Toward NNLL Resummation for Single- and Di-Hadron Production in Hadronic Collisions

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Outline

- QCD Threshold Resummation
- Di-Hadron Production

Hinderer, FR, Sterman, Vogelsang - PRD '15

- Single-Inclusive Hadron Production *in preparation*
- Conclusions

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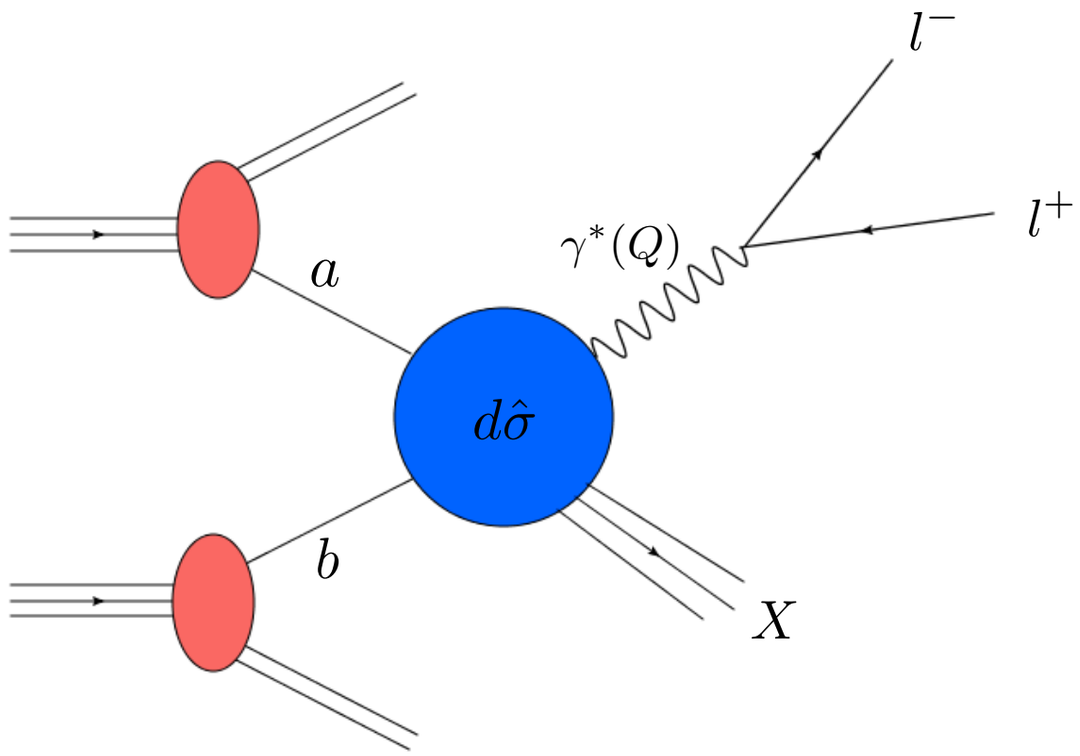
Drell-Yan process

Factorized hadronic scattering

(color singlet $q\bar{q} \to \gamma^*$)



universal PDFs

(up to power corrections $1/Q^2$)

$$Q^4 \frac{d\sigma}{dQ^2} = \sum_{a,b} \int dx_a dx_b \; f_a(x_a,\mu)\, f_b(x_b,\mu)\; \omega_{ab}\left(z = \frac{Q^2}{\hat{s}}, \alpha_s(\mu), \frac{Q}{\mu}\right) + \ldots$$

hard scattering part

$$\omega_{ab} = \omega_{ab}^{(\mathrm{LO})} + \frac{\alpha_s}{\pi}\omega_{ab}^{(\mathrm{NLO})} + \left(\frac{\alpha_s}{\pi}\right)^2 \omega_{ab}^{(\mathrm{NNLO})} + \ldots$$

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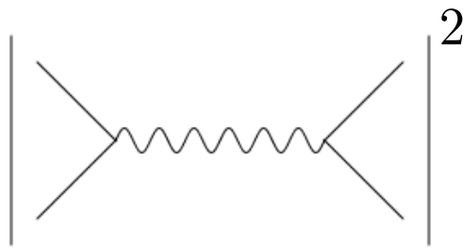
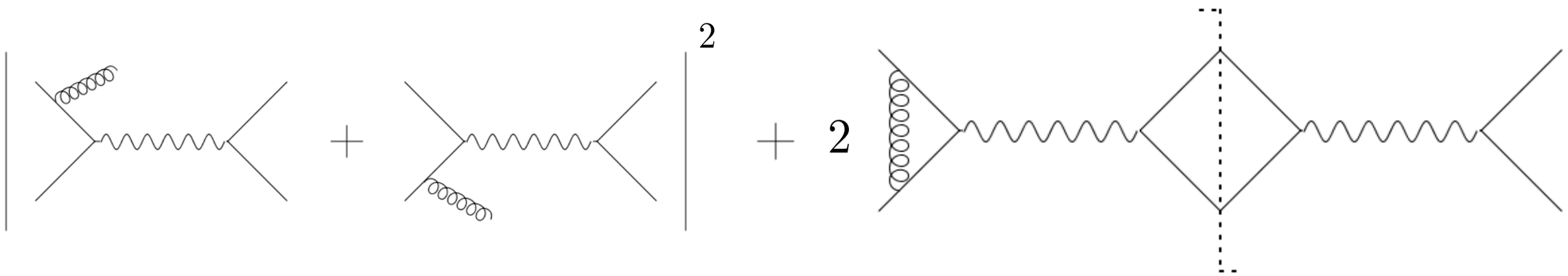
$$\omega_{ab} = \omega_{ab}^{(\mathrm{LO})} + \frac{\alpha_s}{\pi}\omega_{ab}^{(\mathrm{NLO})} + \left(\frac{\alpha_s}{\pi}\right)^2 \omega_{ab}^{(\mathrm{NNLO})} + \dots$$

LO:



$$\omega_{q\bar{q}}^{(\mathrm{LO})} \sim \delta(1-z) \qquad z = \frac{Q^2}{\hat{s}}$$

NLO:



$$\omega_{q\bar{q}}^{(\mathrm{NLO})} \sim \frac{\alpha_s}{2\pi} C_F \left[4(1+z^2) \left(\frac{\ln(1-z)}{1-z} \right)_+ - 2\frac{1+z^2}{1-z}\ln z + \left(\frac{2}{3}\pi^2 - 8\right)\delta(1-z) + 2P_{qq}(z)\ln\frac{Q^2}{\mu^2} \right]$$

spoils perturbative convergence for $z \to 1$ even if $\alpha_s \ll 1$

$$\int_0^1 dz\, h(z)\, [\, g(z)\,]_+ \equiv \int_0^1 dz\, [h(z) - h(1)]\, g(z)$$

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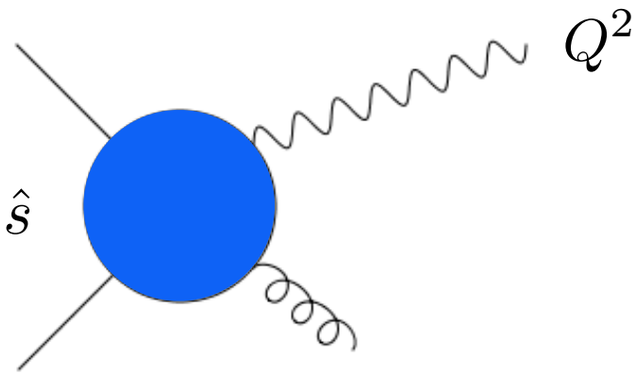
Threshold Logarithms

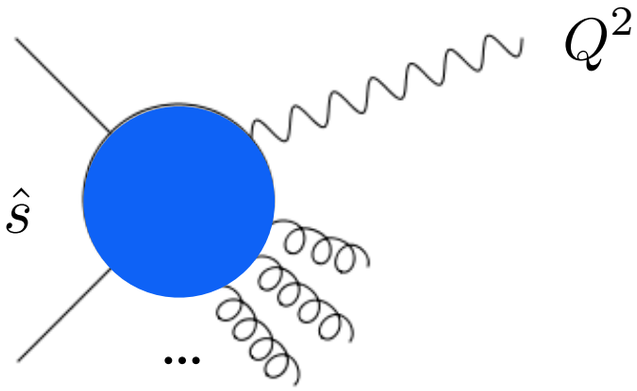
$$z = \frac{Q^2}{\hat{s}} \to 1 \qquad \text{partonic threshold}$$

NLO:



$$\omega_{q\bar{q}}^{(\mathrm{NLO})} \sim \alpha_s \left(\frac{\ln(1-z)}{1-z} \right)_+ + \dots$$

N$^{\mathrm{k}}$LO:



$$\omega_{q\bar{q}}^{(\mathrm{N^kLO})} \sim \alpha_s^k \left(\frac{\ln^{2k-1}(1-z)}{1-z} \right)_+ + \dots$$

- Threshold: the partonic process uses all the available energy $\hat{s}$ to produce the outgoing lepton pair with Q^2
- Origin: suppression of real gluon emission for $z \to 1$, while virtual corrections are allowed
- Logarithms may spoil perturbative series, unless taken into account to all orders

⟶ Threshold resummation

Sterman `81; Catani, Trentadue `89

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Mellin Transform Space

- Drell-Yan

$$\int_0^1 d\tau\, \tau^{N-1} \frac{d\sigma}{dQ^2} \sim \sum_{ab} \left(\int_0^1 dx_a\, x_a^N\, f_a \right) \left(\int_0^1 dx_b\, x_b^N\, f_b \right) \underbrace{\left(\int_0^1 dz\, z^{N-1}\, \omega_{ab} \right)}_{= \tilde{\omega}_{ab}}$$

where

$$\tau = \frac{Q^2}{S} = z\, x_a\, x_b$$

- Threshold Logarithms

$$\alpha_s^k \left(\frac{\ln^{2k-1}(1-z)}{1-z} \right)_+ \quad \to \quad \alpha_s^k \ln^{2k} N$$

large logarithms in N

Accuracy of Resummation

$$\mathcal{O}(\alpha_s^k): \qquad C_{kn} \times \alpha_s^k \ln^n \bar{N}, \quad \text{where } n \leq 2k$$

Fixed Order

LO	1					
NLO	$\alpha_s L^2$	$\alpha_s L$	α_s			
NNLO	$\alpha_s^2 L^4$	$\alpha_s^2 L^3$	$\alpha_s^2 L^2$	$\alpha_s^2 L$	α_s^2	
...	...	...	...	...	...	
N^{k}LO	$\alpha_s^k L^{2k}$	$\alpha_s^k L^{2k-1}$	$\alpha_s^k L^{2k-2}$	$\alpha_s^k L^{2k-3}$	$\alpha_s^k L^{2k-4}$	...

$L = \ln \bar{N}$

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Accuracy of Resummation

$$\mathcal{O}(\alpha_s^k): \qquad C_{kn} \times \alpha_s^k \ln^n \bar{N}, \quad \text{where } n \leq 2k$$

Fixed Order

Resummation

LO	1					
NLO	$\alpha_s L^2$	$\alpha_s L$	α_s			
NNLO	$\alpha_s^2 L^4$	$\alpha_s^2 L^3$	$\alpha_s^2 L^2$	$\alpha_s^2 L$	α_s^2	
...	...	...	...	...	...	
N^{k}LO	$\alpha_s^k L^{2k}$	$\alpha_s^k L^{2k-1}$	$\alpha_s^k L^{2k-2}$	$\alpha_s^k L^{2k-3}$	$\alpha_s^k L^{2k-4}$	...

LL NLL NNLL

$L = \ln \bar{N}$

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Resummed result

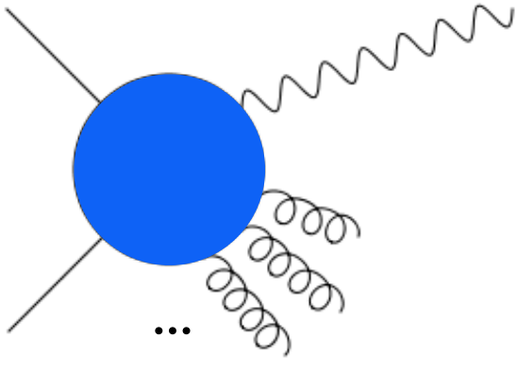
Resummation relies on factorization of

- QCD matrix elements for n-gluon emission in the soft limit
- phase space in Mellin space

in Mellin space: exponentiation of eikonal diagrams

$$\tilde{\omega}_{q\bar{q}}^{(\mathrm{res})}(N) = H_q \exp\left\{2\int_0^1 dz \frac{z^{N-1}-1}{1-z}\left[\int_{\mu_F^2}^{(1-z)^2Q^2}\frac{d\mu^2}{\mu^2}A_q(\alpha_s(\mu)) + D_q(\alpha_s((1-z)Q))\right]\right\}$$

matching coefficient



$$A_i(\alpha_s) = \frac{\alpha_s}{\pi}A_i^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 A_i^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 A_i^{(3)} + \mathcal{O}(\alpha_s^4)$$

$$D_i(\alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^2 D_i^{(2)} + \mathcal{O}(\alpha_s^3)\,,$$

$\overline{\mathrm{MS}}$ scheme

Sterman `81; Gatheral `83; Frenkel, Taylor `84

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Expansion up to NNLL

LL

$$\tilde{\omega}_{q\bar{q}}^{(\mathrm{res})}(N) \sim \exp\left[+2C_F \frac{\alpha_s}{\pi} \ln^2 \bar{N} + \ldots\right]$$

enhancement!

NNLL

$$\tilde{\omega}_{q\bar{q}}^{(\mathrm{res})}(N) \sim \exp\left\{h^{(1)}(\lambda) \ln \bar{N} + h^{(2)}\left(\lambda, \frac{\mu_R}{Q}, \frac{\mu_F}{Q}\right) + \alpha_s(\mu_R)\, h^{(3)}\left(\lambda, \frac{\mu_R}{Q}, \frac{\mu_F}{Q}\right)\right\}$$

LL NLL NNLL

$$\lambda = \alpha_s(\mu_R^2) b_0 \ln \bar{N}$$

$$h^{(1)}(\lambda) = \frac{A_q^{(1)}}{2\pi b_0 \lambda}[2\lambda + (1-2\lambda)\ln(1-2\lambda)]$$

$$h^{(2)} = \ldots$$

$$h^{(3)} = \ldots$$

Catani, Mangano, Nason, Trentadue `96; Vogt `01

Matching and Minimal Prescription

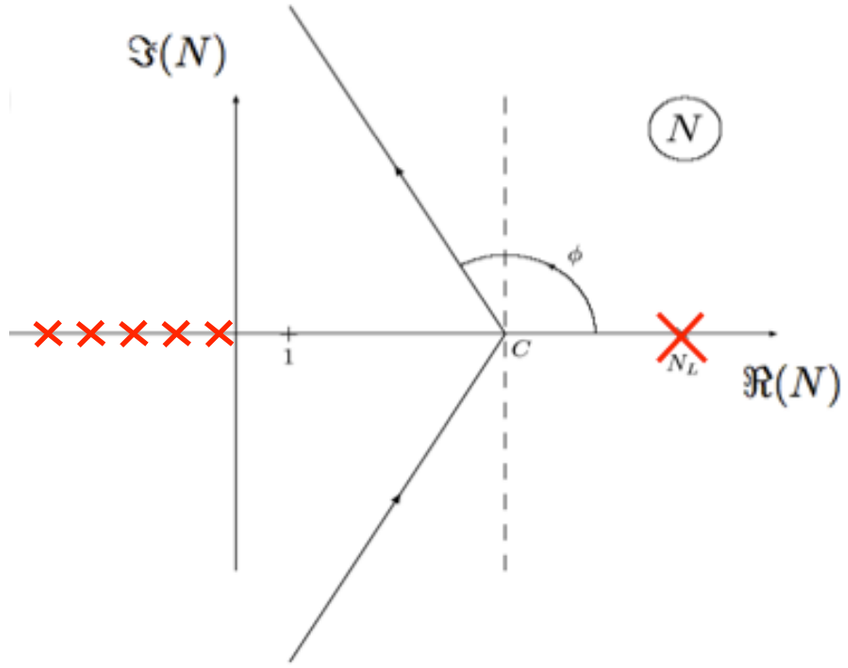
- Matching procedure (avoiding double counting)

$$d\sigma^{\rm match} = \left(d\sigma^{\rm resum} + d\sigma^{\rm NLO}\right) - d\sigma^{\rm resum} \Big|_{\mathcal{O}(\alpha_s)}$$

- Inverse Transformation

$$\sigma^{\rm res} = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dN\, \tau^{-N}\, \tilde{\sigma}^{\rm res}(N)$$

Choosing the contour to the left of the Landau pole



Catani, Mangano, Nason, Trentadue `96

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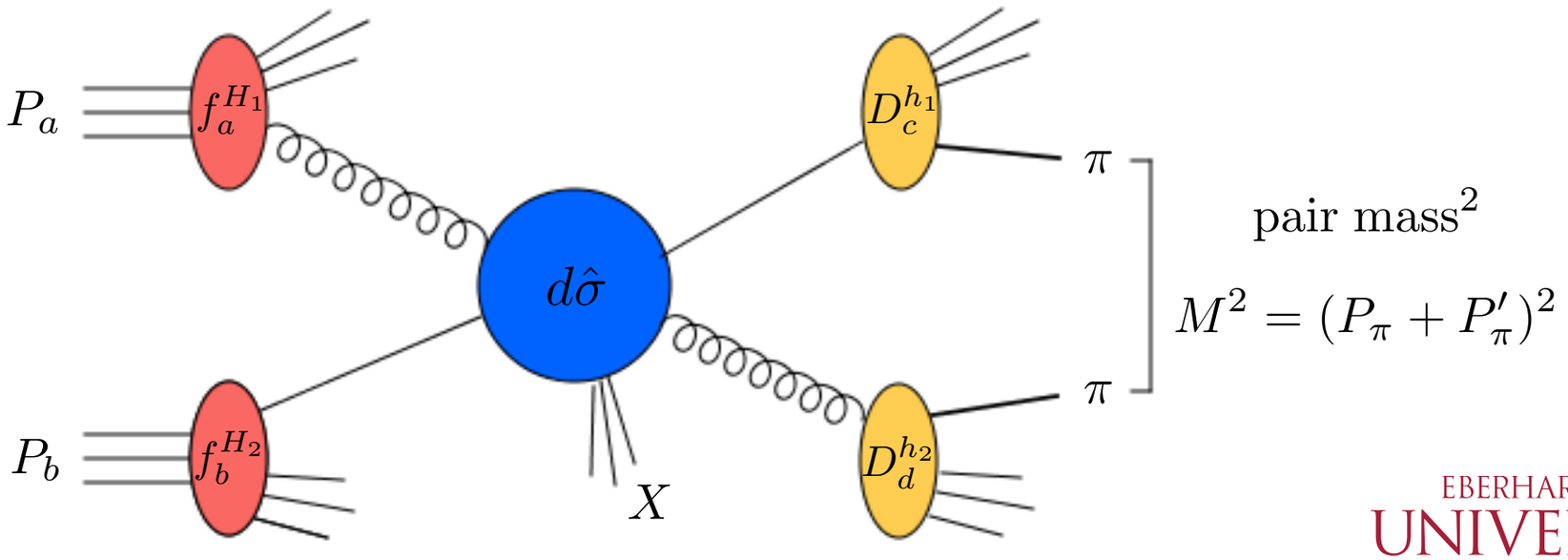
Di-Hadron Production

Why di-hadron production? $H_1 + H_2 \to h_1 + h_2 + X$

- Rather simple structure compared to jet cross sections;
  ``full QCD extension´´ of Drell-Yan
- Exisiting data from various experiments not very well described by NLO *Owens `02*
  data from fixed target: NA24, E711, E706
  collider regime: CCOR, RHIC (planned)
- Previous NLL results from *Almeida, Sterman, Vogelsang `09*
- Starting point for extension toward NNLL for other processes



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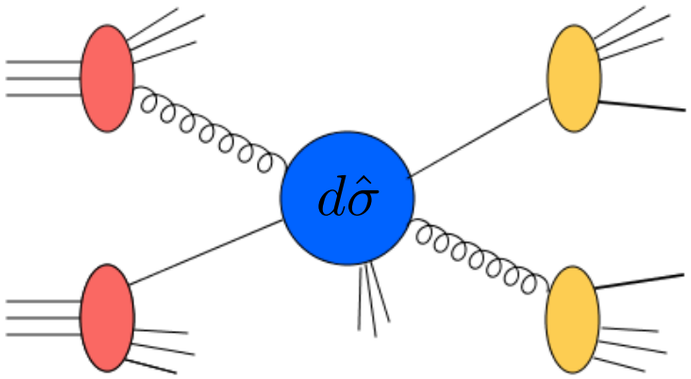
Cross section

$$H_1(P_a) + H_2(P_b) \to h_1(P_c) + h_2(P_d) + X$$



$M^2 = (P_c + P_d)^2$

Define $\Delta\eta = \frac{1}{2}(\eta_1 - \eta_2)$ $\bar{\eta} = \frac{1}{2}(\eta_1 + \eta_2)$

$$M^4 \frac{d\sigma^{H_1 H_2 \to h_1 h_2 X}}{dM^2 d\Delta\eta d\bar{\eta}} = \sum_{abcd} \int_0^1 dx_a dx_b dz_c dz_d \, f_a^{H_1}(x_a, \mu_F) f_b^{H_2}(x_b, \mu_F) \, z_c D_c^{h_1}(z_c, \mu_F) z_d D_d^{h_2}(z_d, \mu_F)$$
$$\times \omega_{ab\to cd}\left(\hat{\tau}, \Delta\eta, \hat{\eta}, \alpha_s(\mu_R), \frac{\mu_R}{\hat{m}}, \frac{\mu_F}{\hat{m}}\right)$$

where $\hat{m}^2 = \frac{M^2}{z_c z_d}$ $\hat{\tau} = \frac{\hat{m}^2}{\hat{s}}$ $\hat{\eta} = \bar{\eta} - \frac{1}{2}\ln\left(\frac{x_a}{x_b}\right)$

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Mellin and Fourier transforms

Write cross section as

$$M^4 \frac{d\sigma^{H_1 H_2 \to h_1 h_2 X}}{dM^2 d\Delta\eta d\bar{\eta}} = \sum_{cd} \int_0^1 dz_c\, dz_d\, z_c\, D_c^{h_1}(z_c, \mu_F)\, z_d\, D_d^{h_2}(z_d, \mu_F)\, \Omega_{H_1 H_2 \to cd}\left(\tau', \Delta\eta, \bar{\eta}, \alpha_s(\mu_R), \frac{\mu_R}{\hat{m}}, \frac{\mu_F}{\hat{m}}\right)$$

where $\tau' = \dfrac{\hat{m}^2}{S} = \hat{\tau} x_a x_b$ (cf. Drell-Yan)

$$\Omega_{H_1 H_2 \to cd}\left(\tau', \Delta\eta, \bar{\eta}, \alpha_s(\mu_R), \frac{\mu_R}{\hat{m}}, \frac{\mu_F}{\hat{m}}\right) \equiv \sum_{ab} \int_0^1 dx_a\, dx_b\, f_a^{H_1}(x_a, \mu_F)\, f_b^{H_2}(x_b, \mu_F)\, \omega_{ab \to cd}\left(\hat{\tau}, \Delta\eta, \hat{\eta}, \alpha_s(\mu_R), \frac{\mu_R}{\hat{m}}, \frac{\mu_F}{\hat{m}}\right)$$

Taking moments

$$\int_{-\infty}^{\infty} d\bar{\eta}\, e^{i\nu\bar{\eta}} \int_0^1 d\tau'\, (\tau')^{N-1}\, \Omega_{H_1 H_2 \to cd}\left(\tau', \Delta\eta, \bar{\eta}, \alpha_s(\mu_R), \frac{\mu_R}{\hat{m}}, \frac{\mu_F}{\hat{m}}\right)$$
$$= \sum_{ab} \tilde{f}_a^{H_1}(N+1+i\nu/2, \mu_F) \tilde{f}_b^{H_2}(N+1-i\nu/2, \mu_F)\, \tilde{\omega}_{ab \to cd}\left(N, \nu, \Delta\eta, \alpha_s(\mu_R), \frac{\mu_R}{\hat{m}}, \frac{\mu_F}{\hat{m}}\right)$$

Almeida, Sterman, Vogelsang `09

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Threshold limit

Partonic threshold set by $\hat{\tau} = \dfrac{\hat{m}^2}{\hat{s}} \to 1$

where all the available energy is used to produce the pair

$$\tilde{\omega}_{ab\to cd}\left(N,\nu,\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right) \equiv \int_{-\infty}^{\infty} d\hat{\eta}\, \mathrm{e}^{i\nu\hat{\eta}} \int_0^1 d\hat{\tau}\, \hat{\tau}^{N-1}\, \omega_{ab\to cd}\left(\hat{\tau},\Delta\eta,\hat{\eta},\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)$$

$$\omega_{ab\to cd} = \left(\frac{\alpha_s}{\pi}\right)^2 \left[\omega^{\mathrm{LO}}_{ab\to cd} + \frac{\alpha_s}{\pi}\,\omega^{\mathrm{NLO}}_{ab\to cd} + \left(\frac{\alpha_s}{\pi}\right)^2 \omega^{\mathrm{NNLO}}_{ab\to cd} + \dots\right]$$

LO: $\omega^{\mathrm{LO}}_{ab\to cd}\left(\hat{\tau},\Delta\eta,\hat{\eta}\right) = \delta\left(1-\hat{\tau}\right)\,\delta\left(\hat{\eta}\right)\,\omega^{(0)}_{ab\to cd}(\Delta\eta)$

NLO: $$\omega^{\mathrm{NLO}}_{ab\to cd}\left(\hat{\tau},\Delta\eta,\hat{\eta}\right) = \delta\left(\hat{\eta}\right)\left[\omega^{(1,0)}_{ab\to cd}(\Delta\eta)\,\delta(1-\hat{\tau}) + \omega^{(1,1)}_{ab\to cd}(\Delta\eta)\left(\frac{1}{1-\hat{\tau}}\right)_+ + \omega^{(1,2)}_{ab\to cd}(\Delta\eta)\left(\frac{\log(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right] + \omega^{\mathrm{reg,NLO}}_{ab\to cd}\left(\hat{\tau},\Delta\eta,\hat{\eta}\right)$$

This structure holds true to all orders

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Threshold Resummation toward NNLL

$$\tilde{\omega}^{\text{resum}}_{ab\to cd}\left(N,\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right) = \xi_R\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\xi_F^{abcd}\left(\alpha_s(\mu_R),\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\Delta_a^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_b^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_c^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_d^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\text{Tr}\left\{H\left(\Delta\eta,\alpha_s(\mu_R)\right)\mathcal{S}_N^{\dagger}\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)S\left(\alpha_s(\hat{m}/\bar{N}),\Delta\eta\right)\mathcal{S}_N\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\right\}_{ab\to cd}$$

Refactorization near threshold

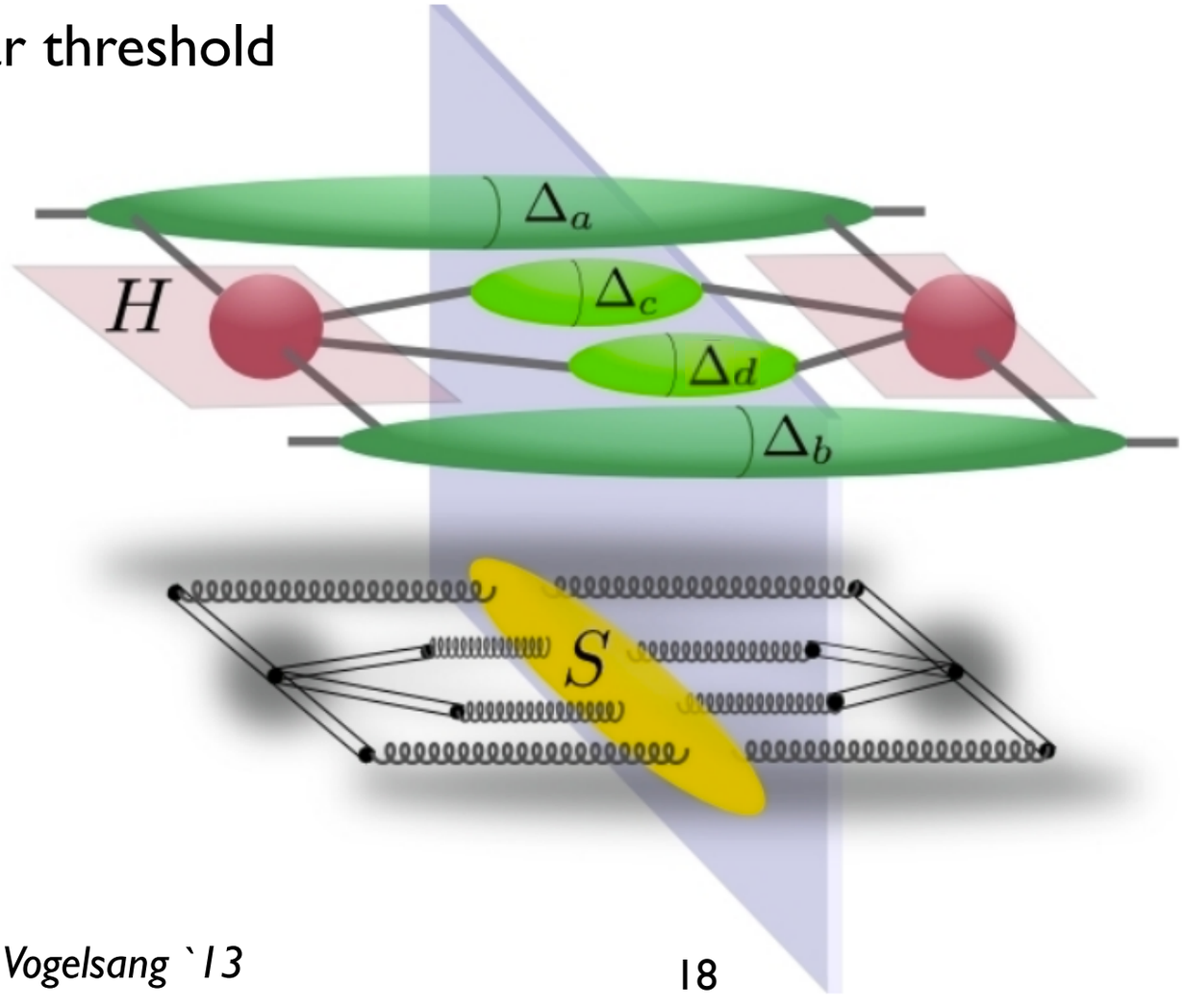

Figure adapted from
de Florian, Pfeuffer, Schäfer, Vogelsang `13

Kidonakis, Oderda, Sterman `98
Bonciani, Catani, Mangano, Nason `98
Banfi, Salam, Zanderighi
Dokshitzer, Marchesini
Almeida, Sterman, Vogelsang `09

$$\tilde{\omega}^{\text{resum}}_{ab\to cd}\left(N,\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right) = \xi_R\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\xi_F^{abcd}\left(\alpha_s(\mu_R),\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\Delta_a^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_b^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_c^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_d^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\mathrm{Tr}\left\{H\left(\Delta\eta,\alpha_s(\mu_R)\right)\mathcal{S}_N^\dagger\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)S\left(\alpha_s(\hat{m}/\bar{N}),\Delta\eta\right)\mathcal{S}_N\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\right\}_{ab\to cd}$$

Radiative factors for all 4 colored partons
taking into account soft + collinear gluon emissions

$$\Delta_i^N\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right) = R_i(\alpha_s(\mu_R))\,\exp\left\{\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\left[\int_{\mu_F^2}^{(1-z)^2\hat{m}^2}\frac{d\mu^2}{\mu^2}A_i(\alpha_s(\mu))+D_i(\alpha_s((1-z)\hat{m}))\right]\right\}$$

similar to DY

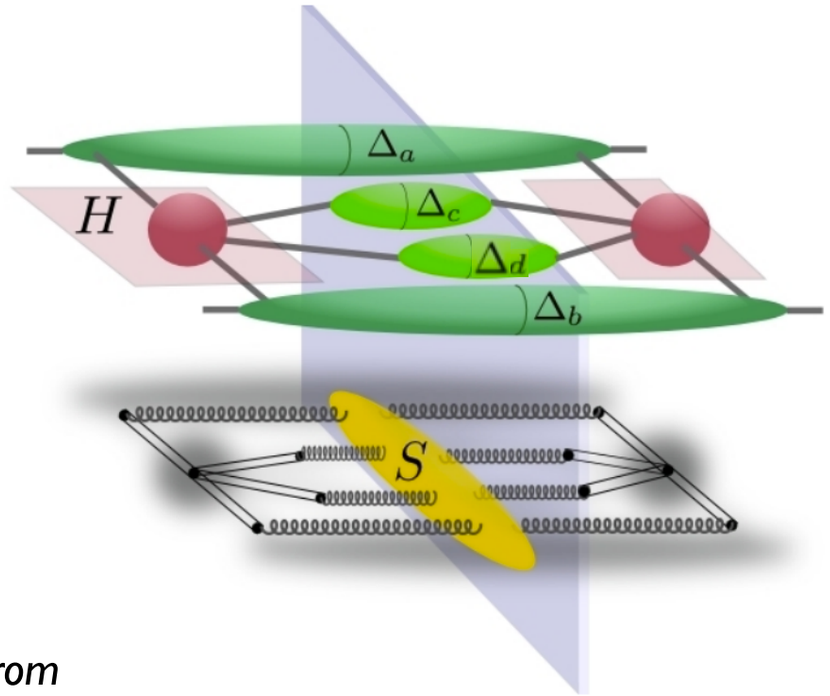

Figure adapted from
de Florian, Pfeuffer, Schäfer, Vogelsang `13

$$\tilde{\omega}^{\text{resum}}_{ab\to cd}\left(N,\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right) = \xi_R\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\xi_F^{abcd}\left(\alpha_s(\mu_R),\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\Delta_a^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_b^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_c^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_d^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\mathrm{Tr}\left\{H\left(\Delta\eta,\alpha_s(\mu_R)\right)\mathcal{S}_N^\dagger\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)S\left(\alpha_s(\hat{m}/\bar{N}),\Delta\eta\right)\mathcal{S}_N\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\right\}_{ab\to cd}$$

this part depends on the scattering angle

Trace taken in the space of color exchange operators $\mathrm{Tr}\{H\mathcal{S}_N^\dagger S\mathcal{S}_N\}_{ab\to cd}$ with

Hard-scattering matrix $H_{ab\to cd}\left(\Delta\eta,\alpha_s\right)=\left(\frac{\alpha_s}{\pi}\right)^2\left[H^{(0)}_{ab\to cd}\left(\Delta\eta\right)+\frac{\alpha_s}{\pi}H^{(1)}_{ab\to cd}\left(\Delta\eta\right)+\mathcal{O}(\alpha_s^2)\right]$

Soft matrix $$S_{ab\to cd}\left(\alpha_s(\hat{m}/\bar{N}),\Delta\eta\right)=S^{(0)}_{ab\to cd}+\frac{\alpha_s(\hat{m}/\bar{N})}{\pi}S^{(1)}_{ab\to cd}\left(\Delta\eta\right)+\mathcal{O}(\alpha_s^2)$$

$$\alpha_s(\hat{m}/\bar{N})=\frac{\alpha_s(\mu_R)}{1-2\lambda}$$

$\ln N$-dependence beyond NLL!

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$$\tilde{\omega}^{\text{resum}}_{ab\to cd}\left(N,\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right) = \xi_R\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\xi_F^{abcd}\left(\alpha_s(\mu_R),\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\Delta_a^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_b^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_c^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_d^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\text{Tr}\left\{H\left(\Delta\eta,\alpha_s(\mu_R)\right)\mathcal{S}_N^\dagger\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)S\left(\alpha_s(\hat{m}/\bar{N}),\Delta\eta\right)\mathcal{S}_N\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\right\}_{ab\to cd}$$

this part depends on the scattering angle

Trace taken in the space of color exchange operators $\text{Tr}\{H\mathcal{S}_N^\dagger S\mathcal{S}_N\}_{ab\to cd}$ with

Wide angle soft gluons

$$\mathcal{S}_{N,ab\to cd}\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right) = \mathcal{P}\exp\left[\frac{1}{2}\int_{\hat{m}^2}^{\hat{m}^2/\bar{N}^2}\frac{d\mu^2}{\mu^2}\Gamma_{ab\to cd}\left(\Delta\eta,\alpha_s(\mu)\right)\right]$$

$$\Gamma_{ab\to cd}\left(\alpha_s,\Delta\eta\right) = \frac{\alpha_s}{\pi}\Gamma^{(1)}_{ab\to cd}\left(\Delta\eta\right) + \left(\frac{\alpha_s}{\pi}\right)^2\Gamma^{(2)}_{ab\to cd}(\Delta\eta) + \mathcal{O}(\alpha_s^3)$$

Kidonakis, Sterman `97
Aybat, Dixon, Sterman `06

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Expansion of the resummed cross section up to NLO

$$\tilde{\omega}^{\text{resum}}_{ab\to cd}\left(N,\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right) = \left(\frac{\alpha_s(\mu_R)}{\pi}\right)^2\left[\text{Tr}\{H^{(0)}S^{(0)}\}_{ab\to cd}\left\{1+2b_0\alpha_s(\mu_R)\ln\frac{\mu_R^2}{\hat{m}^2}\right.\right.$$

$$\left.+\frac{\alpha_s(\mu_R)}{\pi}\sum_{i=a,b,c,d}\left(A_i^{(1)}\ln^2\bar{N}+\frac{1}{4}A_i^{(1)}\zeta(2)+\left(A_i^{(1)}\ln\bar{N}+\frac{1}{2}B_i^{(1)}\right)\ln\frac{\mu_F^2}{\hat{m}^2}\right)\right\}$$

$$\left.+\frac{\alpha_s(\mu_R)}{\pi}\text{Tr}\Big\{-\left[H^{(0)}(\Gamma^{(1)})^\dagger S^{(0)}+H^{(0)}S^{(0)}\Gamma^{(1)}\right]\ln\bar{N}+\left\{H^{(1)}S^{(0)}+H^{(0)}S^{(1)}\right\}_{ab\to cd}\right]+\mathcal{O}(\alpha_s^4)$$

tower

3rd only a combined expression appears that can be extracted from NLO

$$\left(\frac{\alpha_s}{\pi}\right)^k\frac{\sum_i A_i^{(1)}}{(k-1)!}\,\text{Tr}\left\{H^{(1)}S^{(0)}+H^{(0)}S^{(1)}\right\}_{ab\to cd}\ln^{2k-2}\bar{N}$$

Almeida, Sterman, Vogelsang `09
de Florian, Vogelsang `05; `09
de Florian, Hinderer, Mukherjee, FR, Vogelsang `14

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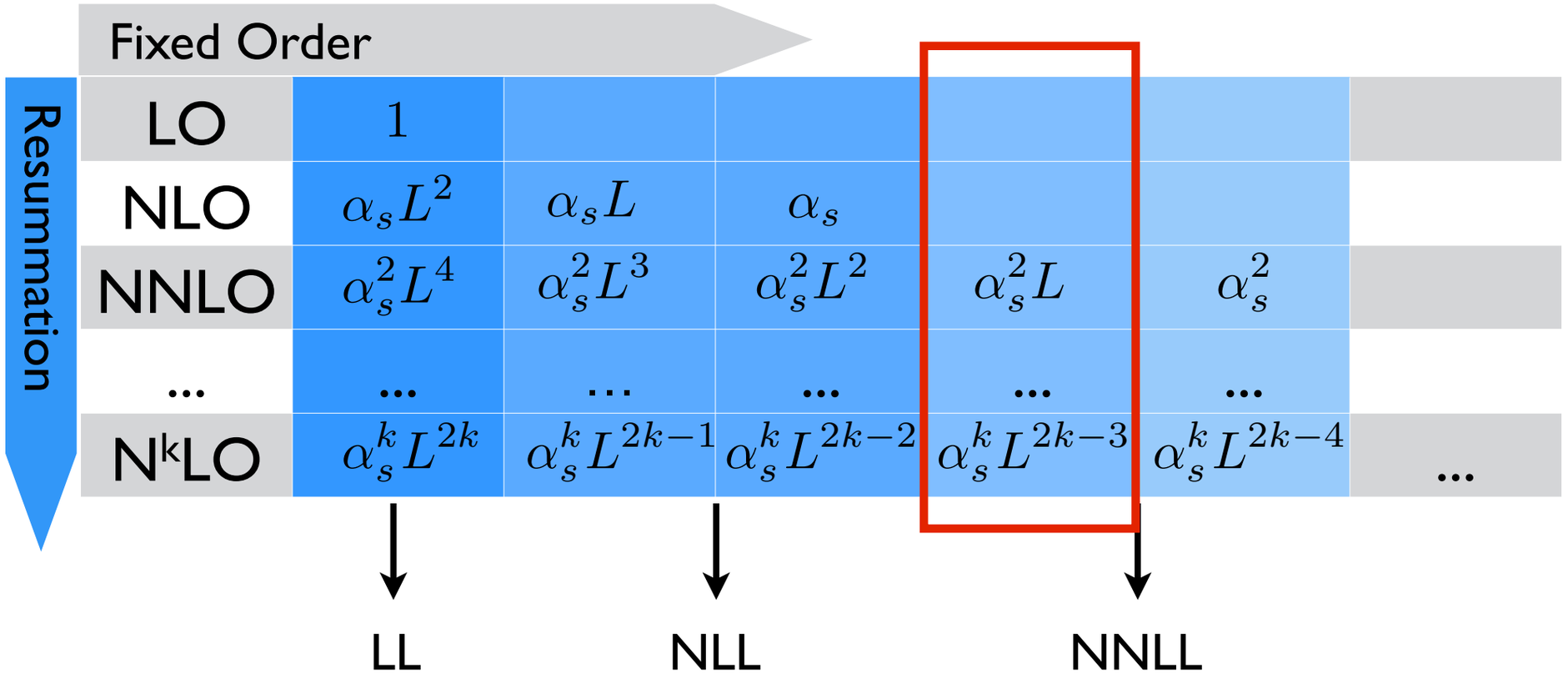
tower

3rd only a combined expression appears that can be extracted from NLO

$$\left(\frac{\alpha_s}{\pi}\right)^k \frac{\sum_i A_i^{(1)}}{(k-1)!} \mathrm{Tr}\left\{H^{(1)}S^{(0)} + H^{(0)}S^{(1)}\right\}_{ab\to cd} \ln^{2k-2}\bar{N}$$

4th $S^{(1)}$, $H^{(1)}$ appear in various combinations with the anomalous dimension matrices $\Gamma^{(1)}$

5th also the matrices $S^{(2)}$, $H^{(2)}$ are needed



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Hard-Scattering Matrix

$qq' \to qq'$
$qq \to qq$
$q\bar{q} \to q\bar{q}$
$q\bar{q} \to q'\bar{q}'$
$q\bar{q}' \to q\bar{q}'$
$q\bar{q} \to gg$
$gg \to q\bar{q}$
$qg \to qg$
$gg \to gg$

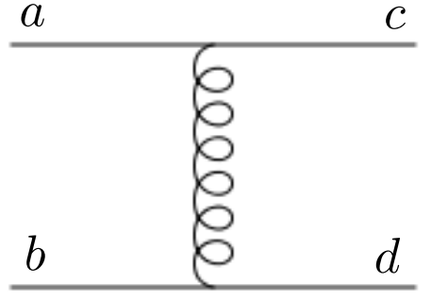
We consider the process $q(p_1,a)q'(p_2,b) \to q(p_3,c)q'(p_4,d)$

t-channel octet-singlet basis

$$\mathcal{C}_1 \equiv T^g_{ca}T^g_{db} = \frac{1}{2}\left(\delta_{ad}\,\delta_{bc} - \frac{1}{N_c}\,\delta_{ac}\,\delta_{bd}\right)$$

$$\mathcal{C}_2 \equiv \delta_{ac}\delta_{bd}$$

Decompose the LO amplitude as $M^{(0)} = \sum_I h_I^{(0)}\mathcal{C}_I$



⟶ construct LO matrix

$$(H^{(0)})_{IJ} \equiv \frac{1}{4N_c^2}\,h_I^{(0)}h_J^{(0)*} = \begin{pmatrix} \frac{2}{N_c^2}\frac{s^2+u^2}{t^2} & 0 \\ 0 & 0 \end{pmatrix}$$

Color traced result obtained via $\frac{1}{4N_c^2}\sum_{a,b,c,d=1}^{N_c}|M^{(0)}|^2 = \mathrm{Tr}[H^{(0)}S^{(0)}]$

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Hard-Scattering Matrix

NLO: Decompose 1-loop amplitude $M^{(1),\mathrm{virt}} = \sum_I \tilde{h}_I^{(1)} \mathcal{C}_I$

1-loop helicity amplitudes from *Kunszt, Signer, Trocsanyi `94; Bern, Kosower `92*

$$\begin{aligned}\mathcal{A}_{1\mathrm{loop}}^{\lambda\lambda'} &= \left(\delta_{ad}\,\delta_{bc} - \frac{1}{N_c}\,\delta_{ac}\,\delta_{bd}\right) a_{4;1}^{\lambda\lambda'} + \delta_{ad}\,\delta_{bc}\,a_{4;2}^{\lambda\lambda'} \\ &= \mathcal{C}_1 \times 2\left(a_{4;1}^{\lambda\lambda'} + a_{4;2}^{\lambda\lambda'}\right) + \mathcal{C}_2 \times \frac{1}{N_c} a_{4;2}^{\lambda\lambda'}\,.\end{aligned}$$

⟶ construct $(\tilde{H}^{(1)})_{IJ} \equiv \frac{1}{4N_c^2}\left(\tilde{h}_I^{(1)} h_J^{(0)*} + h_I^{(0)} \tilde{h}_J^{(1)*}\right)$

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Hard-Scattering Matrix

$$H^{(1)} = \begin{pmatrix} \mathcal{R}e\left\{\frac{1}{2N_c^2}\left[\frac{s^2+u^2}{t^2}\left(-4C_F L(t)^2 + 2X_1(s,t,u)L(t) + 2Y\right)\right.\right. & \left(H^{(1)}\right)_{12} = \left(H^{(1)}\right)^*_{21} \\ \left.\left. + \frac{s^2}{t^2}(C_A - 4C_F)Z(s,t,u) - \frac{u^2}{t^2}(2C_A - 4C_F)Z(u,t,s)\right]\right\} & \\ \frac{1}{2N_c^2}\left[\frac{s^2+u^2}{t^2}X_2(s,t,u)L(t) - \frac{s^2}{t^2}\frac{C_F}{2C_A}Z(s,t,u) + \frac{u^2}{t^2}\frac{C_F}{2C_A}Z(u,t,s)\right] & 0 \end{pmatrix}_{2\times 2}$$

with $X_1(s,t,u) = 6C_F - 4\pi b_0 + 8C_F[L(s) - L(u)] - 2C_A[2L(s) - L(t) - L(u)]$

$X_2(s,t,u),\ Y,\ Z(s,t,u)$ similarly

and $L(t) = \log\frac{-t}{s},\ L(u) = \log\frac{-u}{s},\ L(s) = -i\pi$

see also Kelley, Schwartz `11; Broggio et al. `14

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Hard-Scattering Matrix

$$H^{(1)} = \begin{pmatrix} \mathcal{R}e\left\{\frac{1}{2N_c^2}\left[\frac{s^2+u^2}{t^2}\left(-4C_F L(t)^2 + 2X_1(s,t,u)L(t) + 2Y\right)\right.\right. & \left(H^{(1)}\right)_{12} = \left(H^{(1)}\right)^*_{21} \\ \left.\left. + \frac{s^2}{t^2}(C_A - 4C_F)Z(s,t,u) - \frac{u^2}{t^2}(2C_A - 4C_F)Z(u,t,s)\right]\right\} & \\ \frac{1}{2N_c^2}\left[\frac{s^2+u^2}{t^2}X_2(s,t,u)L(t) - \frac{s^2}{t^2}\frac{C_F}{2C_A}Z(s,t,u) + \frac{u^2}{t^2}\frac{C_F}{2C_A}Z(u,t,s)\right] & 0 \end{pmatrix}_{2\times 2}$$

with $X_1(s,t,u) = 6C_F - 4\pi b_0 + 8C_F[L(s) - L(u)] - 2C_A[2L(s) - L(t) - L(u)]$

$X_2(s,t,u),\ Y,\ Z(s,t,u)$ similarly

and $L(t) = \log\frac{-t}{s},\ L(u) = \log\frac{-u}{s},\ L(s) = -i\pi$

Entirely new feature at NNLL!

Both $H^{(1)}$ and $\Gamma^{(1)}$ have an imaginary part contributing to the cross section in the 4^{th} tower

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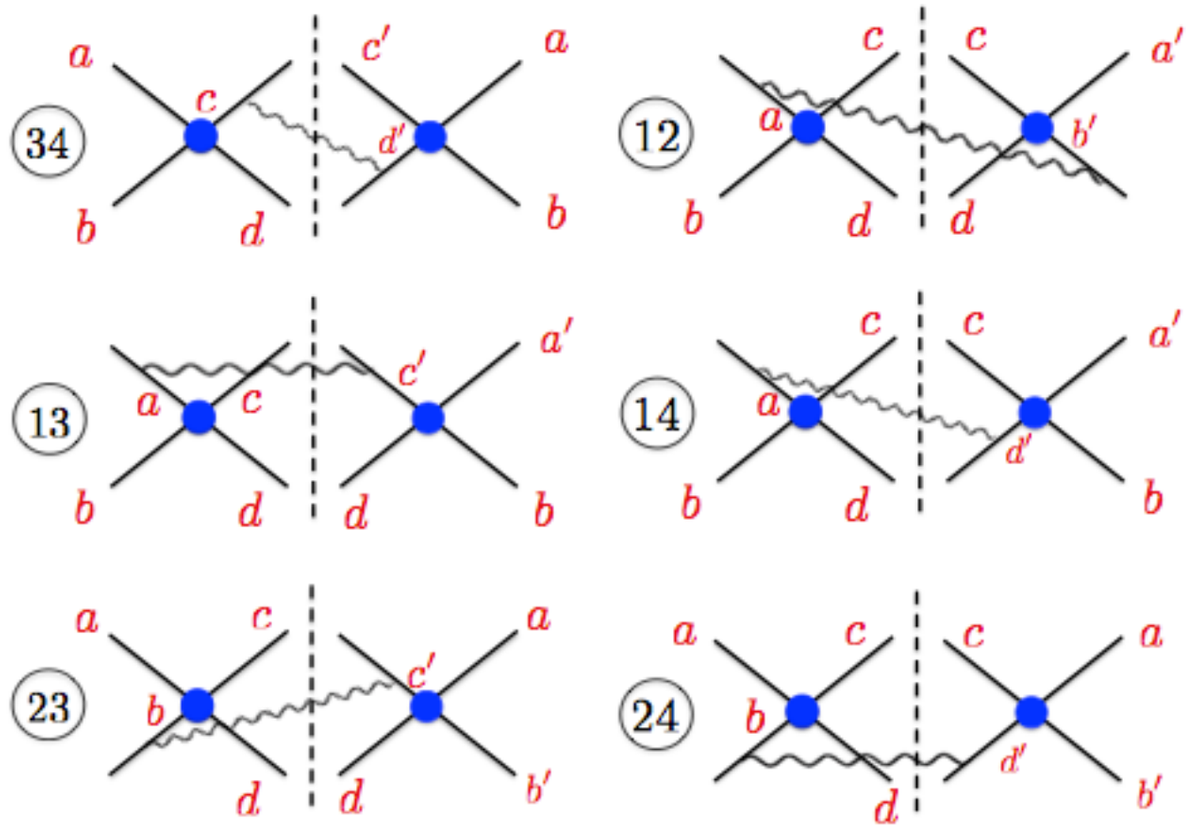
Soft Matrix

We consider the process $q(p_1) + q'(p_2) \to q(p_3) + q'(p_4) + g(k)$

in the eikonal approximation



Each diagram has the structure

$$\sum_{IJ} h_I^{(0)} h_J^{(0)*} \left(\mathcal{R}_{ij}\right)_{JI} I_{ij}$$

For example, we find

$$\mathcal{R}_{12} = \mathcal{R}_{34} = \frac{C_F}{2} \begin{pmatrix} -1 & N_c \\ N_c & 0 \end{pmatrix}$$

$$\longrightarrow \quad \sum_{ij} \mathcal{R}_{ij}\, I_{ij} = \frac{C_F}{2} \begin{pmatrix} \frac{1}{2}(I_{13}+I_{24}) - I_{12} - I_{34} - \frac{N_c^2-2}{2}(I_{14}+I_{23}) & N_c(I_{12}+I_{34}-I_{14}-I_{23}) \\ N_c(I_{12}+I_{34}-I_{14}-I_{23}) & -2N_c^2(I_{13}+I_{24}) \end{pmatrix}$$

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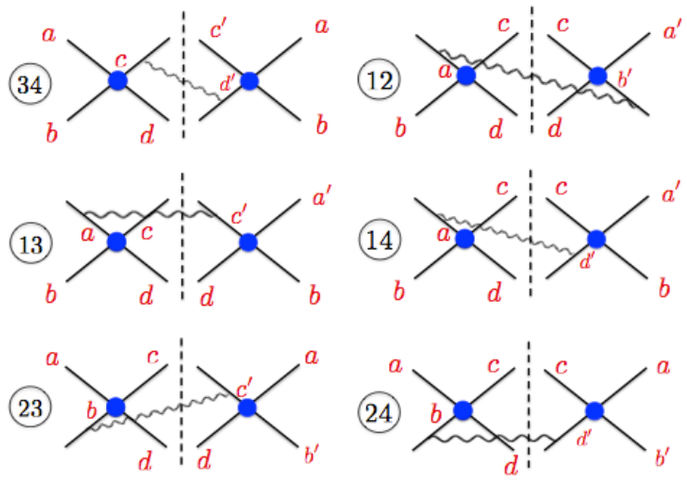
Soft Matrix

The integrals are essentially given by eikonal facors integrated over the gluon phase space



$$I_{ij} = -\frac{\alpha_s}{\pi}\frac{s}{4\pi}e^{\varepsilon\gamma_E}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\int_0^1 d\hat{\tau}\,\hat{\tau}^{-\varepsilon}(1-\hat{\tau})^{1-2\varepsilon}\int d\Omega\,\frac{p_i\cdot p_j}{(p_i\cdot k)(p_j\cdot k)}$$

→ The soft matrix is extracted in Mellin space

$$S^{(1)} = \frac{C_F}{2}\begin{pmatrix} \mathrm{Li}_2\left(-\frac{u}{t}\right)+(2-N_c^2)\,\mathrm{Li}_2\left(-\frac{t}{u}\right) & -2N_c\,\mathrm{Li}_2\left(-\frac{t}{u}\right) \\ -2N_c\,\mathrm{Li}_2\left(-\frac{t}{u}\right) & -4N_c^2\,\mathrm{Li}_2\left(-\frac{u}{t}\right) \end{pmatrix}$$

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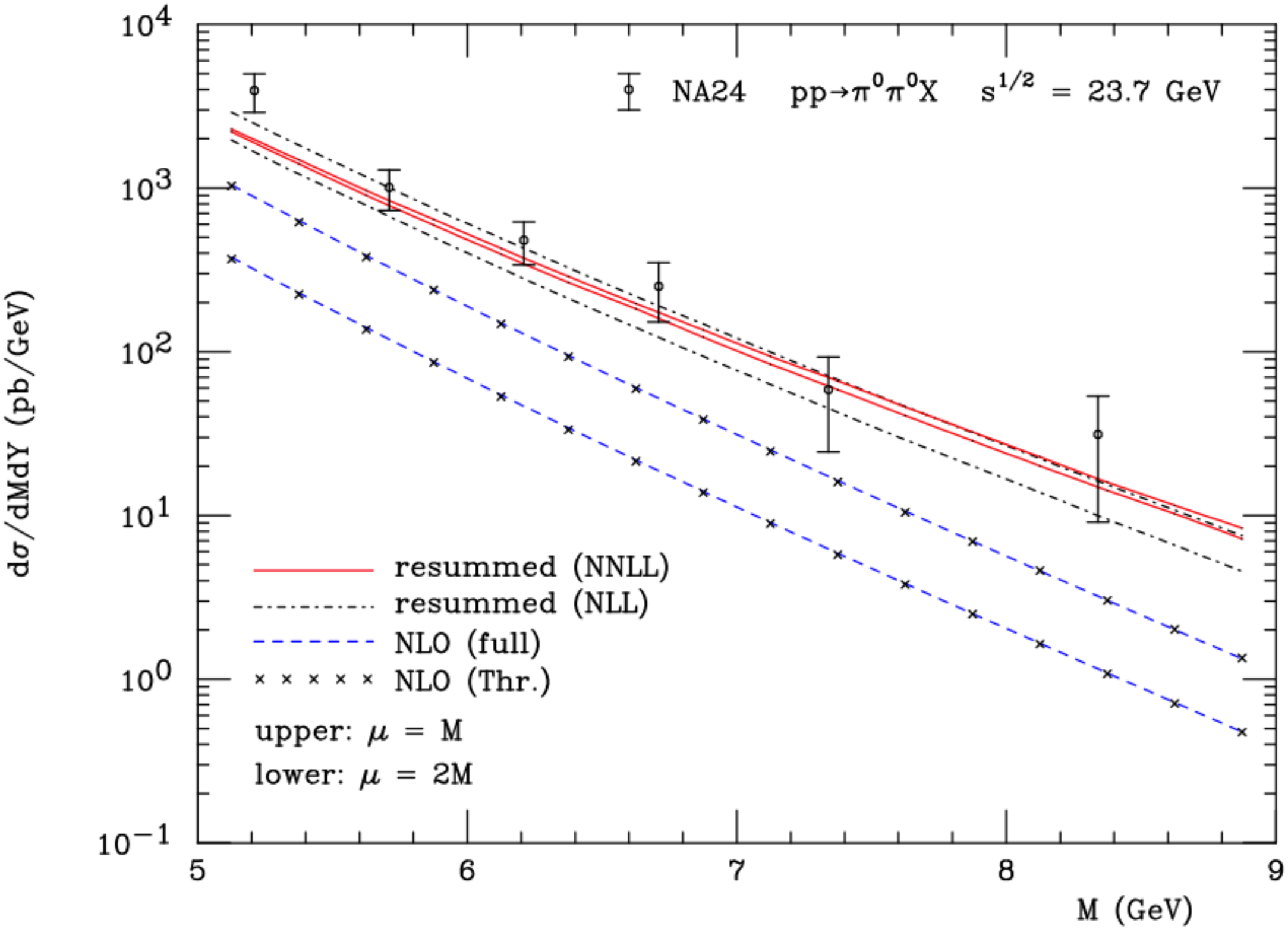

NA24

CTEQ6 NLO PDFs, DSS FFs

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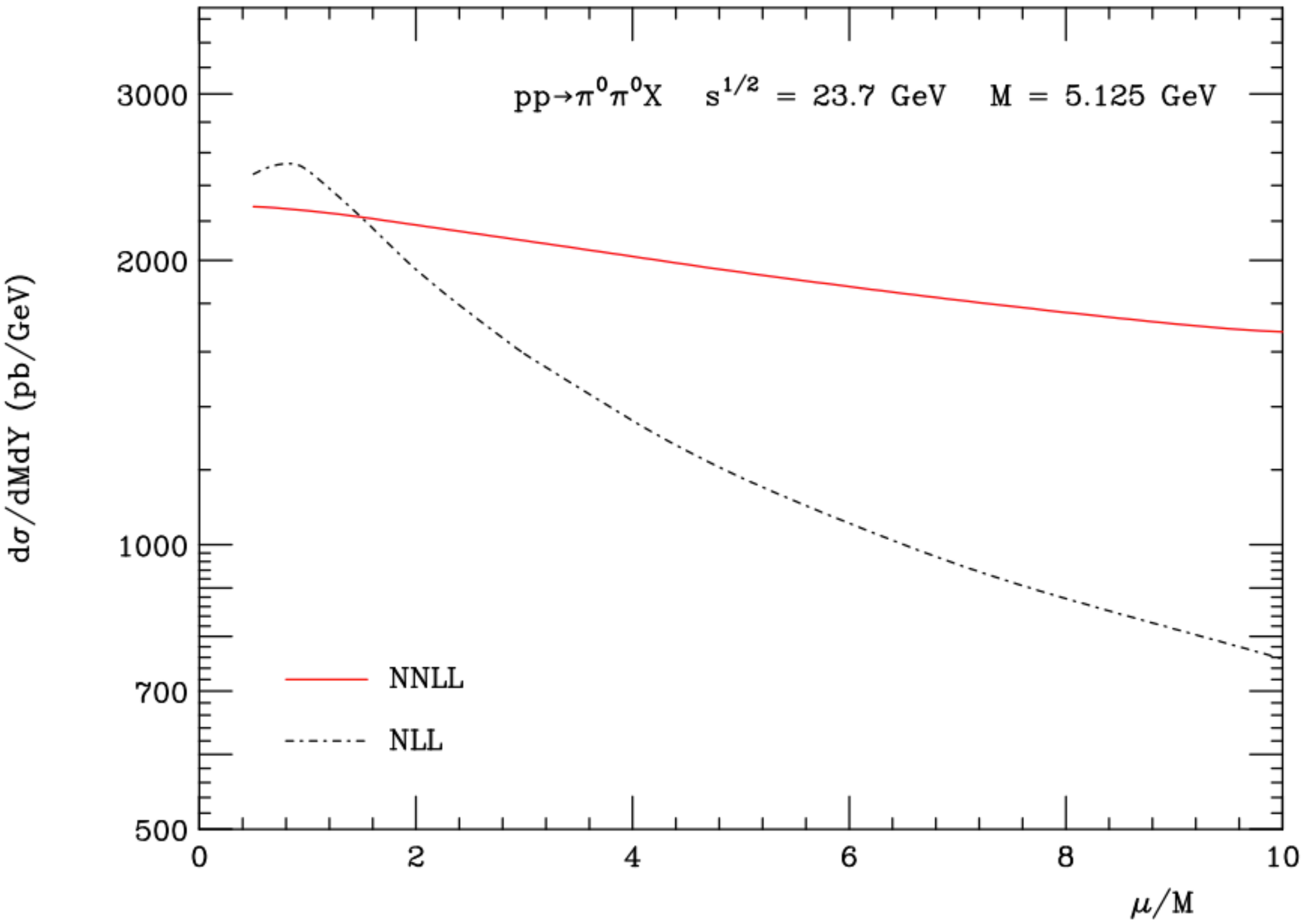

Scale dependence

CTEQ6 NLO PDFs, DSS FFs

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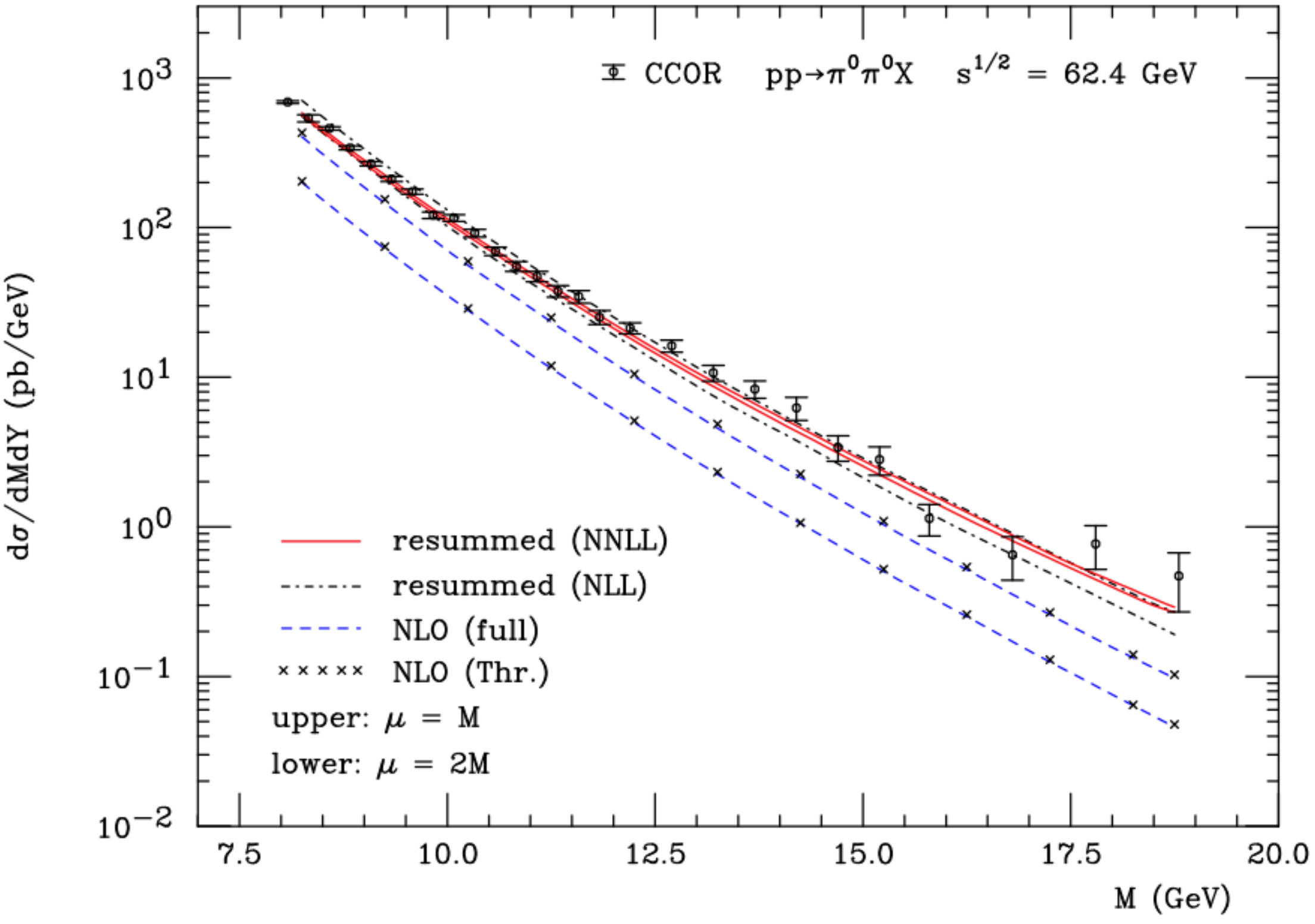

CCOR

CTEQ6 NLO PDFs, DSS FFs

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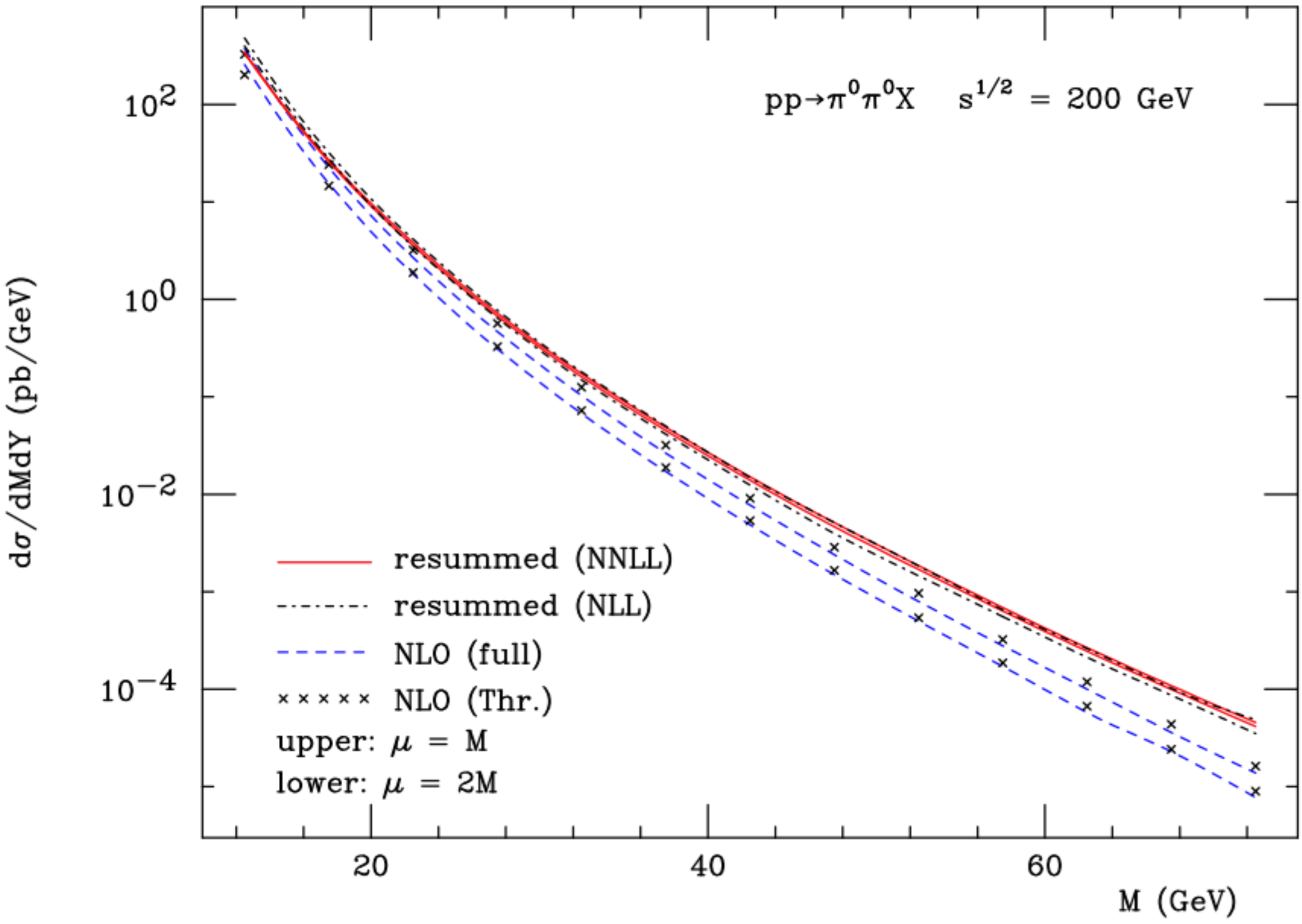

RHIC

CTEQ6 NLO PDFs, DSS FFs

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Outline

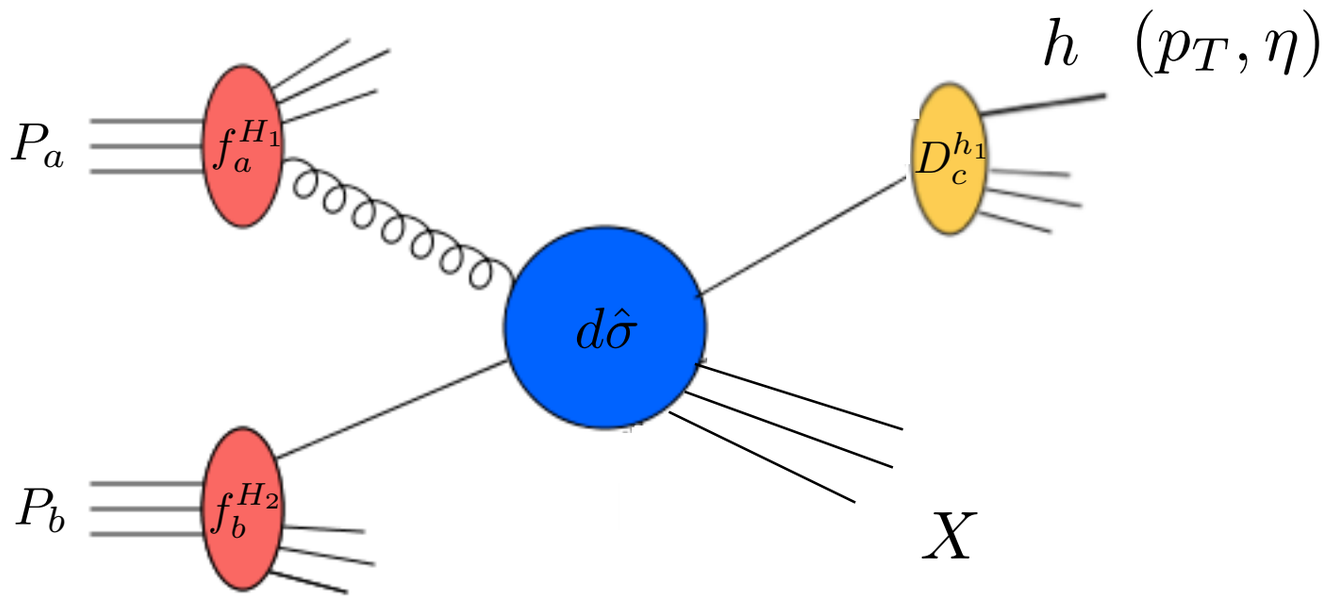
- QCD Threshold Resummation
- Di-Hadron Production

Hinderer, FR, Sterman, Vogelsang - PRD '15

- Single-Inclusive Hadron Production *in preparation*
- Conclusions

Single-Inclusive Hadron Production

$$H_1 + H_2 \to h + X$$



- Data from various experiments (fixed target, RHIC, LHC)
- Relevant for the extraction of FFs (e.g. DSS analysis *de Florian, Sassot, Epele, Hernández-Pinto, Stratmann '14*)
- NLL resummation for the rapidity integrated cross section *de Florian, Vogelsang '05*

⟶ toward NNLL resummation for the double differential cross section

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Double differential cross section

$$\frac{p_T^3 d\sigma}{dp_T d\eta} = \sum_{a,b,c} \int_{x_a^{\min}}^1 dx_a \int_{x_b^{\min}}^1 dx_b \int_x^1 dz_c \frac{\hat{x}_T^4 z_c^2}{8v} f_a(x_a,\mu_F) f_b(x_b,\mu_F) D_c^h(z_c,\mu_F) \frac{\hat{s} d\hat{\sigma}_{ab\to cX}}{dv dw}$$

Hadronic scaling variable $\quad x_T = \dfrac{2p_T}{\sqrt{s}}$

$$x = \frac{x_T \cosh\hat{\eta}}{\sqrt{x_a x_b}} \qquad \hat{x}_T = \frac{x_T}{z_c\sqrt{x_a x_b}}$$

The usual variables
for single-inclusive kinematics

$$v = 1 - \frac{\hat{x}_T}{2} e^{-\hat{\eta}} \qquad w = \frac{1}{v}\frac{\hat{x}_T}{2} e^{\hat{\eta}}$$

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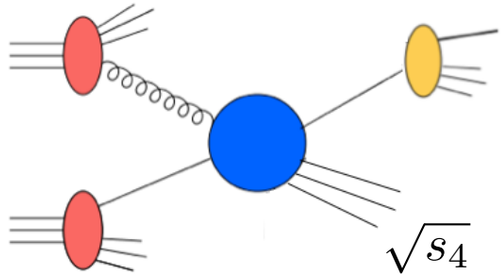
Double differential cross section

$$\frac{p_T^3 d\sigma}{dp_T d\eta} = \sum_{a,b,c} \int_{x_a^{\min}}^1 dx_a \int_{x_b^{\min}}^1 dx_b \int_x^1 dz_c \frac{\hat{x}_T^4 z_c^2}{8v} f_a(x_a,\mu_F) f_b(x_b,\mu_F) D_c^h(z_c,\mu_F) \frac{\hat{s} d\hat{\sigma}_{ab\to cX}}{dv dw}$$

Invariant mass of the unobserved/ recoiling partonic final state:

$$s_4 = \hat{s} + \hat{t} + \hat{u} = \hat{s} v (1-w) = \hat{s}(1 - \hat{x}_T \cosh\hat{\eta})$$



Threshold set for $s_4/\hat{s} \to 0,\ w \to 1$

$$\hat{\sigma}_{ab\to cX}(v,w) = \left(\frac{\alpha_s}{\pi}\right)^2 \left[\hat{\sigma}^{\rm LO}_{ab\to cX}(v) + \frac{\alpha_s}{\pi}\hat{\sigma}^{\rm NLO}_{ab\to cX}(v,w) + \mathcal{O}(\alpha_s^2)\right]$$

$$\alpha_s^k \left(\frac{\ln^{2k-1}(1-w)}{1-w}\right)_+$$

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Taking into account the rapidity dependence *de Florian, Pfeuffer, Schäfer, Vogelsang '14*

$$\frac{p_T^3 d\sigma}{dp_T d\eta} = \sum_{a,b,c} \int_{x_a^{\min}}^1 dx_a \int_{x_b^{\min}}^1 dx_b \int_x^1 dz_c \frac{\hat{x}_T^4 z_c^2}{8v} f_a(x_a,\mu_F) f_b(x_b,\mu_F) D_c^h(z_c,\mu_F) \frac{\hat{s} d\hat{\sigma}_{ab\to cX}}{dvdw}$$

$$= \sum_{a,b,c} \int_0^1 dx_a \int_0^1 dx_b f_a(x_a,\mu_F) f_b(x_b,\mu_F) \times \int_{\mathcal{C}} \frac{dN}{2\pi i} (x^2)^{-N} (D_c^h)^{2N+3}(\mu_F) \tilde{\omega}^{2N}(v)$$

where $$(D_c^h)^N(\mu_F) = \int_0^1 dz\, z^{N-1} D_c^h(z,\mu_F)$$

$$\tilde{\omega}^N(v) = 2\int_0^1 d\frac{s_4}{\hat{s}} \left(1 - \frac{s_4}{\hat{s}}\right)^{N-1} \frac{\hat{x}_T^4}{8v} \frac{\hat{s} d\hat{\sigma}_{ab\to cX}}{dvdw}$$

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Resummed Cross Section

unobserved final state parton cf. DIS

$$\tilde{\omega}^{N,\mathrm{res}}_{ab\to cd}\left(v,\alpha_s(\mu_R^2),\frac{\mu_R^2}{s},\frac{\mu_F^2}{s}\right) = \xi_R\left(\alpha_s(\mu_R^2),\frac{\mu_R^2}{s}\right)\xi_F^{abc}\left(\alpha_s(\mu_R^2),\frac{\mu_R^2}{s},\frac{\mu_F^2}{s}\right)$$

$$\times\,\Delta_a^{N_a}\left(\alpha_s(\mu_R^2),\frac{\mu_R^2}{s},\frac{\mu_F^2}{s}\right)\Delta_b^{N_b}\left(\alpha_s(\mu_R^2),\frac{\mu_R^2}{s},\frac{\mu_F^2}{s}\right)\Delta_c^{N}\left(\alpha_s(\mu_R^2),\frac{\mu_R^2}{s},\frac{\mu_F^2}{s}\right)J_d^N\left(\alpha_s(\mu_R^2),\frac{\mu_R^2}{s}\right)$$

$$\times\,\mathrm{Tr}\left\{H\left(v,\alpha_s(\mu_R^2)\right)\mathcal{S}_N^\dagger\left(v,\alpha_s(\mu_R^2),\frac{\mu_R^2}{s}\right)S\left(\alpha_s(s/\bar{N}^2),v\right)\mathcal{S}_N\left(v,\alpha_s(\mu_R^2),\frac{\mu_R^2}{s}\right)\right\}_{ab\to cd}$$

new soft matrix

essentially the same hard-scattering matrix as for the di-hadrons

rapidity dependence for initial state radiative factors

$$N_a = vN$$
$$N_b = (1-v)N$$

see also Catani, Grazzini, Torre '13

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Soft Matrix

$qq' \to qq'$

$$S^{(1)}_{qq'\to qq'} = -\frac{C_F}{2}\left([\ln(1-v)+\ln v]^2 - 2\,\zeta(2)\right) S^{(0)}_{qq'\to qq'} + C_F \ln(1-v)\ln v \begin{pmatrix} 1 & -C_A \\ -C_A & 0 \end{pmatrix}$$

$gg \to gg$

$$S^{(1)}_{gg\to gg} = -\frac{C_A}{2}\left([\ln(1-v)+\ln v]^2 - 2\,\zeta(2)\right) S^{(0)}_{gg\to gg} + 3\ln(1-v)\ln v \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 4 & 8 & 0 \\ 0 & 0 & 0 & 2 & 4 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 8 & 0 & 20 & 12 \\ 0 & 0 & 0 & 0 & 0 & 6 & 12 & 36 \end{pmatrix}$$

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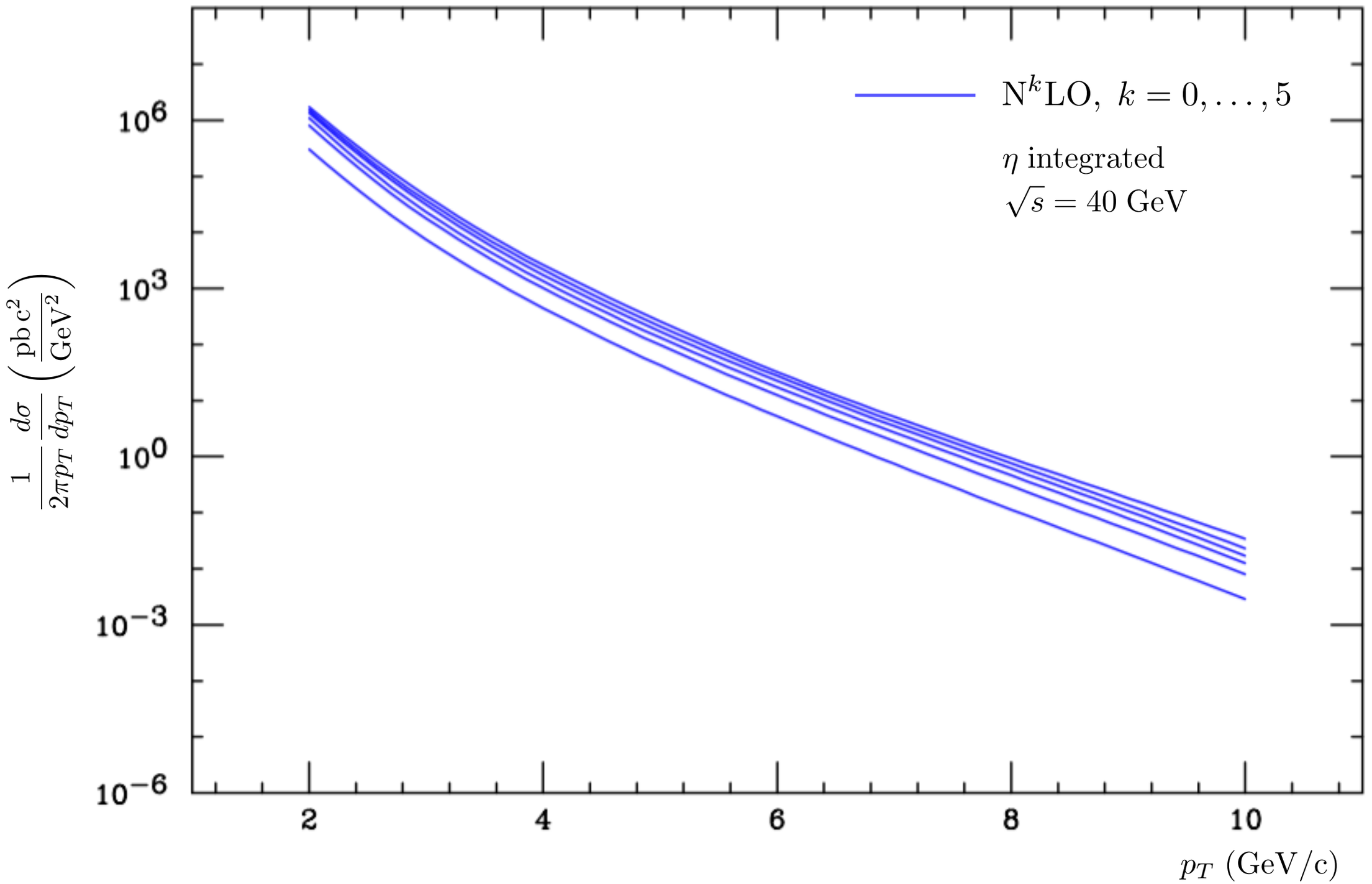

preliminary

using MSTW PDFs and DSS FFs

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Outline

- QCD Threshold Resummation
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Hinderer, FR, Sterman, Vogelsang - PRD '15

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Conclusions

- Extension of threshold resummation for QCD hard scattering toward NNLL accuracy
- Single- and di-hadron production
- Future extension to single- and di-jet production, photons…
- Relevant for longitudinal spin asymmetries

Backup

- Logarithms emphasized by parton distributions

$$d\sigma \sim \int_\tau^1 \frac{dz}{z} \mathcal{L}_{q\bar{q}}\left(\frac{\tau}{z}\right) \omega_{q\bar{q}}(z) \qquad \text{with} \qquad \tau = \frac{Q^2}{S}$$

$z = 1$ relevant,
in particular as $\tau \to 1$

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$$\tilde{\omega}^{\text{resum}}_{ab\to cd}\left(N,\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right) = \xi_R\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\,\xi_F^{abcd}\left(\alpha_s(\mu_R),\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\Delta_a^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_b^{N+1}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_c^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)\Delta_d^{N+2}\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}},\frac{\mu_F}{\hat{m}}\right)$$

$$\times\,\mathrm{Tr}\left\{H\left(\Delta\eta,\alpha_s(\mu_R)\right)\,\mathcal{S}_N^{\dagger}\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\,S\left(\alpha_s(\hat{m}/\bar{N}),\Delta\eta\right)\mathcal{S}_N\left(\Delta\eta,\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right)\right\}_{ab\to cd}$$

Factorization scale dependence

$$\ln\xi_F^{abcd}\left(\alpha_s(\mu_R),\frac{\mu_F}{\hat{m}}\right) = -\frac{1}{2}\sum_{i=a,b,c,d}\int_{\mu_F^2}^{\hat{m}^2}\frac{d\mu^2}{\mu^2}\frac{\alpha_s(\mu)}{\pi}B_i^{(1)}$$

Renormalization scale dependence

$$\ln\xi_R\left(\alpha_s(\mu_R),\frac{\mu_R}{\hat{m}}\right) = 2\int_{\mu_R^2}^{\hat{m}^2}\frac{d\mu^2}{\mu^2}\beta(\alpha_s(\mu))$$

Kulesza, Sterman, Vogelsang `02
Kidonakis, Owens `00

no $\ln N$-dependence!

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Contribution of the Imaginary Parts

Entirely new feature at NNLL!

Both $\Gamma^{(1)},\ H^{(1)}$ have complex contributions, e.g.

- $\Gamma^{(1)}_{qq'\to qq'} = \begin{pmatrix} \frac{1}{N_c}(2S-T-U)+2C_FU & 2(U-S) \\ \frac{C_F}{N_c}(U-S) & 2C_FT \end{pmatrix}$ with $S=-i\pi,\quad T=\ln\left(\frac{-t}{s}\right),\quad U=\ln\left(\frac{-u}{s}\right)$
- $\left(H^{(1)}\right)_{12} = \left(H^{(1)}\right)^*_{21}$ with $L(s)=-i\pi$

Define $M \equiv \mathrm{e}^{\Gamma^\dagger} S \mathrm{e}^{\Gamma}$ and decompose $M = M_R + iM_I$, $H = H_R + iH_I$

Both $M,\ H$ are hermitian matrices. We may write the trace part as

$$\mathrm{Tr}\{HM\} = \mathrm{Tr}\{H_RM_R\} - \mathrm{Tr}\{H_IM_I\}$$

Starts to play a role at $\mathrm{N^3LO}$ in the 4^{th} tower

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Scale dependence at NNLO

Perturbative expansion of the partonic cross section

$$\omega = \sum_{k=0}^{\infty} \alpha_s^{k+2}(\mu_R)\, \omega^{(k)}(\mu_R)$$

Truncation at some fixed $k = n$, the renormalization scale uncertainty is of the order $\mathcal{O}(\alpha_s(\mu_R)^{n+3})$

At NNLO, one finds

$$\begin{aligned}
\omega|_{\text{NNLO}} &= \alpha_s^2(\mu_R)\, \omega^{(0)} + \alpha_s^3(\mu_R)\, \omega^{(1)}(\mu_R) + \alpha_s^4(\mu_R)\, \omega^{(2)}(\mu_R) \\
&= \alpha_s^2(\mu_R)\, \omega^{(0)} + \alpha_s^3(\mu_R) \left(\omega'^{(1)} + 2b_0\, L\, \omega^{(0)}\right) \\
&+ \alpha_s^4(\mu_R) \left(\omega'^{(2)} + 3b_0\, L\, \omega'^{(1)} + (3b_0^2\, L^2 + 2b_1\, L)\, \omega^{(0)}\right)
\end{aligned}$$

where $L \equiv \ln(\mu_R^2/\hat{m}^2)$

$\ln^2 \bar{N},\ \ln \bar{N},\ 1$

3^{rd}	4^{th}	5^{th} tower
✓	✓	✗

Glover `03

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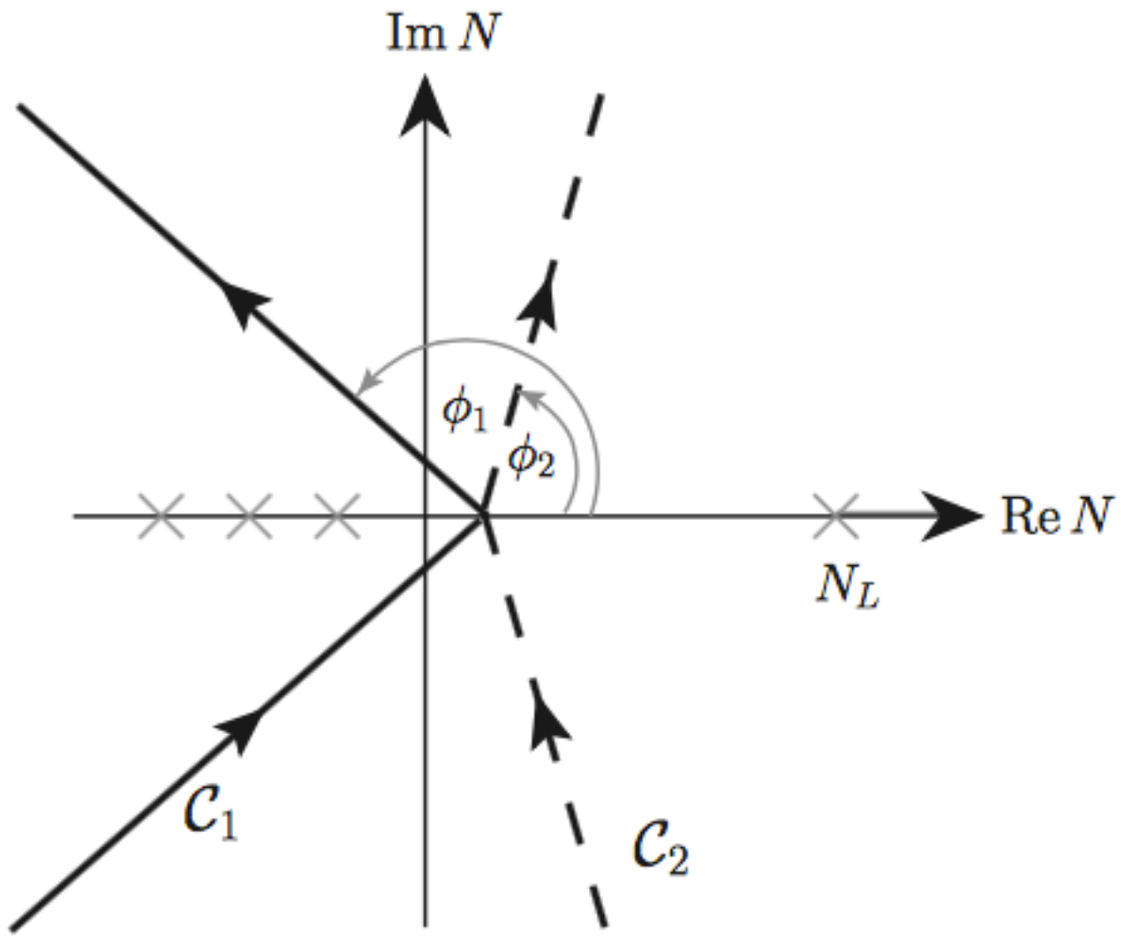

Figure taken from *de Florian, Pfeuffer, Schäfer, Vogelsang '14*