D^{[1]} - Z_2^{[1]}\right)\left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2}\right)\right.$$

$$\left. + \underbrace{\left(Z_D^{[2]} - Z_2^{[2]} - Z_2^{[1]}Z_D^{[1]} + Z_2^{[1]}Z_2^{[1]}\right)\Delta^{[0]}}_{\text{pure UV}}\right] + a_s^3 \ldots \quad \text{2-loop}$$

Perturbative structure of D

$$D = \Delta^{[0]} + \underbrace{\left(\overbrace{\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2}}^{\text{IR-good}} + \overbrace{\left(Z_D^{[1]} - Z_2^{[1]}\right)\Delta^{[0]}}^{\text{pure UV}} \right)}_{\text{1-loop}}$$

$$\underbrace{+ \left[\Delta^{[2]} - \frac{S^{[1]}\Delta^{[1]}}{2} - \frac{S^{[2]}\Delta^{[0]}}{2} + \frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8} + \overbrace{\left(Z_D^{[1]} - Z_2^{[1]}\right)\underbrace{\left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2}\right)}_{\text{IR-good}}}^{\text{UV-sub-divergences subtraction}} + \underbrace{\left(Z_D^{[2]} - Z_2^{[2]} - Z_2^{[1]}Z_D^{[1]} + Z_2^{[1]}Z_2^{[1]}\right)\Delta^{[0]}}_{\text{pure UV}} \right]}_{\text{2-loop}} + a_s^3 \ldots$$

IR-good implies cancelation of mass-divergences and rapidity divergences, but not collinear divergences,which to be cancel by matching procedure

Regularizations

- Massless quarks
- On-shell incoming partons

dimensional regularization $d = 4 - 2\epsilon$

- UV divergences
- Other IR divergences (mass-divergences) $(\lambda, \lambda, \lambda)$
- Collinear divergences $(\lambda^2, 1, \lambda)$
- Rapidity divergences $(\lambda, \lambda^{-1}, 1)$

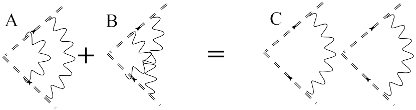
"δ-regularization"

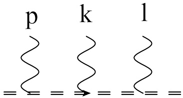
δ-regularization

In original EIS approach the rapidity divergences were regularized as

$$\frac{1}{k^\pm + i0} \longrightarrow \frac{1}{k^\pm + i\delta^\pm}, \qquad \delta^\pm \to +0.$$

At two-loop such regularization violates exponentiation, and may result to non-cancelation of divergences.



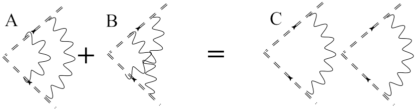


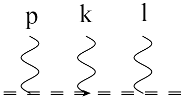
$$= \frac{1}{(p^+ + i\delta)(p^+ + k^+ + i\delta)(p^+ + k^+ + l^+ + i\delta)}$$

Within original δ-regularization, the exponentiation is broken

$$\mathrm{Diag}_A + \mathrm{Diag}_B = \frac{\mathrm{Diag}_C^2}{2} + \delta^+ \underbrace{\int \frac{d^dk}{k^2}\frac{d^dl}{l^2}\frac{1}{(k^+ + l^+)k^+l^+(k^- + l^-)k^-}}_{\frac{1}{\delta^+}\ \text{divergent}}$$

- That can result to artificial singularities in δ
- To incomplete cancelation of $\ln\delta$, that will cause problems at higher loops.





$$= \frac{1}{(p^+ + i\delta)(p^+ + k^+ + 2i\delta)(p^+ + k^+ + l^+ + 3i\delta)}$$

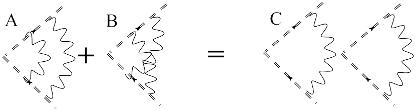
δ-regularization preserving exponentiation

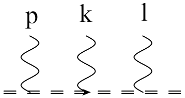
The regularization should be implemented on the level of operator

$$P \exp\left[-ig\int_0^\infty d\sigma A_\pm(\sigma n)\right] \longrightarrow P\exp\left[-ig\int_0^\infty d\sigma A_\pm(\sigma n) e^{-\delta^\pm|\sigma|}\right]$$

Then exponentiation is exact

$$\text{Diag}_A + \text{Diag}_B = \frac{\text{Diag}_C^2}{2}$$





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Then exponentiation is exact

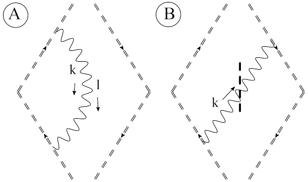
$$\text{Diag}_A + \text{Diag}_B = \frac{\text{Diag}_C^2}{2}$$

In any form, δ-regularization violate gauge-invariance linearly, beware of linearly divergent integrals.

- Is there any regularization <u>with scale</u> for light-like half-infinite Wilson lines without any problem?

How does it work at one-loop

Soft factor



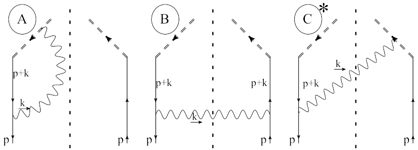
Notation: $$\boldsymbol{\delta} = -2\delta^+\delta^-$$
$$\boldsymbol{B} = \frac{b_T^2}{4}$$
$$L_\delta = \ln\left(\boldsymbol{B}\boldsymbol{\delta}e^{2\gamma_E}\right)$$

$$\begin{aligned}\text{Diag}_A &= -2C_F\boldsymbol{\delta}^{-\epsilon}\Gamma^2(\epsilon)\Gamma(1-\epsilon)\\ \text{Diag}_B &= -2C_F\Big[\underbrace{\boldsymbol{B}^\epsilon\Gamma(-\epsilon)(L_\delta-\psi(-\epsilon)-\gamma_E)}_{\text{UV finite}}-\boldsymbol{\delta}^{-\epsilon}\Gamma^2(\epsilon)\Gamma(1-\epsilon)\Big]\end{aligned}$$

- Only single logarithm of $\boldsymbol{\delta}$ remains at all orders of ϵ-expansion.
- Expression is $\frac{1}{\epsilon^2}$ divergent, it is collinear singularity.

How does it work at one-loop

Collinear matrix element



Notation:

$$\boldsymbol{\lambda}_\delta = \ln\left(\frac{\delta^+}{p^+}\right)$$

$$\mathbf{L}_b = \ln\left(\frac{b_T^2}{4}\frac{\mu^2}{e^{-2\gamma_E}}\right)$$

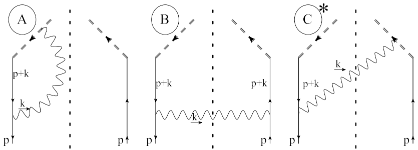
$$\Delta^{[1]} = \frac{C_F}{z^2}\Bigg[\frac{-1}{\epsilon}\left(\frac{2(1+z^2)}{1-z}\right)_+ + \delta(\bar{z})\frac{3+4\boldsymbol{\lambda}_\delta}{\epsilon} + 2\left(\bar{z} - \mathbf{L}_b\frac{1+z^2}{1-z}\right)_+ + \delta(\bar{z})(1+3\mathbf{L}_b+4\mathbf{L}_b\boldsymbol{\lambda}_\delta) + ..\Bigg]$$

- Only single logarithm ($\boldsymbol{\lambda}_\delta$) at all orders of ϵ-expansion.
- $\boldsymbol{\lambda}_\delta$ comes from singularity at $z \to 0$

$$\Delta(z) = \frac{1}{z}\left(z \lim_{\delta\to 0}\Delta(z)\right)_+ + \delta(1-z)\lim_{\delta\to 0}\int_0^1 dx x\Delta(x)$$

How does it work at one-loop

Renormalization



Notation: $\boldsymbol{\lambda}_\delta = \ln\left(\frac{\delta^+}{p^+}\right)$

$$\mathrm{Diag}_A \quad = \quad 0, \qquad \text{"no-scale-in-dimensional-regularization"}$$

IR-singularity exactly cancel UV-singularity

$$\mathrm{Diag}_{A+A^*}\Big|_{UV} \quad = \quad C_F \frac{4}{\epsilon}(1+\boldsymbol{\lambda}_\delta)\delta(\bar z), \qquad \mathrm{SF}_{A+A^*}\Big|_{UV} = a_s C_F \left(\frac{-4}{\epsilon^2} - 4\frac{\ln\left(\frac{2\delta^+\delta^-}{\mu^2}\right)}{\epsilon}\right).$$

Combining it together at $\delta^+\delta^- = \left(\frac{\delta^+}{p^+}\right)^2\zeta$ we find renormalization constant

$$Z_D^{[1]} = -C_F\left(\frac{2}{\epsilon^2} + \frac{4+2\ln\left(\frac{\mu^2}{2\zeta}\right)}{\epsilon}\right).$$

How does it work at one-loop

$$D^{[1]}=\overbrace{\Delta^{[1]}-\frac{S^{[1]}}{2}}^{\delta\text{ cancels}}-Z_2^{[1]}+Z_D^{[1]} = \frac{C_F}{z^2}\Big[\frac{-2}{\epsilon}\left(\frac{(1+z^2)}{1-z}\right)_+ +2\left(\bar z-\mathbf{L}_b\frac{1+z^2}{1-z}\right)_+ +\delta(\bar z)\left(-\boldsymbol{L}_b^2+2\boldsymbol{L}_b\boldsymbol{L}_\zeta+3\boldsymbol{L}_b+1-\frac{\pi^2}{6}\right)\Big].$$

- In our set of regularization d is scaleless, i.e. all diagrams are equal to zero.
- Thus, d is pure UV-renormalization constant (DGLAP kernel)

$$d^{[1]}=-2\frac{C_F}{\epsilon}\left(\frac{1+z^2}{1-z}\right)_+$$

- The singularities at $\epsilon\to 0$ cancel is $\Delta^{[1]}-\frac{d^{[1]}}{z^{2-\epsilon}}$

One-loop result

$$C^{[1]}=\frac{C_F a_s}{z^2}\Big[-2\mathbf{L}_{b/z}P_{qq}(z)+2\bar z+\delta(\bar z)\left(-\boldsymbol{L}_b^2+2\boldsymbol{L}_b\boldsymbol{L}_\zeta+3\boldsymbol{L}_b-\frac{\pi^2}{6}\right)\Big].$$

- Coincide with [Collins,textbook]

Two-loop: general structure

Counting powers of $\ln\delta$

- Logarithm of soft-factor must be proportional to single $\ln(\delta^+\delta^-)$, otherwise definition of individual TMDs impossible.

$$S = \exp\left[A\ln(\delta^+\delta^-) + B\right] = 1 + \underbrace{S^{[1]}}_{C_F \ln\delta} + \underbrace{S^{[2]}}_{C_F^2 \ln^2\delta + C_F C_A \ln\delta + C_F N_F \ln\delta}$$

- $\Delta^{[1]} \sim C_F\left[(...)_+ + \delta(\bar{z})(\ln\delta + ..)\right]$

$$D^{[2]} = \Delta^{[2]} - \frac{S^{[1]}\Delta^{[1]}}{2} - \frac{S^{[2]}\Delta^{[0]}}{2} + \frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8} + ...$$

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- $\Delta^{[1]} \sim C_F\left[(...)_+ + \delta(\bar{z})(\ln\delta + ..)\right]$

$C_F C_A$ and $C_F N_f$ part

$$D^{[2]} = \Delta^{[2]} - \cancel{\frac{S^{[1]}\Delta^{[1]}}{2}} - \frac{S^{[2]}\Delta^{[0]}}{2} + \cancel{\frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8}} + ...$$

- Structure of $\Delta^{[2]} \sim C_A$ and $\sim N_f$ part should be

$$(\text{free of } \ln\delta)_+ + \delta(1-z)\,(\text{linear in } \ln\delta)$$

- Rather straightforward cancelation, can be traced diagram-by-diagram.

Two-loop: general structure

Counting powers of $\ln\delta$

- Logarithm of soft-factor must be proportional to single $\ln(\delta^+\delta^-)$, otherwise definition of individual TMDs impossible.

$$S=\exp\left[A\ln(\delta^+\delta^-)+B\right]=1+\underbrace{S^{[1]}}_{C_F\ln\delta}+\underbrace{S^{[2]}}_{C_F^2\ln^2\delta+C_FC_A\ln\delta+C_FN_F\ln\delta}$$

- $\Delta^{[1]}\sim C_F\left[(...)_++\delta(\bar z)(\ln\delta+..)\right]$

C_F^2 part

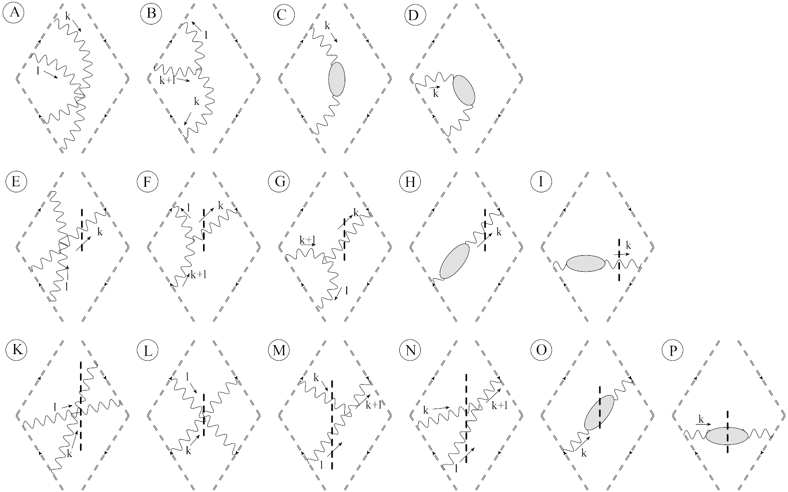
$$D^{[2]}=\Delta^{[2]}-\underbrace{\frac{S^{[1]}\Delta^{[1]}}{2}}_{\ln\delta\times(..)_++\delta(\bar z)(\ln^2\delta+..)}-\underbrace{\frac{S^{[2]}\Delta^{[0]}}{2}+\frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8}}_{\delta(\bar z)(\ln^2\delta+..)}+....$$

- Structure of $\Delta^{[2]}\sim C_F^2$ part should be

$$(\ln\delta+..)_++\delta(1-z)\left(\ln^2\delta+\ln\delta+..\right)$$

- Complected cancelation between higher ϵ terms of products of one-loop expressions.

Soft factor evaluation

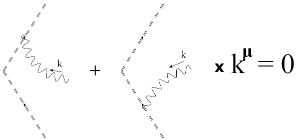


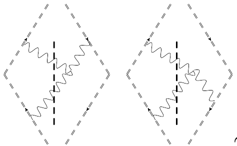
- Only web-diagrams (since exponentiation is preserved)
- Complex-conjugated and mirror-conjugated diagrams to be added.

Problems of δ-regularizations

Violation and restoration of gauge invariance

$$k_\mu \left(\frac{n^\mu}{k^+ - i\delta^+} - \frac{\bar{n}^\mu}{k^- + i\delta^-} \right) \sim \delta,$$



- Diagrams with gluon-polarization and F and M.
-  $\sim \delta(b_T)$ part is an artifact of regularization.

Attention point!

- δ-regularization can regularize soft divergences, e.g. diag F. Check and remove by hands.

Cancelation of δ^ϵ

- The general form for the diagram is $\boldsymbol{\delta} = -2\delta^+\delta^-$, $\boldsymbol{B} = b_T^2/4$

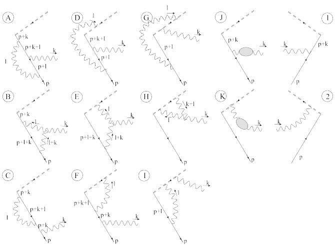
$$\text{Diag} = \underbrace{\boldsymbol{\delta}^{2\epsilon}a(\epsilon) + \boldsymbol{\delta}^\epsilon\boldsymbol{B}^\epsilon b(\epsilon) + \boldsymbol{B}^{2\epsilon}[f_2(\epsilon)\ln^2(\boldsymbol{\delta})}_{\text{To cancel}} + f_1(\epsilon)\ln(\boldsymbol{\delta}) + f_0(\epsilon)]$$

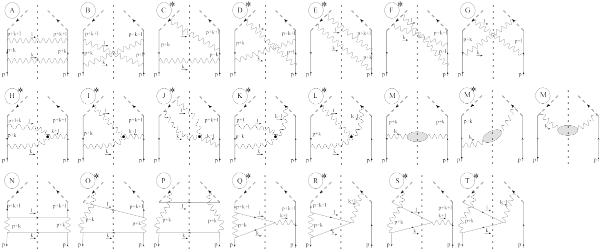
- The direct calculation confirms the cancelation

Final result takes the form

$$S = \exp\left[a_s\boldsymbol{B}^\epsilon\left(w_1(\epsilon)\ln\left(\delta^+\delta^-\right) + r_1(\epsilon)\right) + a_s^2\boldsymbol{B}^{2\epsilon}\left(w_2(\epsilon)\ln\left(\delta^+\delta^-\right) + r_2(\epsilon)\right) + a_s^3...\right]$$

$\Delta^{[2]}$ evaluation





All virtual-virtual diagrams equal to zero.

Evaluation: technical details

$$\text{typical expression} = \int \frac{dl\, dk}{(2\pi)^{2d}} \delta\left(\frac{\bar{z}}{z}p^+ - k^+\right) e^{i(kb)_T} F\left[k^+, l^+, \delta, ...\right]$$

- Introduce
$$1 = \int_{-\infty}^{\infty} d\omega\; p^+ \delta(\omega p^+ - l^+)$$
- All eikonal propagator, turns to $\frac{1}{\omega + i\delta}$, $\frac{1}{\bar{z} - i\delta z}$, etc.
- Loop integrals are simple: (3 integrals for VR, 6+3 integrals for RR)
- Collect diagrams to cancel mass-divergences.
- Integrate over ω, and z (if necessary) (mathematica+hands) (most difficult part: phases)
$$\Delta(z) = \frac{1}{z}\left(z \lim_{\delta\to 0}\Delta(z)\right)_+ + \delta(1-z) \lim_{\delta\to 0}\int_0^1 dx x\Delta(x)$$

Cancelation of divergences at C_F

Cancelation of divergences in C_F^2-part is shown

- Individually diagrams diverge as $\ln^4 \delta$
  - $\ln^4 \delta$ and $\ln^3 \delta$ cancel in sum of diagrams for $\Delta^{[2]}$
  - $\ln^2 \delta$ and $\ln \delta$ cancel in the combination

$$\Delta_{CF}^{[2],+} - \frac{S^{[1]}\Delta^{[1]}}{2}\Big|_{``plus''} \qquad \text{free of } \boldsymbol{\lambda}_\delta.$$

$$\Delta_{CF}^{[2],\delta} - \frac{S^{[1]}\Delta^{[1]}}{2} + \frac{S^{[1]}S^{[1]}\Delta^{[0]}}{8} \qquad \text{free of } \boldsymbol{\lambda}_\delta,$$

- The higher orders of ϵ-expansion ($\sim \epsilon, \epsilon^2$) of 1-loop cancel
- Renormalization after soft function multiplication $Z(\Delta S)$

Cancelation of divergences at N_f

- All divergences cancel (a la 1-loop)

Cancelation of divergences at C_A

- $\Delta_{C_A}^{+}$ is free of $\ln \delta$ (as it should be)
- $\Delta_{C_A}^{\delta}$ is under evaluation.
- "Surprises" are not expected.

Conclusion

- The first calculation of $b_T \to 0$ matching coefficient is (nearly) done.
- The results to be used for NNLO analysis of SIDIS.

$$
\begin{aligned}
\mathrm{Part}_{N_f} \qquad & \frac{4P_{qq}}{27}\left(28+30\mathbf{L}_b+9\mathbf{L}_b^2\right)-\frac{8\mathbf{L}_b\left(\left(z^2+1\right)\ln z-\bar{z}z\right)}{3(1-z)}\\
&+2\frac{\left(1+z^2\right)\ln^2 z}{3(1-z)}+\frac{4\ln z\left(z^2-9z-2\right)}{9(1-z)}+4\frac{7z-5}{9}\\
&+\delta(\bar{z})\Bigg[-\frac{4}{3}\left(-\frac{2}{3}\mathbf{L}_b^3+\mathbf{L}_b^2\mathbf{L}_\zeta-\frac{\pi^2}{3}\mathbf{L}_b\right)-\frac{20}{9}\left(3\mathbf{L}_b+2\mathbf{L}_b\mathbf{L}_\zeta\right)\\
&-\frac{112}{27}\mathbf{L}_\zeta+\frac{2}{9}\mathbf{L}_b^2+\left(-\frac{832}{81}+\frac{5\pi^2}{9}+\frac{28\zeta_3}{9}\right)\Bigg]
\end{aligned}
$$