What are the Low-Q and Large-\textit{x} Boundaries of Collinear QCD Factorization Theorems?

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Introduction

- QCD complexities
  - Non-Abelian
  - Confinement
- Can only be solved analytically in the simplest of cases.
- Use Factorization theorems to simplify the calculation.
Introduction

- **Factorization:**
  - Method of disentangling the physics at different space-time scales by taking the asymptotically large limit of some physical energy

- **Useful in QCD:**
  - Asymptotic freedom allows short-distance processes to be calculated using perturbative calculations
    - Factorize to separate perturbative part from non-perturbative part
Introduction

- Example: Collinear Factorization in Deeply Inelastic Scattering (DIS)
  - Assume that $Q \gg m$ where $Q = \sqrt{-q^2}$ and $m$ is a generic mass scale on the order of a hadron mass.
Introduction

- Want to explore physics at lower $Q$ (~ a few GeV) and larger $x_{bj} (\gtrsim 0.5)$
  - Interplay of perturbative and nonperturbative
- For example DIS at moderately low momentum transfer ($Q \sim 1 - 2$ GeV)
  - $Q \gg m$ is not an accurate assumption
  - But $\alpha_s/\pi \lesssim 0.1$ so can still use perturbative calculations.
Introduction

- Proposed techniques for extending QCD factorization to lower energies and/or larger $x_{bj}$:
  - Target mass corrections (Georgi and Politzer, 1976)
  - Large Bjorken-x corrections from re-summation (Sterman, 1987)
  - Higher twist operators (Jaffe and Soldate, 1982)

- Questions arise:
  - Which method would give the most accurate approximation?
  - Are there other corrections that should be included?
Introduction

- What can we do to test how effective these techniques really are?
  - Problem: Non-Abelian nature of QCD leaves “blobs” that cannot be calculated without making approximations.

- There is no reason these techniques can only be applied to QCD.
- They should work for most re-normalizable Quantum Field Theories (QFT).
Introduction

- Use a simple QFT that requires no approximations
  - Perform an exact calculation in this QFT
  - Perform the same calculation after applying a factorization theorem to the QFT
  - Compare results numerically
Simple Model Definition

- Interaction Lagrangian Density:
  \[ \mathcal{L}_{\text{int}} = -\lambda \bar{\Psi}_N \psi_q \phi + \text{h.c.} \]
  - \( \bar{\Psi}_N \): Spin-1/2 “Nucleon” Field with mass \( M \)
  - \( \psi_q \): Spin-1/2 “Quark” Field with mass \( m_q \)
  - \( \phi \): Scalar “Diquark” Field with mass \( m_s \)
  - The nucleon and quark couple to photon while the scalar does not.
Standard Notation in Inclusive DIS

- Inclusive DIS process
- \( e(l) + N(P) \rightarrow e(l') + X(p_x) \)
  - \( l \) and \( l' \) are the initial and final lepton four-momenta
  - \( P \) is the four-momentum of the nucleon
  - \( p_x = p_q + p_s \) is the four-momentum of the inclusive hadronic state

![Diagram of QCD event](image.png)
Standard Notation in Inclusive DIS

- Using Breit frame where
  - Nucleon momentum in +z direction
  - Photon momentum in -z direction
- Using light-front coordinates
  - Four-vector:
    \[ v^\mu = (v^+, v^-, v_T) \]
  - “±” components:
    \[ v^\pm = (v^0 \pm v^z)/\sqrt{2} \]
  - Transverse component:
    \[ v_T \]
Standard Notation in Inclusive DIS

- **Momenta**
  - Nucleon
    \[ P = \left( \frac{Q}{x_n \sqrt{2}}, \frac{x_n M^2}{Q \sqrt{2}}, 0_T \right) \]
  - Photon
    \[ q = l - l' \quad q = \left( -\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, 0_T \right) \]
  - Internal Parton
    \[ k = (k^+, k^-, k_T) \]
  - Final Parton
    \[ k + q \]

- **Where**

  \[ Q \equiv \sqrt{-q^2} \]

  Nachtmann \( x \)

  \[ x_n \equiv -\frac{q^+}{P^+} = \frac{2x_{bj}}{1 + \sqrt{1 + 4x_{bj}^2 M^2/Q^2}} \]

  Bjorken \( x \)

  \[ x_{bj} = \frac{Q^2}{2P \cdot q} \]
The DIS cross section can be written as

\[ E' \frac{d\sigma}{d^3\ell'} = \frac{\alpha^2}{2\pi(s-M^2)Q^4} L_{\mu\nu} W^{\mu\nu} \]

Where

- \( \alpha \) is the electromagnetic fine structure constant
- \( L_{\mu\nu} \) is the leptonic tensor given by
  \[ L_{\mu\nu} = 2 (\ell_\mu \ell'_\nu + \ell'_\mu \ell_\nu - g_{\mu\nu} \ell \cdot \ell') \]
- \( W^{\mu\nu} \) is the hadronic tensor, which in terms of structure functions \( F_1 \) and \( F_2 \) is given by

\[ W^{\mu\nu}(P, q) = \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) F_1(x_n, Q^2) + \left( P^\mu - q^\mu \frac{P \cdot q}{q^2} \right) \left( P^\nu - q^\nu \frac{P \cdot q}{q^2} \right) \frac{F_2(x_n, Q^2)}{P \cdot q} \]
Define Projection Tensors for the Structure Functions

\[ P^{\mu\nu}_1 W_{\mu\nu}(P, q) = F_1(x_n, Q^2), \quad P^{\mu\nu}_2 W_{\mu\nu}(P, q) = F_2(x_n, Q^2) \]

\[ P^{\mu\nu}_1 = -\frac{1}{2} P^{\mu\nu}_g + \frac{2Q^2 x_n^2}{(M_H x_n^2 + Q^2)^2} P^{\mu\nu}_{PP}, \]

\[ P^{\mu\nu}_2 = \frac{12Q^4 x_n^3 (Q^2 - M_H^2 x_n^2)}{(Q^2 + M_H x_n^2)^4} \left( P^{\mu\nu}_{PP} + \frac{(M_H^2 x_n^2 + Q^2)^2}{12Q^2 x_n^2} P^{\mu\nu}_g \right) \]

Where

\[ P^{\mu\nu}_g = g^{\mu\nu}, \quad P^{\mu\nu}_{PP} = P^\mu P^\nu. \]
Exact Kinematics

- Familiar DIS Handbag Diagram
For electromagnetic gauge invariance these diagrams must also be included.
Exact Kinematics

To demonstrate the calculations it is convenient to organize the hadronic tensor by separating the integrand into factors as follows:

\[ W^{\mu\nu}(P,q) = \sum_{j \in \text{graphs}} \int \frac{dk^+ dk^- d^2k_T}{(2\pi)^4} [\text{Jac}] T_j^{\mu\nu} [\text{Prop}]_j \delta(k^- - k_{sol}^-) \delta(k^+ - k_{sol}^+) \]

Where

- \( j \) refers to Figures A, B, and C
- \([\text{Prop}]_j\) is the denominators of the internal propagators in Figure \( j \)
- \( T_j^{\mu\nu} \) is the appropriate Dirac trace for Figure \( j \)
- \([\text{Jac}]\) is the appropriate jacobian factor to isolate \( k^- \) and \( k^+ \) in the arguments of the delta functions
Exact Kinematics

- The arguments of the delta functions give the quadratic system

\[(q + k)^2 - m_q^2 = 0,\]
\[(P - k)^2 - m_s^2 = 0.\]

- Solving this system for \(k^+ \equiv \xi P^+\) and \(k^-\) yields two solutions for \(k^-\).

- Only one solution is physically realistic (0 if \(Q\) is taken to infinity).

- The correct solution to the system is

\[k^- = k^\text{sol}^- = \frac{\sqrt{\Delta} - Q^2(1 - x_n) - x_n (m_s^2 - m_q^2 - M^2(1 - x_n))}{2 \sqrt{2} Q (1 - x_n)},\]
\[k^+ = k^\text{sol}^+ = \frac{k_T^2 + m_q^2 + Q(Q + \sqrt{2} k^-)}{\sqrt{2}(Q + \sqrt{2} k^-)},\]

- Where \(\Delta = [Q^2(1 - x_n) - x_n (M^2(1 - x_n) + m_q^2 - m_s^2)]^2\)
\[- 4x_n(1 - x_n)[k_T^2(Q^2 + x_n M^2) - Q^2M^2(1 - x_n) + Q^2 m_s^2 + x_n M^2 m_q^2]\]
Exact Kinematics

- The Jacobian factor is:

\[
[\text{Jac}] = \frac{x_nQ (2k^- + \sqrt{2}Q) - (k_1^2 + m_1^2)x_n(Q^2 + x_n M^2)}{4(1 - x_n)k^- Q^2(\sqrt{2}k^- + 2Q) + 2\sqrt{2}[Q^4(1 - x_n) - (k_1^2 + m_1^2)x_n(Q^2 + x_n M^2)]}
\]

- The propagator factors are:

\[
[\text{Prop}]_A = \frac{1}{(k^2 - m_0^2)^2},
\]

\[
[\text{Prop}]_B = \frac{1}{((P + q)^2 - M^2)^2} = \frac{x_n^2}{(Q^2(1 - x_n) - M^2 x_n^2)^2},
\]

\[
[\text{Prop}]_C = \frac{x_n}{(k^2 - m_0^2)(Q^2(1 - x_n) - M^2 x_n^2)}.
\]
Exact Kinematics

- The Dirac traces are:

\[ T_A^{\mu\nu} = \text{Tr} \left[ (\not{P} + M)(\not{k} + m_q)\gamma^\mu(\not{k} + \not{q} + m_q)\gamma^\nu(\not{k} + m_q) \right], \]

\[ T_B^{\mu\nu} = \text{Tr} \left[ (\not{P} + M)\gamma^\mu(\not{P} + \not{q} + M)(\not{k} + \not{q} + m_q)(\not{P} + \not{q} + M)\gamma^\nu \right], \]

\[ T_C^{\mu\nu} = 2 \text{Tr} \left[ (\not{P} + M)(\not{k} + m_q)\gamma^\mu(\not{k} + \not{q} + m_q)(\not{P} + \not{q} + M)\gamma^\nu \right], \]

- Factor of 2 is for the Hermitian conjugate of Figure C.

- Define the projected quantities:

\[ T_j^g = P^\mu^\nu \ T_{j\mu\nu}, \quad T_j^{PP} = P^\mu^\nu \ P_{PP} \ T_{j\mu\nu} \]
The $p_{\mu}^\nu$ projections with traces evaluated are:

$$ T_A^g = -8 \left[ 2(P \cdot k + m_q M) k \cdot q + (k^2 - 3m_q^2) P \cdot k - 2Mm_q^3 + (m_q^2 - k^2) P \cdot q \right], $$

$$ T_B^g = 8 \left[ 2M^3m_q + P \cdot k (2M^2 - Q^2) - 2(M^2 + Mm_q) Q^2 + 2k \cdot q (M^2 - P \cdot q) + [2(M^2 + Mm_q) + Q^2] P \cdot q \right], $$

$$ T_C^g = -16 \left[ -2(P \cdot k)^2 + k^2 M^2 + (M^2 - m_q M) k \cdot q - M^2m_q^2 + 2Mm_q Q^2 + (m_q^2 - Mm_q) P \cdot q - 2P \cdot k (k \cdot q + Mm_q - Q^2 + P \cdot q) \right], $$
The $P_{PP}^{\mu\nu}$ projections with traces evaluated are:

\[
T_{A}^{PP} = 4 \left[ 4(P \cdot k)^3 + 4(P \cdot k)^2(Mm_q + P \cdot q) \\
- M P \cdot k (3k^2M + 2M k \cdot q - 3Mm_q^2 - 4m_q P \cdot q) \\
- M^3m_q(k^2 + 2k \cdot q - m_q^2) - M^2(k^2 - m_q^2) P \cdot q \right],
\]

\[
T_{B}^{PP} = 4M^2 \left[ P \cdot k (4M^2 + Q^2) + 4M^2(k \cdot q + Mm_q) - Q^2(4M^2 + Mm_q) \\
+ [2k \cdot q + 4(M^2 + Mm_q) - Q^2] P \cdot q \right],
\]

\[
T_{C}^{PP} = 8M \left[ 4M(P \cdot k)^2 + MP \cdot k (2k \cdot q + 4Mm_q - Q^2) \\
- M^2[2M(k^2 + k \cdot q - m_q^2) + m_qQ^2] \\
- [k^2M - (2M + m_q)(2P \cdot k + Mm_q)] P \cdot q \right].
\]
Define the nucleon structure functions as:

\[
F_1 \left( x_n, Q^2 \right) = \int \frac{d^2k_T}{(2\pi)^2} \mathcal{F}_1(x_n, Q^2, k_T^2),
\]

\[
F_2 \left( x_n, Q^2 \right) = \int \frac{d^2k_T}{(2\pi)^2} 2x_n \mathcal{F}_2(x_n, Q^2, k_T^2)
\]

Where

\[
\mathcal{F}_1 \left( x_n, Q^2, k_T^2 \right) = \frac{1}{(2\pi)^2} [\text{Jac}] \sum_j \left( \frac{1}{2} T^q_j + \frac{2Q^2 x_n^2}{(M^2 x_n^2 + Q^2)^2} T^{PP}_j \right) [\text{Prop}, j],
\]

\[
2x_n \mathcal{F}_2 \left( x_n, Q^2, k_T^2 \right) = \frac{1}{(2\pi)^2} \frac{12 Q^4 x_n^3 (Q^2 - M^2 x_n^2)}{(Q^2 + M^2 x_n^2)^4}
\]

\[
\times [\text{Jac}] \sum_j \left( T_j^{PP} - \frac{(M^2 x_n^2 + Q^2)^2}{12 Q^2 x_n^2} T^q_j \right) [\text{Prop}, j].
\]
Exact Kinematics

- The exact kinematics impose an upper bound on $k_T$.
- Start from calculation of $W$ in the center-of-mass frame:

$$W = p_q^0 + p_s^0 \bigg|_{\text{c.m.}} = \sqrt{m_q^2 + k_T^2 + k_z^2} + \sqrt{m_s^2 + k_T^2 + k_z^2} \bigg|_{\text{c.m.}}$$

- $W$ in the Breit frame:

$$W^2 = (P + q)^2 = (p_q + p_s)^2 = M^2 + \frac{Q^2(1 - x_{bji})}{x_{bji}}$$

- Set the two equations for $W$ equal to each other, and solve for $k_T$ with $k_z = 0$

$$k_{T\text{max}} = \sqrt{\frac{x_{bji}(M^2 - (m_q + m_s)^2) + Q^2(1 - x_{bji})}{4x_{bji}[Q^2(1 - x_{bji}) + M^2x_{bji}]}}$$
Collinear Factorization

- Un-approximated hadronic tensor

\[ W^{\mu\nu}(P, q) = \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ H^\mu(k, k') J(k') H^{\nu\dagger}(k, k') L(k, P) \right] \]
Collinear Factorization

- **Factorized Hadronic Tensor**

\[
W^{\mu\nu}(P, q) = \frac{1}{2Q^2} \frac{1}{\mathcal{H}^{\mu\nu}(Q^2)} \left[ \mathcal{H}^{\mu}(Q^2) \mathcal{H}^{\nu}(Q^2) \right] \left( \int \frac{dk^-d^3k_T}{(2\pi)^3} \frac{\gamma^+}{2} L(\hat{k}, P) \right) + O\left(\frac{m^2}{Q^2}\right) W^{\mu\nu}.
\]

- Where
  - \(\hat{k} \equiv (x_b P^+, 0, 0)\), \(\hat{k}' = \hat{k} + q\), and \(\tilde{k} \equiv (\hat{k}^+, k^-, k_T)\)
Collinear Factorization

- Factorization of the simple QFT
  - In the exact calculation, we had to consider these diagrams

- At the large $Q$ limit, Figures B and C are suppressed by powers of $m/Q$
- Only need to factorize Figure A
Collinear Factorization

For a specific structure function

\[ F_i(x_b, Q^2) = \mathcal{H}_i(Q^2) f(x_b) + O\left(\frac{m^2}{Q^2}\right), \quad i = 1, 2, \]

Where

\[ \mathcal{H}_i(Q^2) \equiv P_i^{\mu\nu} \frac{1}{2Q^2} \text{Tr} \left[ H_\nu(Q^2) k^i H^i_\mu(Q^2) \right] \]
**Collinear Factorization**

- The hard functions are
  \[ H(Q^2)\mu = \gamma^\mu, \quad H^\dagger(Q^2)\nu = \gamma^\nu \]

- The projected hard functions are
  \[ H_1(Q^2) = 1, \]
  \[ H_2(Q^2) = \frac{2Q^2 x_{bj} (Q^2 - M^2 x_{bj}^2)}{(Q^2 + M^2 x_{bj}^2)^2} \]
  \[ = 2 x_{bj} \left( 1 + O \left( \frac{M^2 x_{bj}^2}{Q^2} \right) \right) \]
Collinear Factorization

The lower part is given by:

\[ f(x_{bj}) = \int \frac{dk^-d^2k_T}{(2\pi)^3} \left( \frac{1}{k^2 - m_q^2} \right)^2 \Tr \left[ \frac{\gamma^+}{2} (\not{k} + m_q)(\not{P} + M)(\not{k} + m_q) \right] \times (2\pi) \delta_+ \left( (P - \not{k})^2 - m_s^2 \right). \]

Integrating over \( k^- \) yields:

\[ k^- = -\frac{x_{bj} [k_T^2 + m_s^2 + (x_{bj} - 1)M^2]}{\sqrt{2Q(1 - x_{bj})}} \]

The parton virtuality is:

\[ \tilde{k}^2 = -\frac{k_T^2 + x_{bj} [m_s^2 + (x_{bj} - 1)M^2]}{1 - x_{bj}} \]

The \( k_T \)-unintegrated functions \( \mathcal{F}_{1,2} \) (equivalent to what was defined in the exact case are):

\[ \mathcal{F}_1(x_{bj}, Q^2, k_T^2) = \mathcal{F}_2(x_{bj}, Q^2, k_T^2) = \frac{1}{(2\pi)^2} \frac{(1 - x_{bj}) [k_T^2 + (m_q + x_{bj}M)^2]}{[k_T^2 + x_{bj}m_s^2 + (1 - x_{bj}) m_q^2 + x_{bj}(x_{bj} - 1)M^2]^2} \]
Collinear Factorization

- Expanding exact solutions in powers of $1/Q$

\[
\xi = x_{b1} \left[ 1 + \frac{k_T^2 + m_a^2 - x_{b1}^2 M^2}{Q^2} \right. \\
\left. - \frac{x_{b1}^2 M^2 (k_T^2 + m_a^2) + x_{b1} (k_T^2 + m_a^2) (k_T^2 + m_a^2 - M^2) - 2M^4 x_{b1} (x_{b1} - 1)}{Q^4 (x_{b1} - 1)} \right] + O \left( \frac{m^6}{Q^6} \right),
\]

\[
k_1^- = - \frac{x_n}{Q \sqrt{2}} \left[ \frac{k_T^2 + m_a^2 + (x_n - 1) M^2}{1 - x_n} - \frac{x_n (k_T^2 + m_a^2) (k_T^2 + m_a^2)}{Q^4 (x_n - 1)^2} \right] + O \left( \frac{m \cdot m^6}{Q^6} \right),
\]

\[
k^2 = - \frac{k_T^2 + x_n [m_a^2 + (x_n - 1) M^2]}{1 - x_n} \left[ \frac{k_T^2 + m_a^2 + (x_n - 1) M [m_a - (x_n - 1) M]}{Q^4 (x_n - 1)^2} \right] + O \left( \frac{m^2 \cdot m^4}{Q^4} \right).
\]
Comparison Between the Exact Calculation and the Standard Approximation

- Want to choose a set of masses that mimics QCD
  - For $M$, use the proton mass (0.938 GeV)
  - Choose values of $m_q$ and $m_s$ such that $|k|$ is on the order of a few MeV and the $k_T$ distribution peaks at not more than 300 MeV
    - $m_q$ should be on the order of a few MeV
    - $m_s$ is chosen on a case by case basis:
      - In QCD, the remnant mass would grow with $Q$. The mass used here should behave similarly.
      - The mass in the quark-diquark rest frame is constrained
        \[ M - m_q < m_s \leq W(x_{bj}, Q) - m_q \]
  - Solve $v \equiv \sqrt{-k^2}$ at $k_T = 0$ for $m_s$. 
Comparison Between the Exact Calculation and the Standard Approximation

- Plots of exact and approximate $k_T F_1$ for $x_{bj} = 0.6$
Comparison Between the Exact Calculation and the Standard Approximation

- Plot $v \equiv \sqrt{-k^2}$ vs. $k_T$ ($x_{bj} = 0.6$, $m_q = 0.3$ GeV, and $m_s$ corresponding to $v(k_T = 0) = 0.5$ GeV)
Comparison Between the Exact Calculation and the Standard Approximation

- **Integrated Structure Functions**
  - **Exact:**
    \[
    I(x_{bj}, Q) = \int_0^{k_{T_{max}}} dk_T \; k_T \; \mathcal{F}_1^{\text{exact}}(x_{bj}, Q, k_T)
    \]
  - **Approximate:**
    \[
    \hat{I}(x_{bj}, Q, k_{\text{cut}}) = \int_0^{k_{\text{cut}}} dk_T \; k_T \; \mathcal{F}_1^{\text{approx}}(x_{bj}, Q, k_T)
    \]

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<th>(Q = 20\ \text{GeV})</th>
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<td>0.67 0.45 0.49 0.35</td>
<td>0.90 0.88 0.86 0.85</td>
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Our analysis provides a means of clearly defining purely kinematic TMCs.
Expand exact solutions in powers of $m/Q$, but keep only powers of $M/Q$ (assume powers of $k_T/Q$, $m_q/Q$, and $m_s/Q$ are still negligible):

\[
\xi \rightarrow \xi_{\text{TMC}} = x_{bj} \left[ 1 - \frac{x_{bj}^2 M^2}{Q^2} + \frac{2M^4 x_{bj}^4}{Q^4} + \cdots \right] = x_n
\]

\[
k^- \rightarrow k_{\text{TMC}}^{-} = -\frac{x_n \left[ k_T^2 + m_s^2 + (x_n - 1)M^2 \right]}{\sqrt{2Q(1-x_n)}}
\]

\[
k^2 \rightarrow k_{\text{TMC}}^{2} = -\frac{k_T^2 + x_n \left[ m_s^2 + (x_n - 1)M^2 \right]}{1-x_n}
\]

This is equivalent to inserting $x_n$ in place of $x_{bj}$ in the collinear factorized equations for these quantities.

Define purely kinematic TMCs as those corrections obtained from this substitution.
Purely Kinematic TMCs

- Plots of $k_T\mathcal{F}_1$ (exact, approximate, and approximate with $x_{bj} \rightarrow x_n$) ($x_{bj} = 0.6$, $m_q = 0.3$ GeV, and $m_s$ corresponding to $\nu(k_T = 0) = 0.5$ GeV)
Summary of Findings

- Analysis using the simple QFT demonstrates that the most accurate QCD factorization theorem for low-$Q$ and large-$x_{bj}$ would need to account for corrections due to parton mass, parton transverse momentum, and parton virtuality as well as the target mass.

- This type of analysis using a simple QFT can be used as a testing ground for any factorization theorem.

- From this analysis, we can define purely kinematical TMCs as corrections that result from substituting $x_n$ in place of $x_{bj}$ in the collinear factorized formula.