# TMD Evolution: the Small-x Perspective

Yuri Kovchegov
The Ohio State University

based on arXiv:1505.01176 with Matt Sievert

# Outline

- Introduction
  - Classical fields, Wilson lines
  - Non-linear small-x evolution

- TMD evolution
  - Large-x case
  - Small-x: quark TMDs of an unpolarized target
  - Small-x TMDs of a polarized target: an outlook

# Introduction:
# calculations in saturation physics

# Two-step prescription

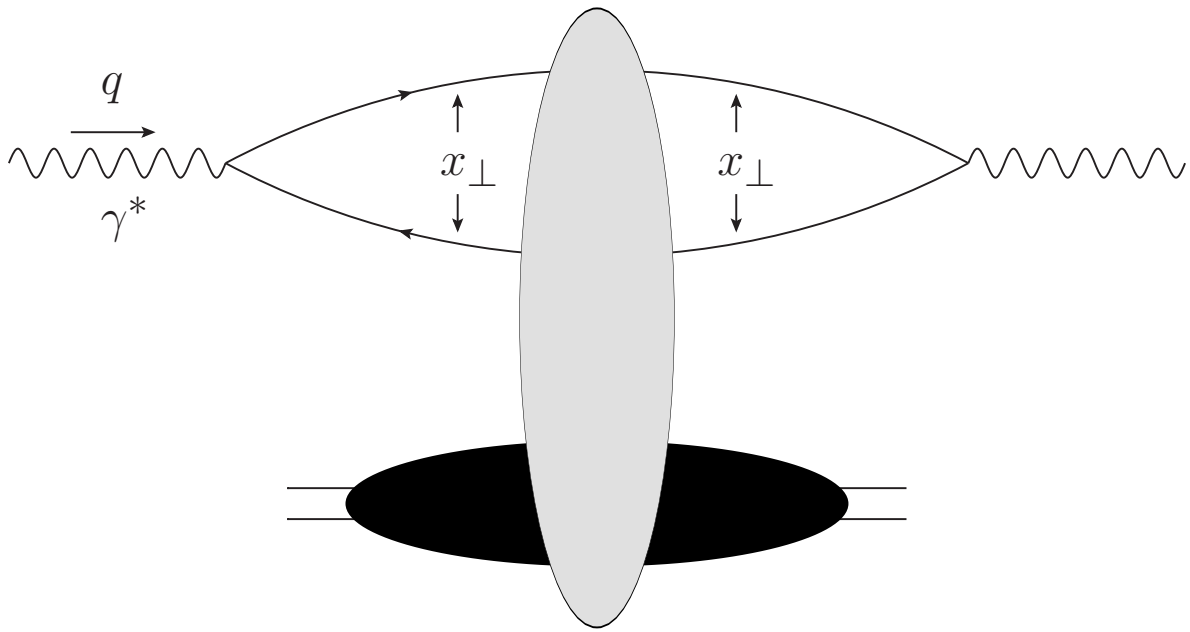
To calculate observables in the saturation picture one has to follow the two-step procedure:

- Calculate the observable in the classical approximation.
- Include nonlinear small-x evolution corrections (BK/JIMWLK), introducing energy-dependence.
- (To compare with experiment, need to at least fix the scale of the running coupling.)

# DIS: Quasi-Classics

# Dipole picture of DIS

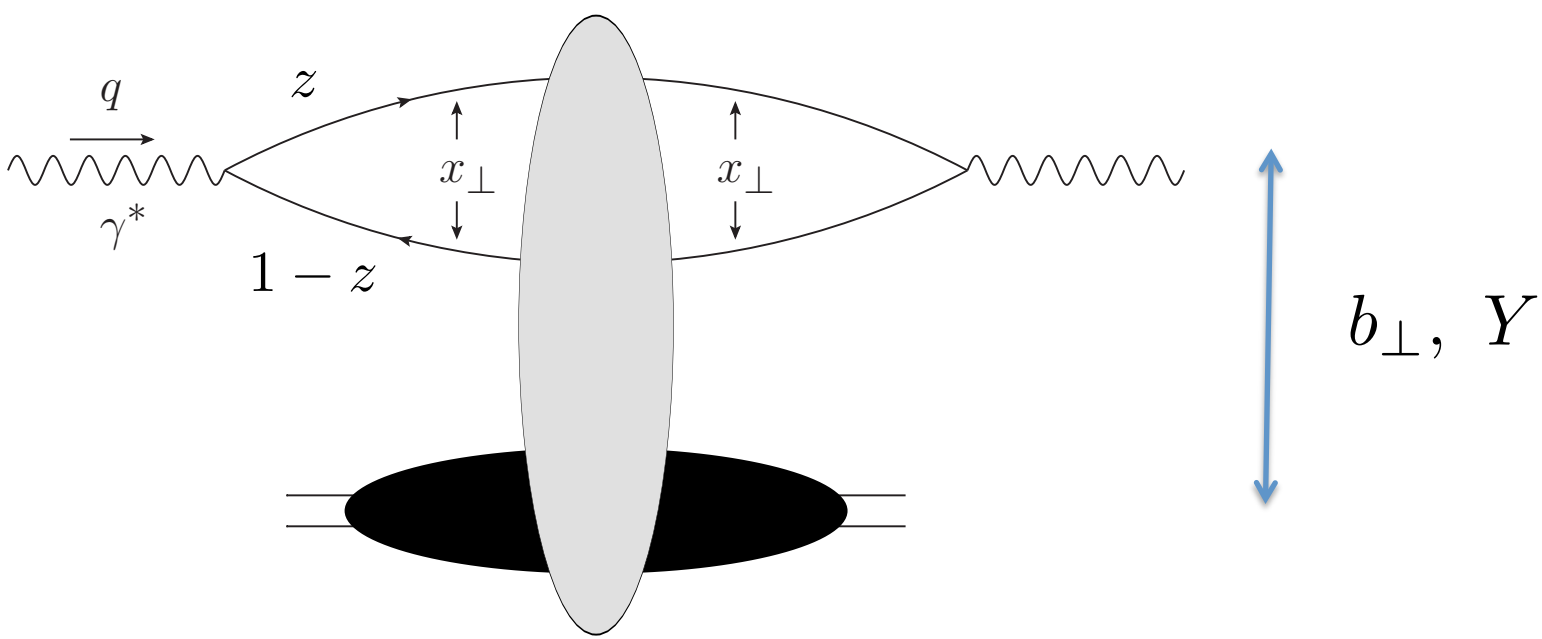
- In the dipole picture of DIS the virtual photon splits into a quark-antiquark pair, which then interacts with the target.
- The total DIS cross section and structure functions are calculated via:



# Dipole Amplitude

- The total DIS cross section is expressed in terms of the (Im part of the) forward quark dipole amplitude N:

$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\,\pi}\, d^2 b_\perp \int_0^1 \frac{dz}{z\,(1-z)}\, |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2\; N(\vec{x}_\perp, \vec{b}_\perp, Y)$$



# Dipole Amplitude

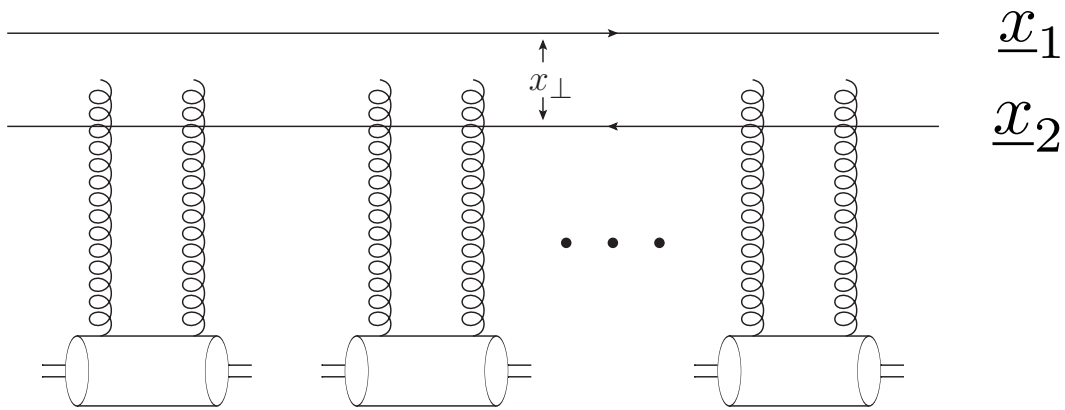
- The quark dipole amplitude is defined by

$$N(\underline{x}_1, \underline{x}_2) = 1 - \frac{1}{N_c} \left\langle \mathrm{tr} \left[ V(\underline{x}_1)\, V^\dagger(\underline{x}_2) \right] \right\rangle$$

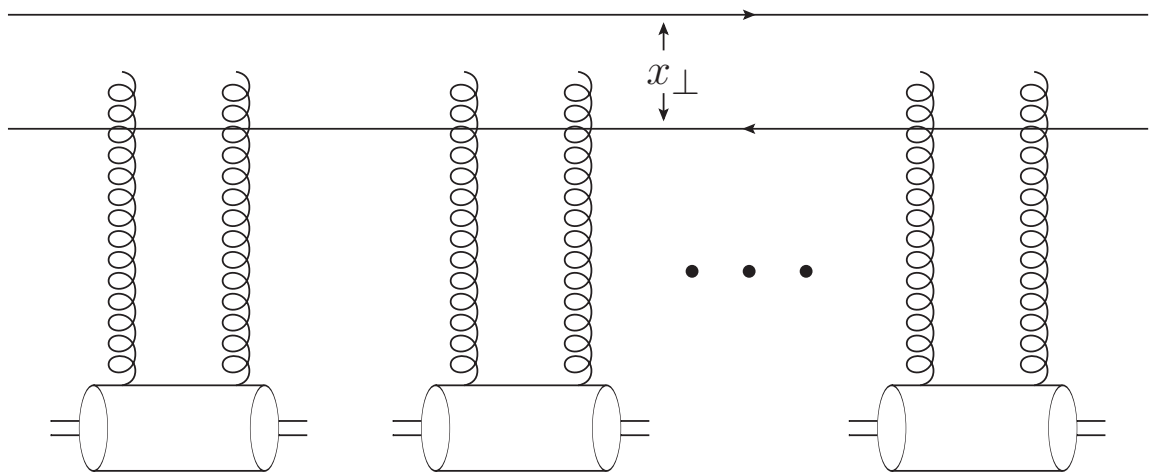
- Here we use the Wilson lines along the light-cone direction

$$V(\underline{x}) = \mathrm{P} \exp \left[ i\, g \int_{-\infty}^{\infty} dx^+\, A^-(x^+, x^- = 0, \underline{x}) \right]$$

- In the classical Glauber-Mueller/McLerran-Venugopalan approach the dipole amplitude resums multiple rescatterings:

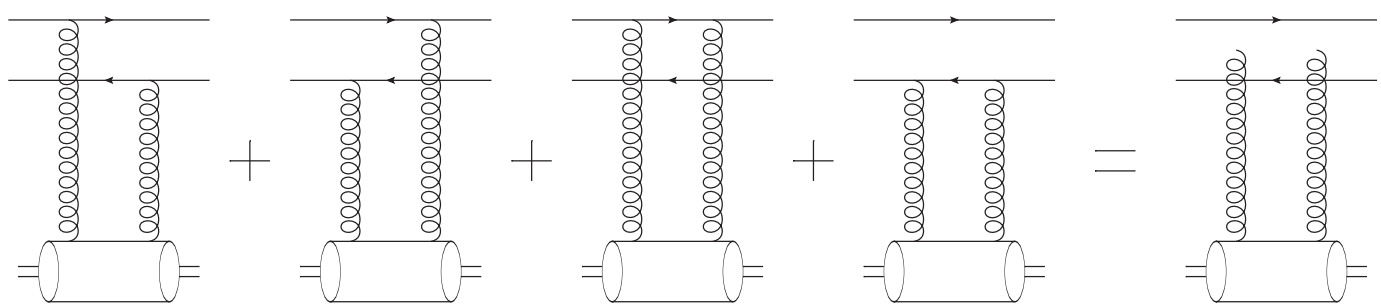


# Quasi-classical dipole amplitude




A.H. Mueller, '90

Lowest-order interaction with each nucleon – two gluon exchange – the same resummation parameter as in the MV model: $\alpha_s^2 A^{1/3}$

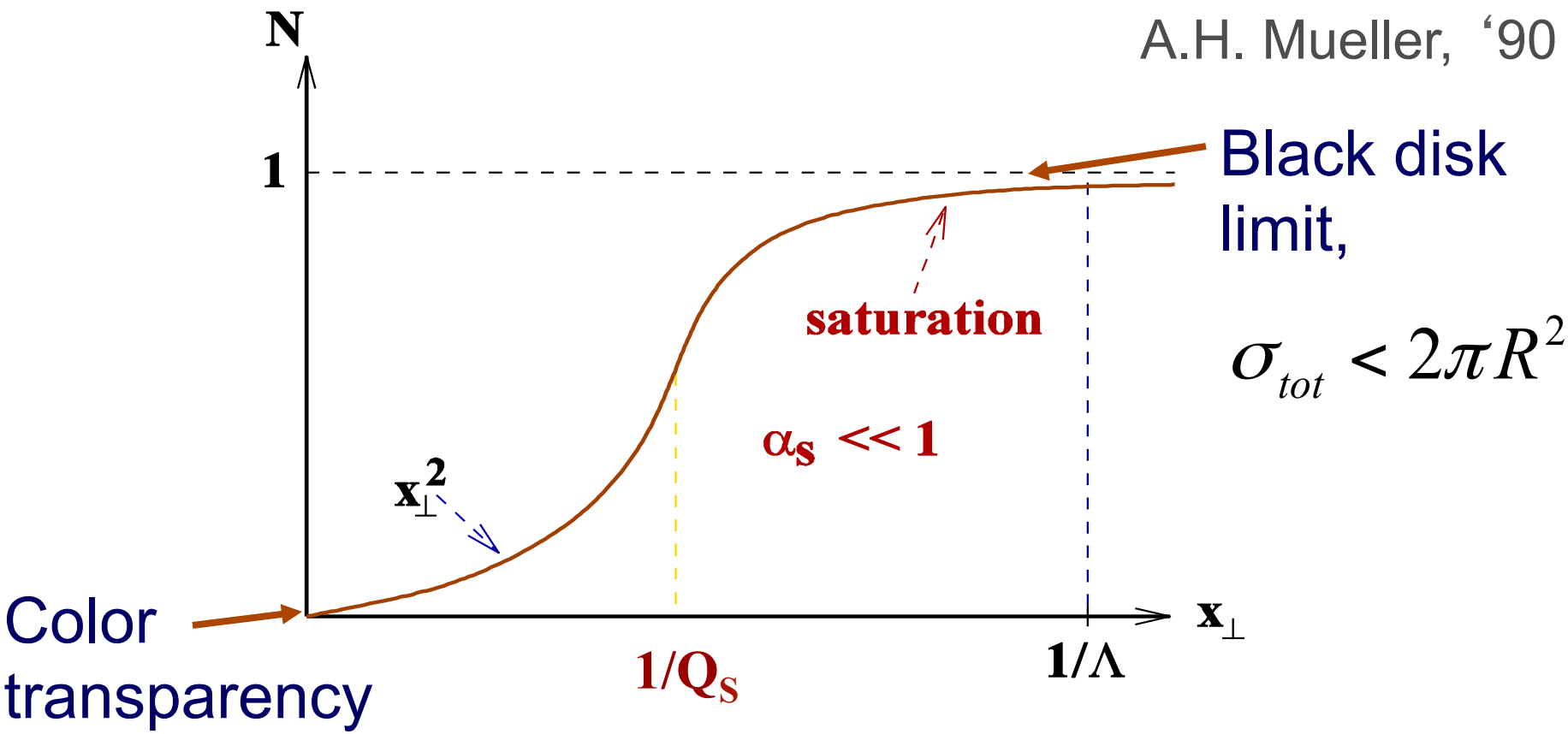


# DIS in the Classical Approximation

The dipole-nucleus amplitude in the classical approximation is

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2\, Q_s^2}{4}\, \ln \frac{1}{x_\perp\, \Lambda}\right]$$

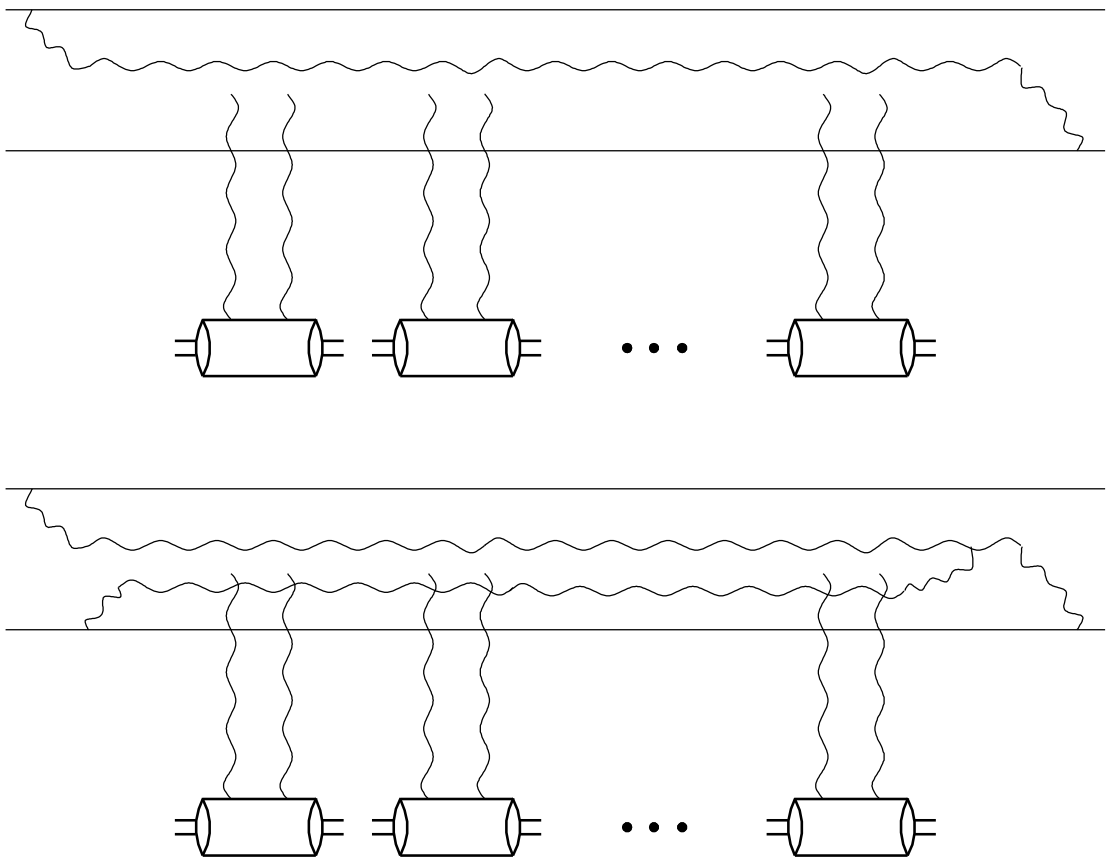
A.H. Mueller, '90



# DIS: Small-x Evolution

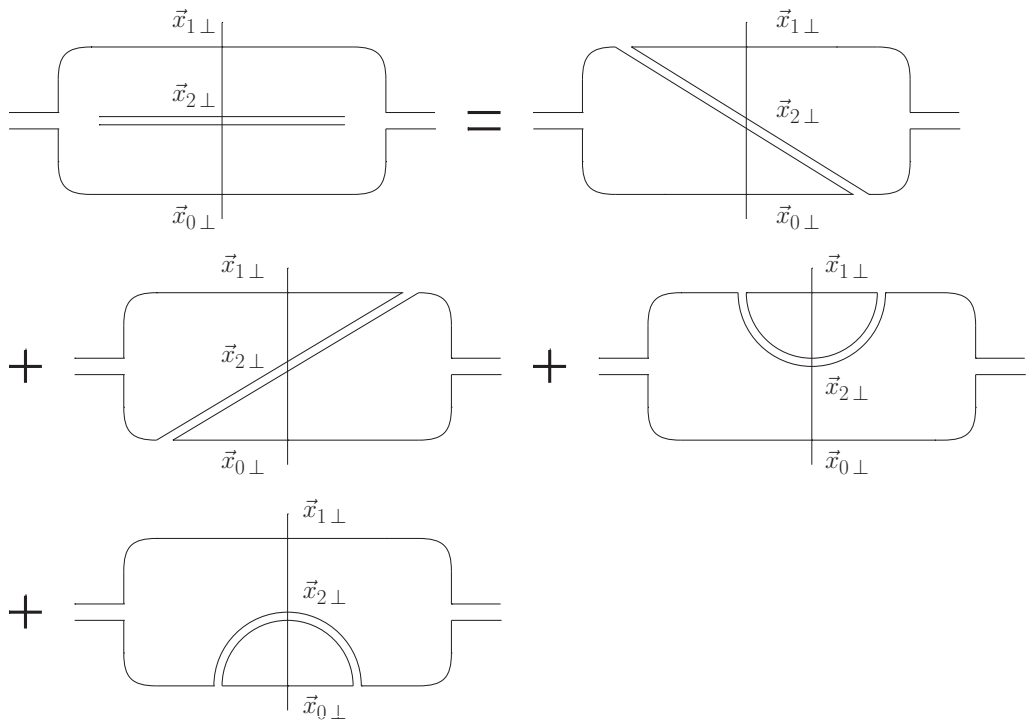
# Dipole Amplitude

- The energy dependence comes in through nonlinear small-x BK/JIMWLK evolution, which resums the long-lived s-channel gluon corrections:



$$\alpha_s \ln \frac{1}{x} \sim \alpha_s Y \sim 1$$

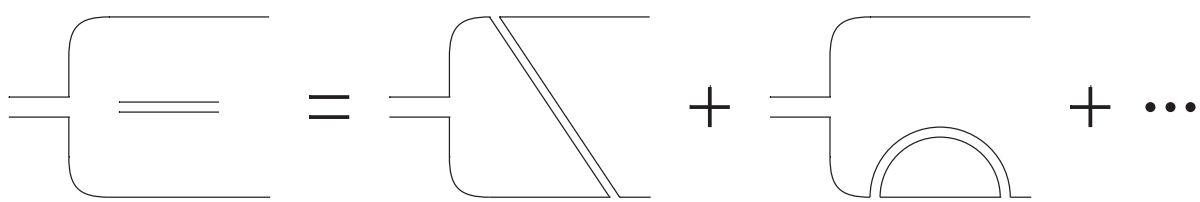
# Notation (Large-$N_C$)




Real emissions in the amplitude squared

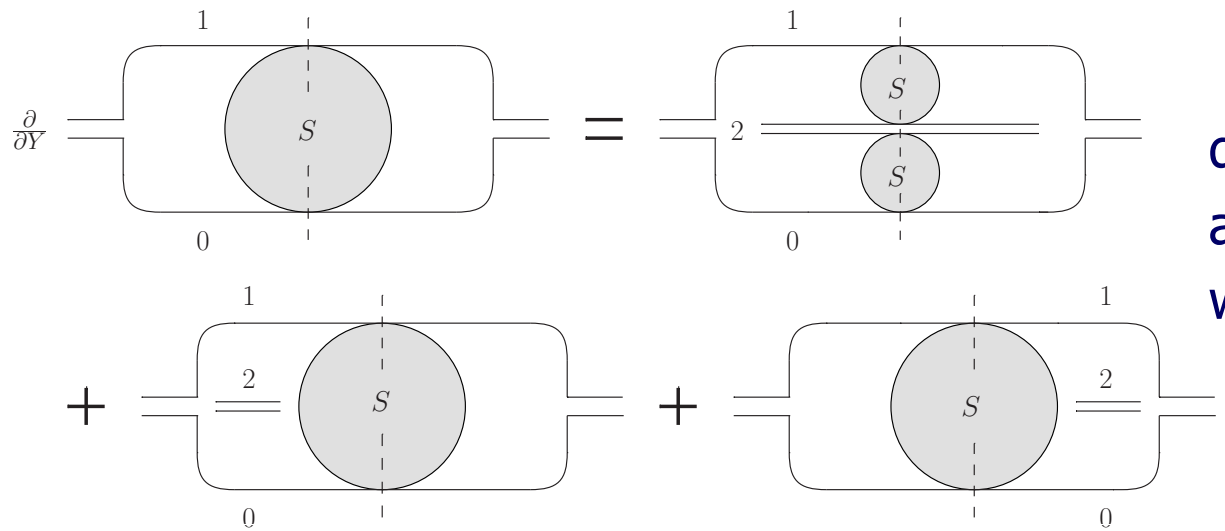
(dashed line – all Glauber-Mueller exchanges at light-cone time =0)

Virtual corrections in the amplitude (wave function)



# Nonlinear Evolution

To sum up the gluon cascade at large-$N_C$ we write the following equation for the dipole S-matrix:



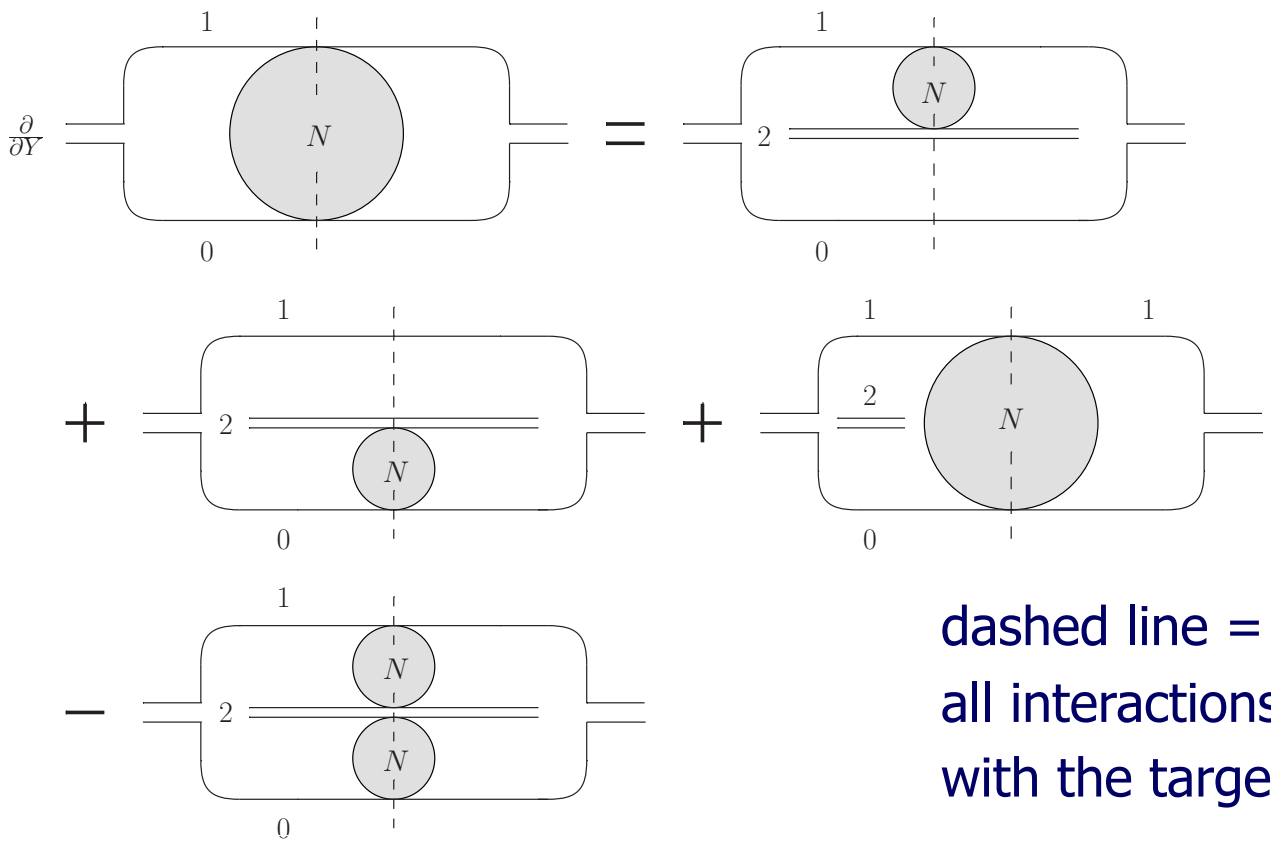

dashed line = all interactions with the target

$$\partial_Y S_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{01}^2}{x_{02}^2\, x_{21}^2} \left[ S_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, S_{\mathbf{x}_2,\mathbf{x}_1}(Y) - S_{\mathbf{x}_0,\mathbf{x}_1}(Y) \right]$$

Remembering that S= 1-N we can rewrite this equation in terms of the dipole scattering amplitude N.

# Nonlinear evolution at large $N_c$

As N=1-S we write

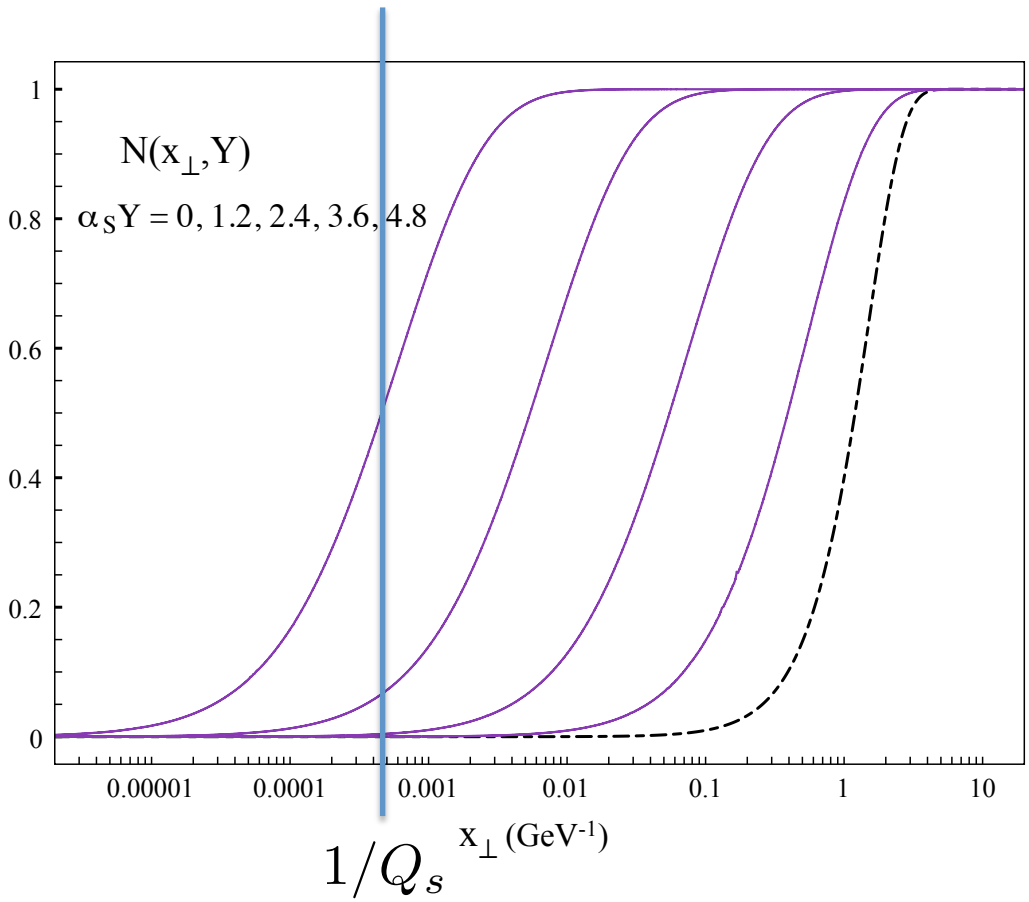



dashed line =
all interactions
with the target

$$\partial_Y N_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s\, N_c}{2\,\pi^2} \int d^2x_2\, \frac{x_{01}^2}{x_{02}^2\, x_{21}^2}\, \left[N_{\mathbf{x}_0,\mathbf{x}_2}(Y) + N_{\mathbf{x}_2,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, N_{\mathbf{x}_2,\mathbf{x}_1}(Y)\right]$$

Balitsky ‘96, Yu.K. ‘99

# Evolution of the Dipole Amplitude with Rapidity

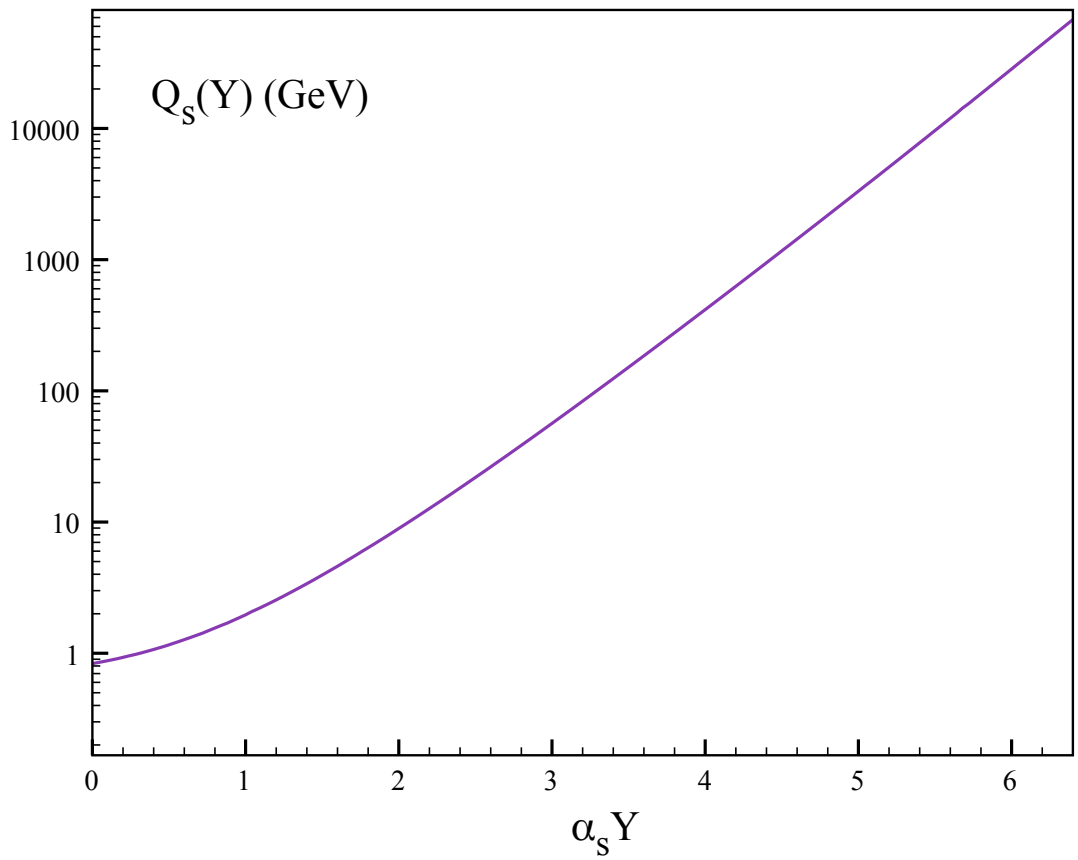



numerical solution
by J. Albacete '03

BK solution preserves the black disk limit (N<1):

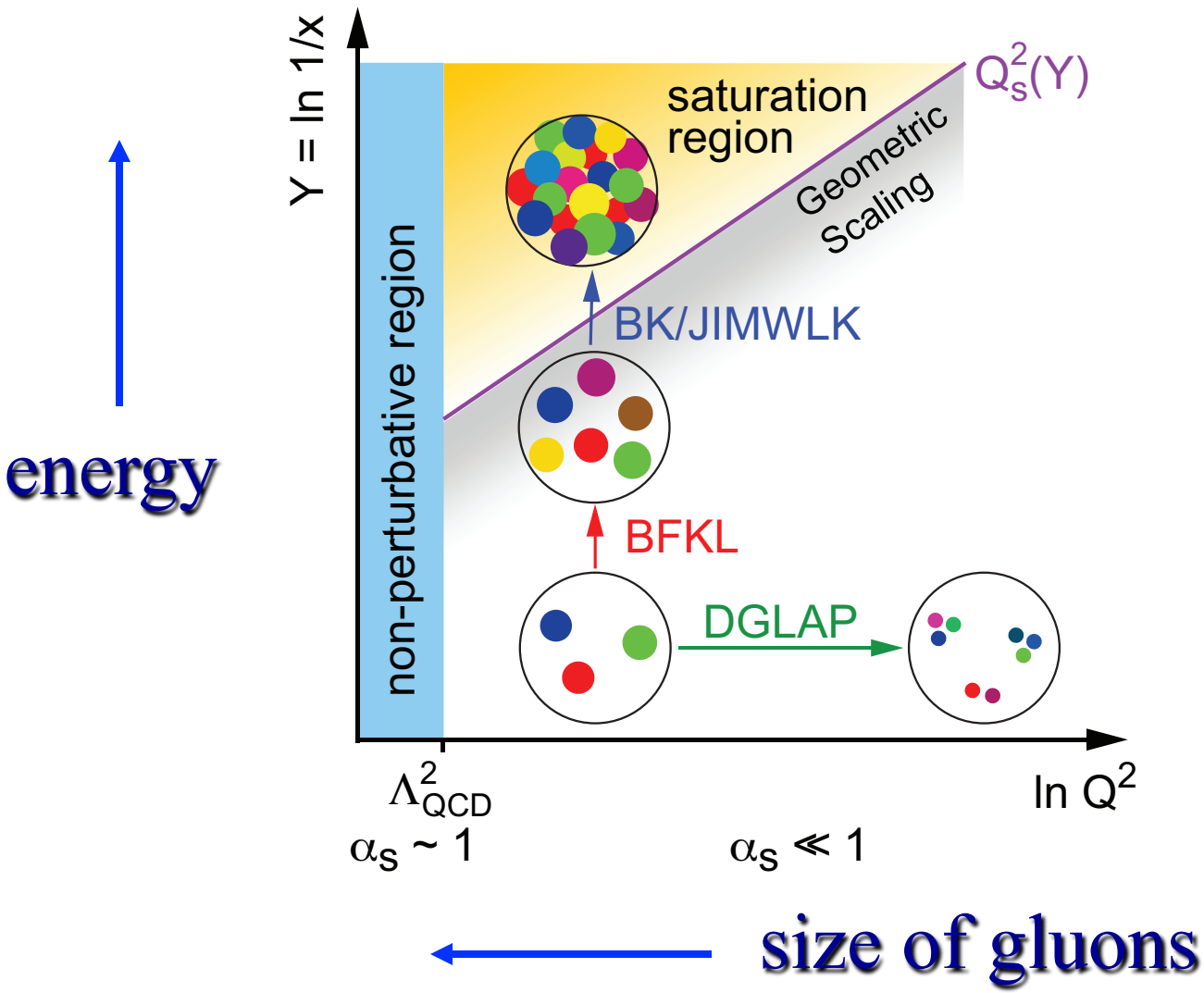
$$\sigma^{q\bar{q}A} = 2 \int d^2b \, N(x_\perp, b_\perp, Y)$$

# Saturation scale




numerical solution by J. Albacete

# Map of High Energy QCD

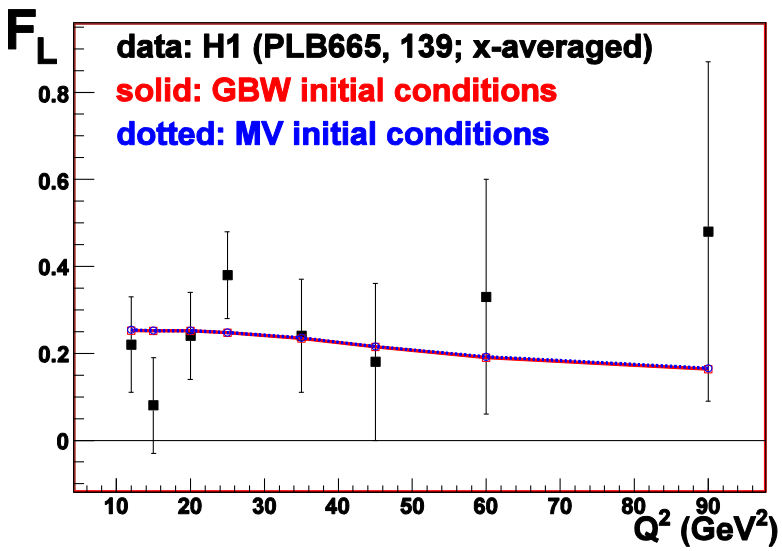


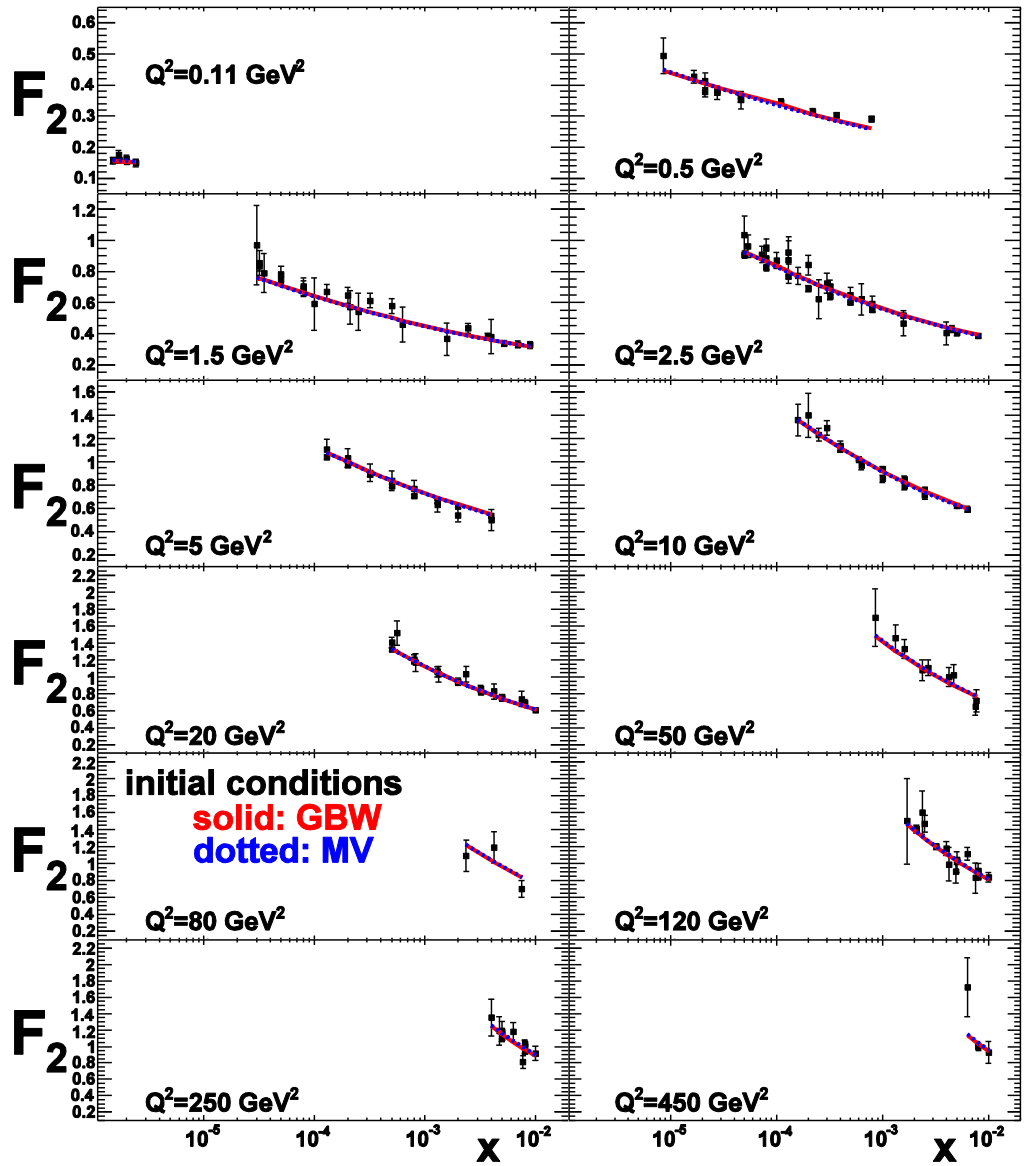
# Comparison of rcBK with HERA F2 Data

DIS structure functions:

$$F_{2,\,L} = \frac{Q^2}{4\,\pi^2\,\alpha_{EM}}\,\sigma^{\gamma * p}_{tot,\,L}$$

from Albacete, Armesto, Milhano, Salgado '09



# TMD Evolution: large-x

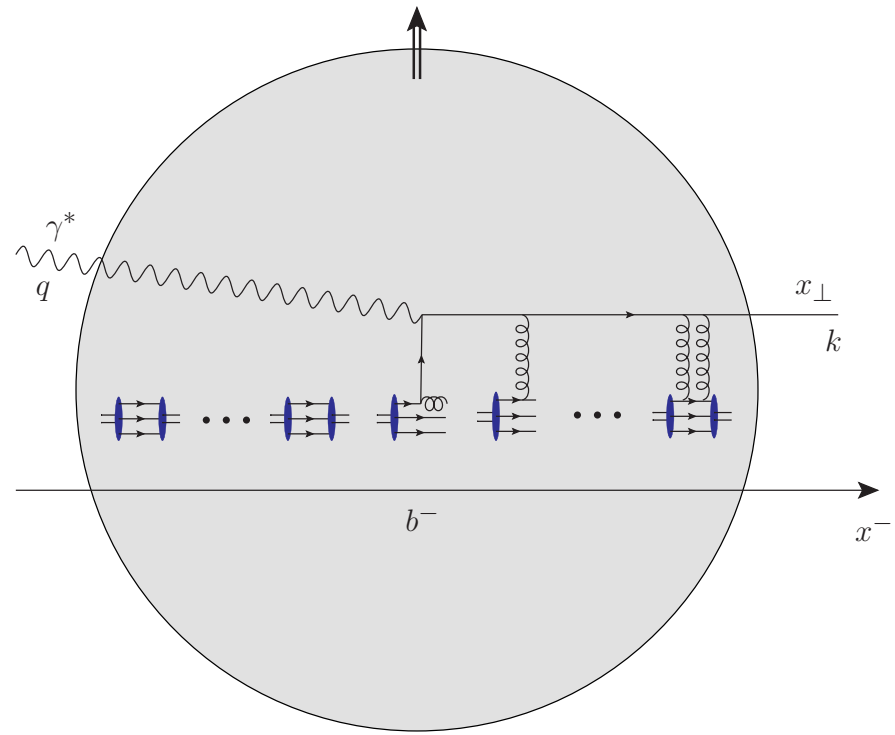
# Quark Production in SIDIS

- Start with inclusive classical quark production cross section in SIDIS (see the talk by M. Sievert).

- The kinematics is standard:

$$s \sim Q^2 \gg \perp^2$$

- The result is




**Wigner distribution**

$$\frac{d\sigma^{\gamma^*+A\to q+X}}{d^2k\,dy} = A \int \frac{dp^+\,d^2p\,db^-}{2(2\pi)^3} \int d^2x\,d^2y\; W\left(p, b^-, \frac{\underline{x}+\underline{y}}{2}\right)$$

$$\times \int \frac{d^2k'}{(2\pi)^2}\, e^{-i\,(\underline{k}-\underline{k}')\cdot(\underline{x}-\underline{y})}\, \frac{d\hat{\sigma}^{\gamma^*+N\to q+X}}{d^2k'\,dy}(p,q)\, D_{\underline{x}\,\underline{y}}[+\infty, b^-]$$

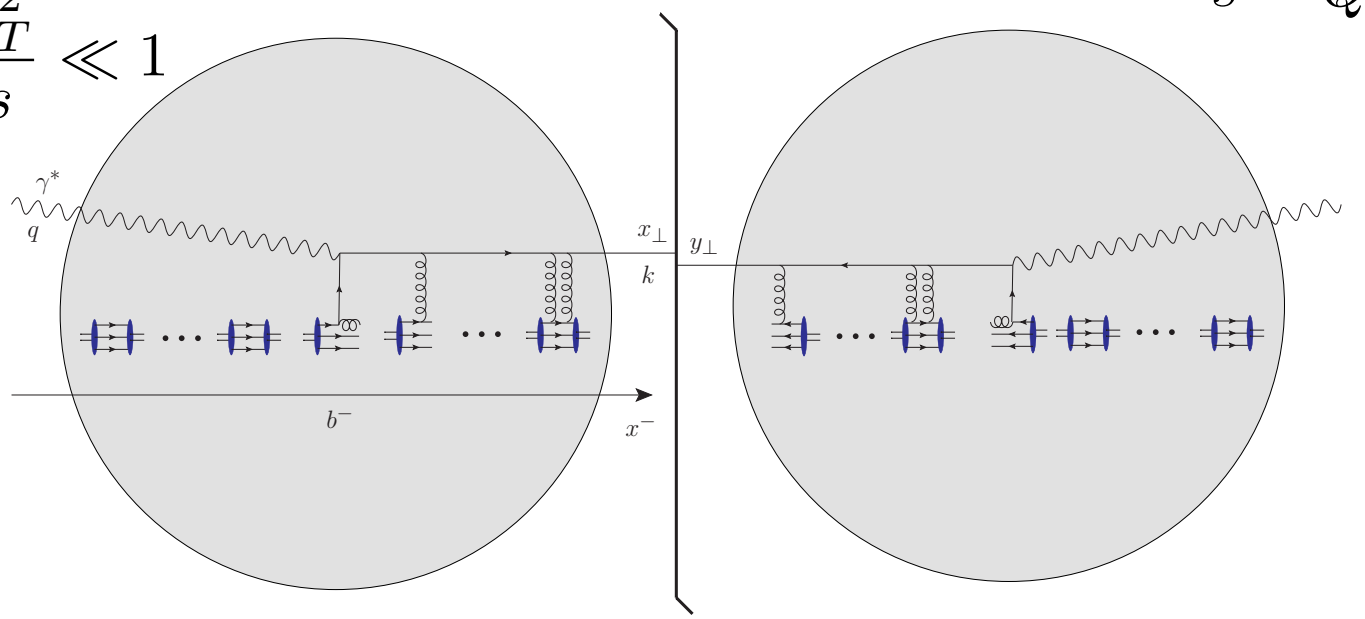
**LO cross sect** **Wilson lines**

# Quark Production

Note: effective x

$$x_{eff} \approx \frac{k_T^2}{s} \ll 1$$

$$s \sim Q^2 \gg \perp^2$$

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**LO cross sect** **Wilson lines**

# Wilson lines

- Here

$$D_{\underline{x}\,\underline{y}}[+\infty, b^-] = \left\langle \frac{1}{N_c} \mathrm{Tr}\left[ V_{\underline{x}}[+\infty, b^-]\, V^\dagger_{\underline{y}}[+\infty, b^-] \right] \right\rangle$$

  is the quark dipole scattering S-matrix with

$$V_{\underline{x}}[b^-\,,\, a^-] \equiv \mathcal{P}\, \exp\left[ \frac{ig}{2} \int_{a^-}^{b^-} dx^- A^+(x^+ = 0, x^-, \underline{x}) \right]$$

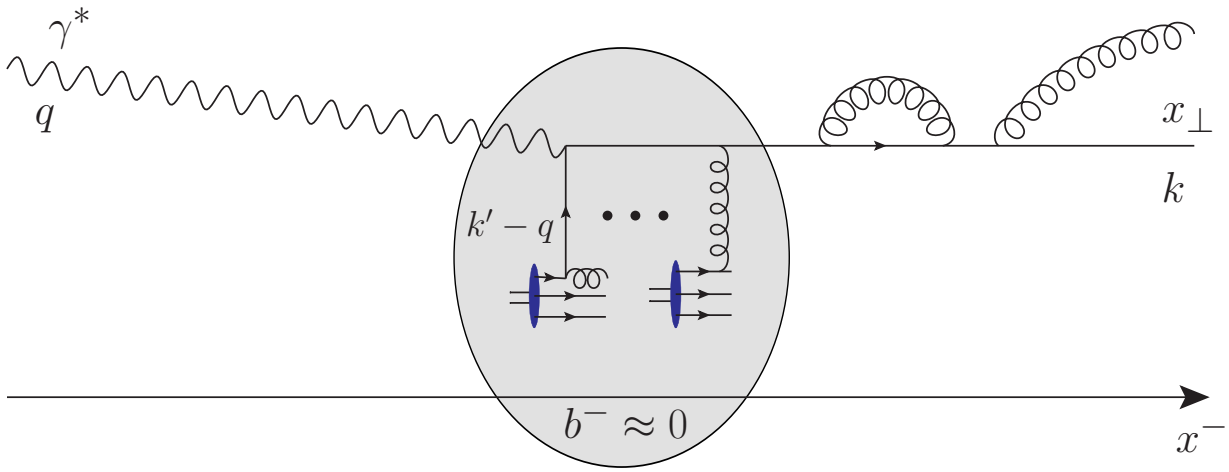
  denoting Wilson lines. This is the standard 'staple' (in a gauge where the link at infinity does not contribute).

- The leading contribution to $D_{xy}$ is x🡐🡒y symmetric and will be denoted $S_{xy}$.

# Evolution Corrections

We work in $A^+ = 0$ gauge (projectile light-cone gauge) and look for emissions giving $\ln s \sim \ln Q^2$.

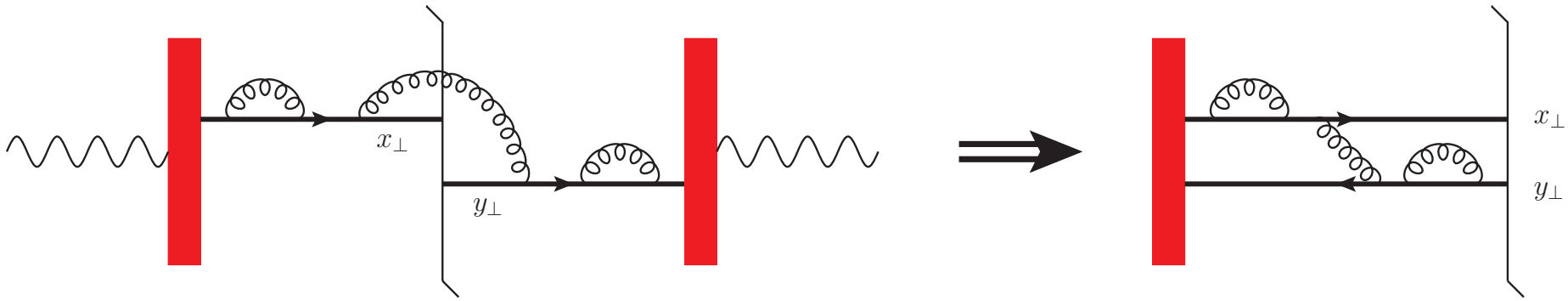
Such emissions and virtual corrections come only from the semi-infinite Wilson line describing the outgoing quark:




How do we sum them up?

# Crossing Symmetry

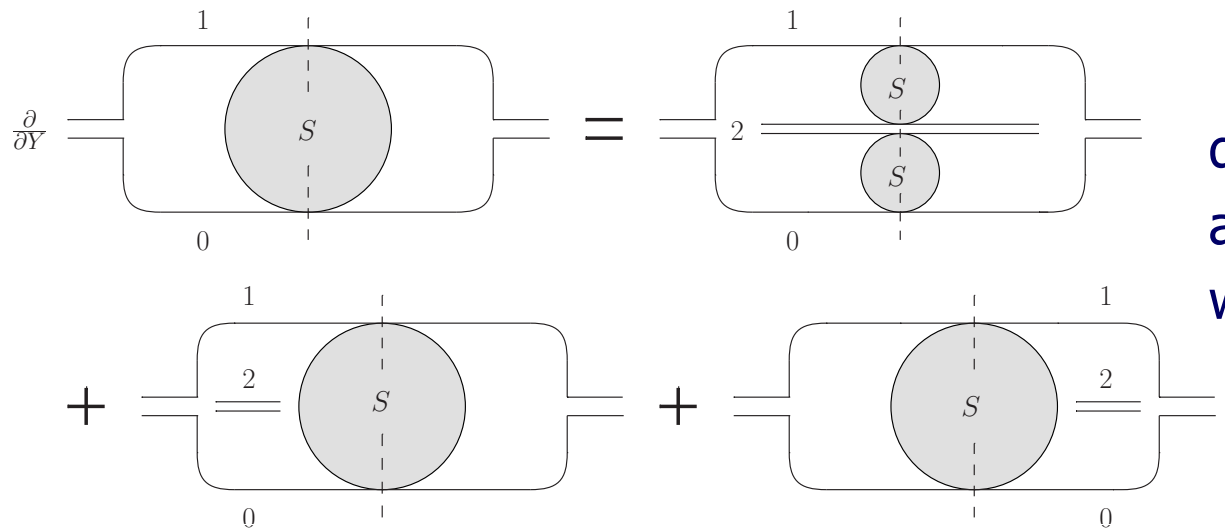
- To sum up evolution corrections for the semi-infinite Wilson line in the amplitude and another one in the cc amplitude, use the "crossing" symmetry to reflect the cc Wilson line into the amplitude (forming the standard SIDIS light-cone staple):




- We end up with a ½ a dipole with, from the standpoint of the dipole evolution, only virtual corrections.

# Nonlinear Evolution

To sum up the gluon cascade at large-$N_C$ we write the following equation for the dipole S-matrix:




dashed line = all interactions with the target

$$\partial_Y S_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{01}^2}{x_{02}^2\, x_{21}^2} \left[ S_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, S_{\mathbf{x}_2,\mathbf{x}_1}(Y) - S_{\mathbf{x}_0,\mathbf{x}_1}(Y) \right]$$

Remembering that S= 1-N we can rewrite this equation in terms of the dipole scattering amplitude N.

# TMD Evolution at large-x

- Keeping only (a half of) the virtual corrections of the dipole evolution we write

$$\partial_Y S_{xy}[\infty^-, b^-](Y) = -\frac{\alpha_s\, C_F}{2\,\pi^2} \int d^2 z_\perp \, \frac{(\underline{x}-\underline{y})^2}{(\underline{x}-\underline{z})^2\,(\underline{z}-\underline{y})^2}\, S_{xy}[\infty^-, b^-](Y)$$

where rapidity is ($s \approx Q^2$)

$$Y = \ln[s\,(\underline{x}-\underline{y})^2] \approx \ln[Q^2\,(\underline{x}-\underline{y})^2]$$

- The z-integral is UV divergent, and has to be regulated by 1/Q. The solution is

$$S_{xy}[\infty^-, b^-](Q^2) = \exp\left(-\int_{Q_0^2}^{Q^2} \frac{d\mu^2}{\mu^2}\, \frac{\alpha_s\, C_F}{\pi}\, \ln[\mu^2\,(\underline{x}-\underline{y})^2]\right) S_{xy}[\infty^-, b^-](Q_0^2)$$

with the initial condition at $Q_0$ given by the quasi-classical expression (see Matt's talk).

# Sudakov Form-Factor

- We reproduced the standard Sudakov form-factor, which is characteristic of the CSS evolution.

$$S_{xy}[\infty^-, b^-](Q^2) = \exp\left( -\int_{Q_0^2}^{Q^2} \frac{d\mu^2}{\mu^2} \frac{\alpha_s C_F}{\pi} \ln[\mu^2 (\underline{x} - \underline{y})^2] \right) S_{xy}[\infty^-, b^-](Q_0^2)$$

- To calculate any TMD of an unpolarized nucleus, simply 'evolve' the classical semi-infinite dipole amplitude using the form-factor above, and use it in the quark-quark correlator (see Matt's talk)

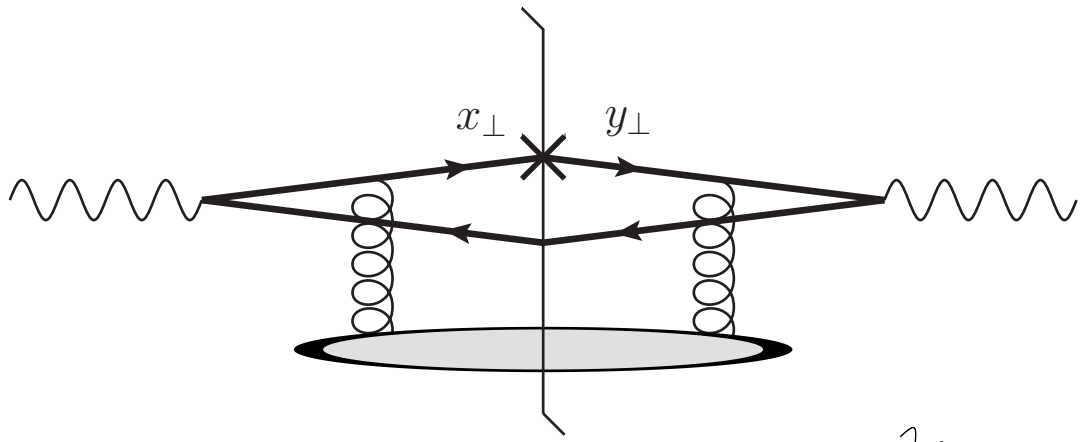
$$\Phi^A(x, \underline{k}; P; Q^2) = \frac{2A\, g_{+-}}{(2\pi)^5} \int d^{2+}p\, d^{2-}b\, d^2r\, d^2k'\, e^{-i(\underline{k} - \underline{k}' - \hat{x}\underline{p})\cdot \underline{r}}$$
$$\times \left( W_{unp}(p, b; P)\, \phi_{unp}(\hat{x}, \underline{k}'; p; Q_0^2) - \hat{W}_{pol,\,\mu}(p, b; P) \hat{\phi}^{\mu}_{pol}(\hat{x}, \underline{k}'; p; Q_0^2) \right) S^{[\infty^-, b^-]}_{(r_T, b_T)}(Q^2)$$

- For gluon TMD evolution simply replace the Casimir, $C_F$ ➔ $N_c$ in S.

# Quark TMD Evolution: small-x, unpolarized nucleus

# Quark Production in SIDIS at Small-x

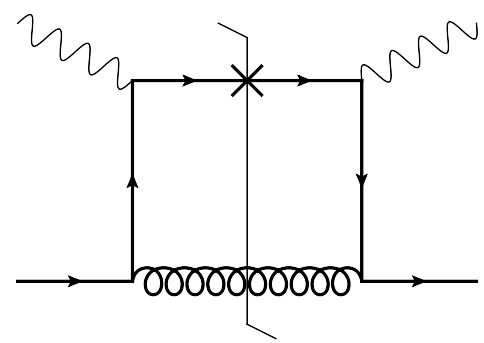
- To find unpolarized target TMDs at small-x it is convenient to start by considering the quark production cross section for SIDIS on an **unpolarized** nucleus.
- The dominant process is different, even at the lowest order:




- Compared to the standard LO process, the one above comes in with an extra factor of

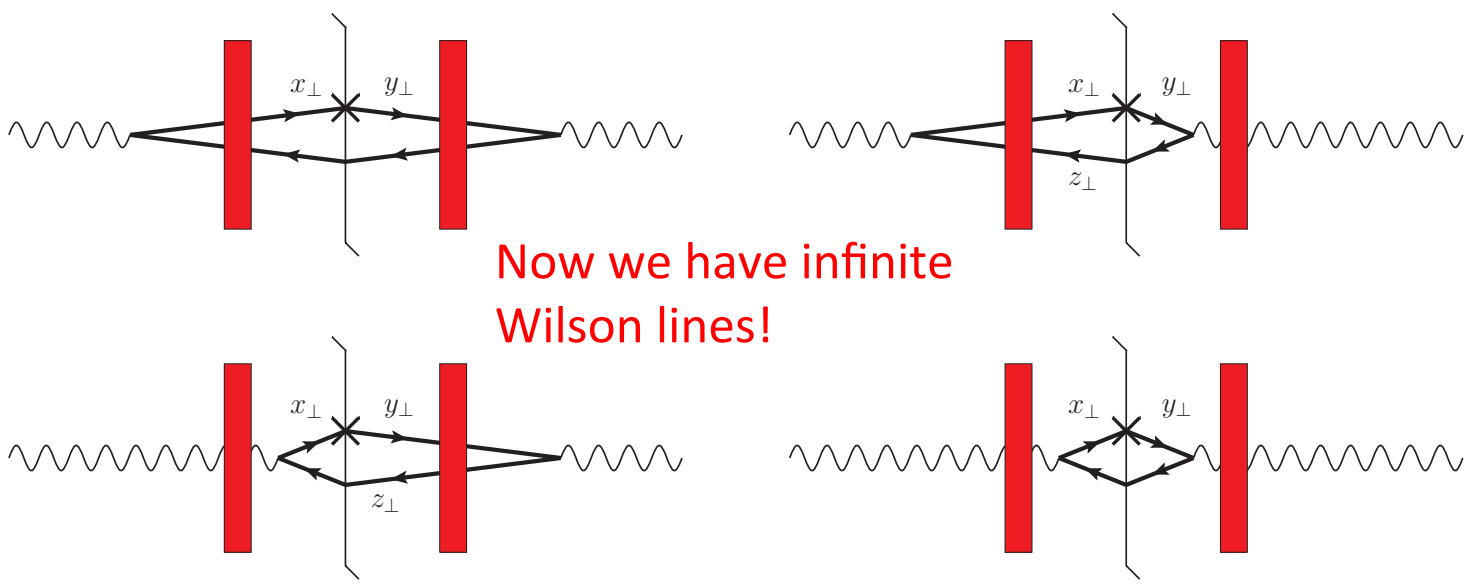
$$\sim \frac{\alpha_s}{x}$$

and is dominant at very low x.



# SIDIS to All Orders

- SIDIS process can now be easily generalized to include all-order interactions with the shock waves:




Now we have infinite Wilson lines!

- The SIDIS cross section is

$$\frac{d\sigma_{T,L}^{SIDIS}}{d^2k_T} = \int_0^1 \frac{dz}{z\,(1-z)} \int \frac{d^2x_\perp\, d^2y_\perp\, d^2z_\perp}{2(2\pi)^3}\, e^{-i\underline{k}\cdot(\underline{x}-\underline{y})}\, \Psi_{T,L}^{\gamma^*\to q\bar{q}}(\underline{x}-\underline{z},z)\, \left[\Psi_{T,L}^{\gamma^*\to q\bar{q}}(\underline{y}-\underline{z},z)\right]^*$$

$$\times \left[S_{x,y}^{[+\infty,-\infty]} - S_{x,z}^{[+\infty,-\infty]} - S_{z,y}^{[+\infty,-\infty]} + 1\right]$$

# Quark TMD Evolution at Small-x

- Taking the large-$Q^2$ limit of the SIDIS cross section we can extract the unpolarized quark TMD out of it:

$$f_1^A(x,k_T) = \frac{2\,N_c}{\pi^3\,x}\int \frac{d^2x_\perp\,d^2y_\perp\,d^2z_\perp}{2(2\pi)^3}\,e^{-i\underline{k}\cdot(\underline{x}-\underline{y})}\,\frac{\underline{x}-\underline{z}}{|\underline{x}-\underline{z}|^2}\cdot\frac{\underline{y}-\underline{z}}{|\underline{y}-\underline{z}|^2}$$
$$\times\,\frac{|\underline{x}-\underline{z}|^4-|\underline{y}-\underline{z}|^4-2\,|\underline{x}-\underline{z}|^2\,|\underline{y}-\underline{z}|^2\,\ln\frac{|\underline{x}-\underline{z}|^2}{|\underline{y}-\underline{z}|^2}}{(|\underline{x}-\underline{z}|^2-|\underline{y}-\underline{z}|^2)^3}\left[S_{x,y}^{[+\infty,-\infty]}-S_{x,z}^{[+\infty,-\infty]}-S_{z,y}^{[+\infty,-\infty]}+1\right]$$

- Since the Wilson lines are now infinite, we have infinite dipoles, whose evolution is given by the BK equation at large-$N_c$:

$$\partial_Y S_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s\,N_c}{2\pi^2}\int d^2x_2\,\frac{x_{01}^2}{x_{02}^2\,x_{21}^2}\,\left[S_{\mathbf{x}_0,\mathbf{x}_2}(Y)\,S_{\mathbf{x}_2,\mathbf{x}_1}(Y)-S_{\mathbf{x}_0,\mathbf{x}_1}(Y)\right]$$

- Initial conditons are given by the quasi-classical Glauber-Mueller formula again.

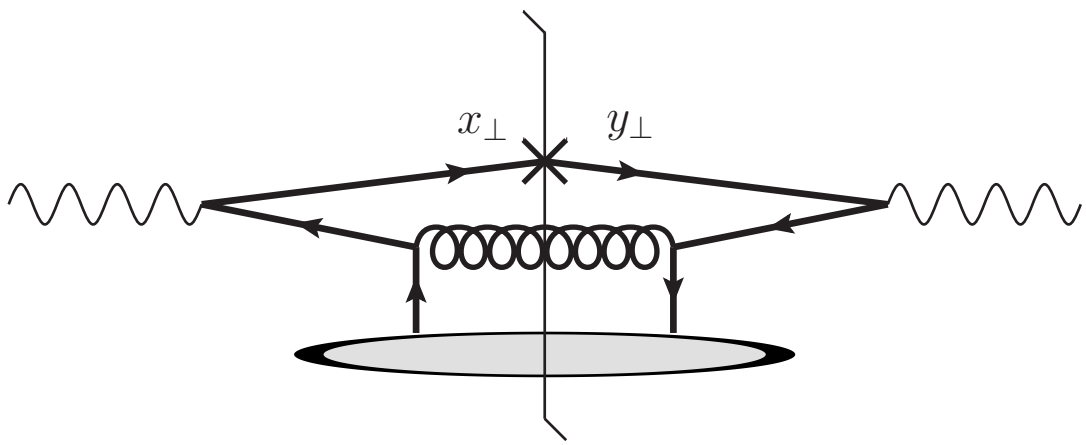
For unpolarized <u>gluon</u> TMD at low-x
see F. Dominguez et al, ‘11; Balitsky & Tarasov ‘15 (next two talks)

# Polarized Quark TMD Evolution at Small-x: an Outline

# Target Spin-Dependent SIDIS

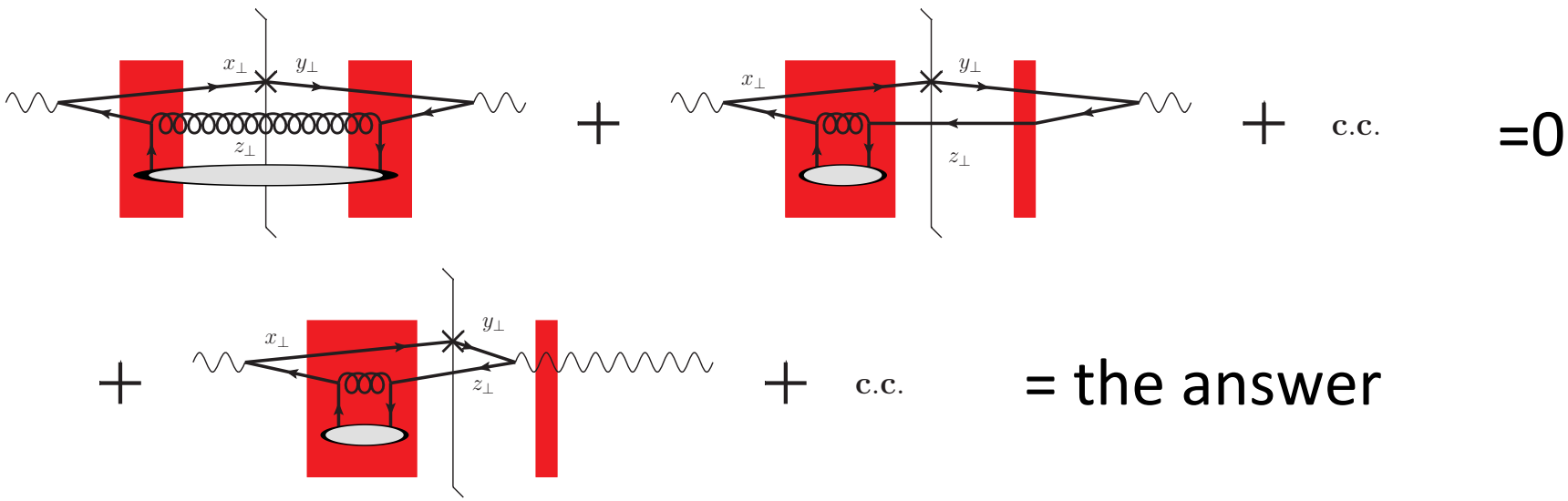
To transfer spin information between the polarized target and the produced quark we either need to exchange quarks in the t-channel, or non-eikonal gluons.

Here's an example of the quark exchange:



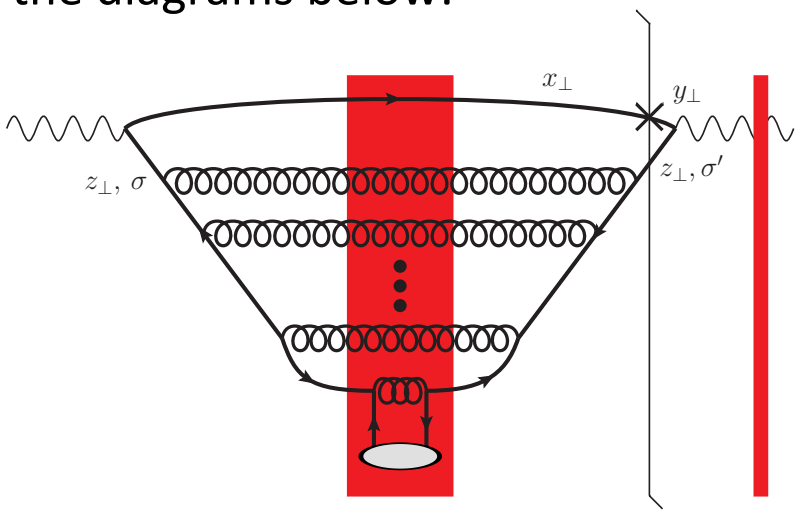
# Target Spin-Dependent SIDIS

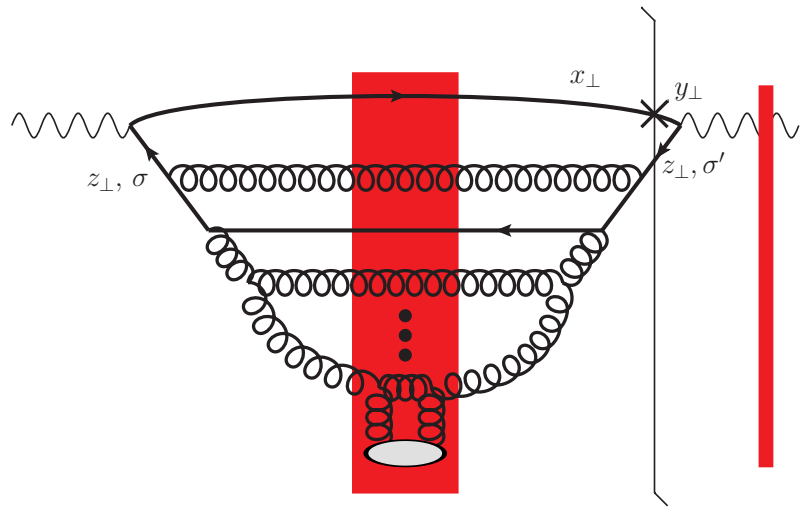
It is straightforward to include multiple shock wave interactions into the polarized SIDIS cross section:



# Small-x Polarized-Quark TMD Evolution

Evolution corrections can be included into the polarized TMDs using the diagrams below:




Interestingly the quark and non-eikonal gluon ladders mix (see the right panel), resulting in a more complicated evolution equation.

# Small-x Polarized-Quark TMD Evolution

- A generic quark-spin dependent TMD can be written as (target spin is $\Sigma$, tagged quark spin is $\lambda$)

$$f_A(x, k_T; \lambda, \Sigma) = -\int \frac{d^2x_\perp\, d^2y_\perp\, d^2z_\perp}{2(2\pi)^3}\, e^{-i\underline{k}\cdot(\underline{x}-\underline{y})} \sum_{\sigma,\sigma'} f_{\sigma\sigma'}(\underline{x}-\underline{z}, \underline{y}-\underline{z}; \lambda) \times \left[R_{x,z}^{\sigma\,\sigma'}(Y) + R_{y,z}^{\sigma\,\sigma'\,*}(Y)\right]$$

where R is the QCD Reggeon amplitude and $f_{\sigma\,\sigma'}$ can be calculated from the light-cone wave function.

- R obeys the equation (Itakura, YK, McLerran, Teaney '03), no gluon ladders ➔ approximation for TMD evolution!

$$\alpha_s \ln^2 s \sim 1$$

$$R_{x,z}^{\sigma\,\sigma'}(Y) = r_{x,z}^{\sigma\,\sigma'}(Y) + \frac{\alpha_s C_F}{2\pi^2} \int \frac{d^2w_\perp}{w_\perp^2} \int_{Y_i}^{\min\{Y, Y-\ln(z_\perp^2/w_\perp^2)\}} dy\, R_{z,w}^{\sigma\,\sigma'}(y)\, S_{x,z}^{[+\infty,-\infty]}(y)$$

# What to Expect

- Without saturation effects, similar evolution for the $g_1$ structure function was considered by Bartels, Ermolaev and Ryskin in '96.

- Including the mixing of quark and gluon ladders, they obtained

$$R^{\sigma\,\sigma'} \sim x^{-z_s\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}}}$$

  with $z_s$ = 3.45 for 4 quark flavors.

- The power is large and negative, and can easily become large enough to make the net power of x smaller than −1 for the realistic strong coupling of the order of $\alpha_s$ = 0.2 − 0.3, resulting in polarized TMDs which actually grow with decreasing x fast enough for the integral of the TMDs over the low-x region to be (potentially) large.

- Can this solve the spin puzzle? To be continued…

# Conclusions

- We have constructed the TMD evolution for

- Large-x, both quark and gluon TMDs, polarized and unpolarized target: we reproduced the standard Sudakov form-factor (cf. CSS evolution).

- Small-x quark TMDs of the unpolarized target: we showed that they evolve with the BK/JIMWLK evolution.

- Small-x polarized-target TMDs appear to evolve with the QCD Reggeon evolution + nonlinear saturation effects. To be studied in more detail (Pitonyak, Sievert, Venugopalan, YK, in preparation).

# Outlook

- Quasi-classical expressions provide initial conditions which can be used in a global fit of all the TMD data.

- All is ready for a global fit of small-x evolved unpolarized-target TMDs.

- The missing part is the polarized-target TMD evolution.