Connecting Different TMD Factorization Formalisms in QCD

Old Dominion University and Jefferson Laboratory

Ted Rogers

• Based on:

Map of Semi-inclusive DIS

Transverse Momentum

Rapidity

Talk by O. Gonzalez

Talk by B. Wang

High $P_T$ Scattering:
Collinear Factorization

TMD Factorization
(with a Fragmentation Function)

Soft Region

Target Region
(Fracture Function)
TMD Factorization

• Non-perturbative transverse momentum
  – Relationship to hadron structure, spin, power corrections etc...
    • Important to have explicit TMD definitions
    • Extract from data and compare across processes to test universality
    • Predict with non-perturbative techniques: lattice, etc...

• Issues in semi-inclusive deeply inelastic scattering
  – Smallish $Q$ is typical
  – Limited range of rapidity and transverse momentum at small $Q$ and/or large $x$
  – Nature of power corrections from hadron masses?

Talks by E. Moffat and W. Melnitchouk
**Phenomenology and Partons**

*Talk by C. Pisano*

Cross Section \( \approx H_f \int d^2k_T F_{f/p}(x, k_T - q_T) D_{h/f}(z, zk_T) \)

*(Schweitzer, Strikman, Weiss, (2013))*

*(Signori, Bacchetta, Radici, Schnell (2013))*

Flavor / Sea
Non-perturbative transverse momentum
- Relationship to hadron structure, spin, power corrections etc...
  - Important to have explicit TMD definitions
  - Extract from data and compare across processes to test universality
  - Predict with non-perturbative techniques: lattice, etc...

Issues in semi-inclusive deeply inelastic scattering
- Smallish Q is typical
- Match to central & target regions
- Limited range of rapidity and transverse momentum at small Q and/or large x
- Nature of power corrections from hadron masses?

*Talks by E. Moffat and J. Guerrero*
TMD Factorization

• Many results exist.
  – Resummation in collinear factorization
  – CSS
  – SCET
  – Sudakov Factors

• Formalisms often appear different on the surface.

• Goals:
  – Map old styles to new.
  – Is there convergence toward a standardized set of definitions (and results) for TMD definitions?
  – Bring diverse results together with consistent conventions (notation, etc)?
Older Formalisms...

- **CSS1** - Multiple redefinitions of factors (starting from TMD definitions) No explicit hard part.
  
  \((\text{Collins, Soper, Sterman (1981-1985)})\)

  - Match to collinear for \(\Lambda_{\text{QCD}} \ll q_T \ll Q\) and \(q_T \approx Q\).

- **Catani, de Florian, Grazzini et al.**

  - Factorization takes a simple form.

  - Large transverse momentum (e.g., \(Y\)-term) results are automatic.

  - Ressummation scheme dependence; shown there is no uniquely defined hard part.
    \((\text{Catani, de Florian, Grazzini (2001)})\)
Drell-Yan

\[
\frac{d\sigma}{dQ^2 dy dq_T^2} = \frac{4\pi^2 \alpha^2}{9Q^2 s} \sum_{j,j_A,j_B} e_j^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \times \left[ \int_{x_A}^1 \frac{d\xi_A}{\xi_A} f_{j_A/A}(\xi_A; \mu_{b*}) \bar{C}_{j/j_A}^{\text{CSS1}, \text{DY}} \left( \frac{x_A}{\xi_A}, b_*; \mu_{b*}^2, \mu_{b*}, C_2, a_s(\mu_{b*}) \right) \right. \\
\times \left. \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{j_B/B}(\xi_B; \mu_{b*}) \bar{C}_{j/j_B}^{\text{CSS1}, \text{DY}} \left( \frac{x_B}{\xi_B}, b_*; \mu_{b*}^2, \mu_{b*}, C_2, a_s(\mu_{b*}) \right) \right] \\
\times \exp \left\{ -\int_{\mu_{b_*}^2}^{\mu^2} \frac{d\mu'}{\mu'} \left[ A_{\text{CSS1}}(a_s(\mu'); C_1) \ln \left( \frac{\mu^2}{\mu'^2} \right) + B_{\text{CSS1}, \text{DY}}(a_s(\mu'); C_1, C_2) \right] \right\} \\
\times \exp \left[ -g_{j_A}^{\text{CSS1}}(x_A, b_T; b_{\text{max}}) - g_{j_B}^{\text{CSS1}}(x_B, b_T; b_{\text{max}}) - g_K^{\text{CSS1}}(b_T; b_{\text{max}}) \ln(Q^2/Q_0^2) \right] \\
+ \text{suppressed corrections.}
\]

\[ \mu_Q \equiv C_2 Q \]
\[ \mu_b \equiv C_1/b_T \]
\[ \mu_{b*} \equiv C_1/b_* \]

No explicit hard part here
Drell-Yan

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\frac{d\sigma}{dQ^2 dy dq_T^2} = \frac{4\pi^2 \alpha^2}{9Q^2 s} \sum_{j_A,j_B} e_j^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{iq_T \cdot b_T} \times \\
\times \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} f_{j_A/A}(\xi_A; \mu_{b*}) C_{j_j/A}^{\text{CSS1},\text{DY}} \left( \frac{x_A}{\xi_A}, b_*; \mu_{b_*}^2, \mu_{b*}, C_2, a_s(\mu_{b*}) \right) \\
\times \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} f_{j_B/B}(\xi_B; \mu_{b*}) C_{j_B/j_B}^{\text{CSS1},\text{DY}} \left( \frac{x_B}{\xi_B}, b_*; \mu_{b_*}^2, \mu_{b*}, C_2, a_s(\mu_{b*}) \right) \\
\times \exp \left\{ -\int_{\mu_{b*}^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left[ A_{\text{CSS1}}(a_s(\mu'); C_1) \ln \left( \frac{\mu^2}{\mu'^2} \right) + B_{\text{CSS1},\text{DY}}(a_s(\mu'); C_1, C_2) \right] \right\} \\
\times \exp \left[ -g_{j_A}^{\text{CSS1}}(x_A, b_T; b_{\text{max}}) - g_{j_B}^{\text{CSS1}}(x_B, b_T; b_{\text{max}}) - g_K^{\text{CSS1}}(b_T; b_{\text{max}}) \ln(Q^2/Q_0^2) \right] \\
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Drell-Yan

\[
\frac{d\sigma}{dQ^2 \, dy \, dq_T^2} = \frac{4\pi^2 \alpha^2}{9Q^2s} \sum_{j,j_A,j_B} e_j^2 \int \frac{d^2b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \\
\times \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} f_{j_A/A}(\xi_A; \mu_{b*}) \, \tilde{C}_{j/j_A}^{CSS1, \, DY} \left( \frac{x_A}{\xi_A}, b_*; \mu_{b*}^2, \mu_{b*}^*, C_2, a_s(\mu_{b*}) \right) \\
\times \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} f_{j_B/B}(\xi_B; \mu_{b*}) \, \tilde{C}_{j/j_B}^{CSS1, \, DY} \left( \frac{x_B}{\xi_B}, b_*; \mu_{b*}^2, \mu_{b*}^*, C_2, a_s(\mu_{b*}) \right) \\
\times \exp \left\{ -\int_{\mu_{b*}^2}^{\mu_{b*}^2} \frac{d\mu'^2}{\mu'^2} \left[ A_{CSS1}(a_s(\mu'); C_1) \ln \left( \frac{\mu_Q^2}{\mu'^2} \right) + B_{CSS1, \, DY}(a_s(\mu'); C_1, C_2) \right] \right\} \\
\times \exp \left[ -g_{j_A/A}^{CSS1}(x_A, b_T; b_{\text{max}}) - g_{j_B/B}^{CSS1}(x_B, b_T; b_{\text{max}}) - g_{K}^{CSS1}(b_T; b_{\text{max}}) \ln(Q^2/Q_0^2) \right] \\
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No explicit hard part here
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\[
\frac{d\sigma}{dQ^2 \, dy \, dq_T^2} = \frac{4\pi^2 \alpha^2}{9Q^2 s} \sum_{j, j_A, j_B} e_j^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \\
	imes \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} f_{j_A/A} (\xi_A; \mu_{b*}) \tilde{C}^{\text{CSS1}, \, \text{DY}}_{j/j_A} \left( x_A, b_*/\mu_{b*}^2, \mu_{b*}^2, C_2, a_s(\mu_{b*}) \right) \\
	imes \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} f_{j_B/B} (\xi_B; \mu_{b*}) \tilde{C}^{\text{CSS1}, \, \text{DY}}_{j/j_B} \left( x_B, b_*/\mu_{b*}^2, \mu_{b*}^2, C_2, a_s(\mu_{b*}) \right) \\
\times \exp \left\{ -\int_{\mu_{b*}^2}^{\mu_Q^2} \frac{d\mu'}{\mu'^2} \left[ A_{\text{CSS1}} (a_s(\mu'); C_1) \ln \left( \frac{\mu_Q^2}{\mu'^2} \right) + B_{\text{CSS1}, \, \text{DY}} (a_s(\mu'); C_1, C_2) \right] \right\} \\
\times \exp \left[ -g_{j_A/A}^{\text{CSS1}} (x_A, b_T; b_{\text{max}}) - g_{j_B/B}^{\text{CSS1}} (x_B, b_T; b_{\text{max}}) - g_K^{\text{CSS1}} (b_T; b_{\text{max}}) \ln(Q^2/Q_0^2) \right] \\
+ \text{suppressed corrections.}
\]

\[\mu_Q \equiv C_2 Q, \quad \mu_b \equiv C_1/b_T, \quad \mu_{b*} \equiv C_1/b_*\]

No explicit hard part here
• Replacement

\[ b_*(b_T) \rightarrow \begin{cases} 
  b_T & b_T \ll b_{\text{max}} \\
  b_{\text{max}} & b_T \gg b_{\text{max}}
\end{cases} \]

• One commonly used functional form.

\[ b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{\text{max}}^2}} \]

\[ \mu_Q \equiv C_2 Q \]
\[ \mu_b \equiv C_1/b_T \]
\[ \mu_{b_*} \equiv C_1/b_* \]
New TMD methods

• Improved TMD function definitions (e.g., CSS2).
  \( (J. \text{ Collins textbook, (2011))} \)
  
  – SCET-based approaches
    • Main differences from CSS2: Implementation of regulators.
    • At least two are equivalent to CSS2
      \( (\text{Echevarria, Idilbi, Scimemi (2012); Collins, TCR (2013)} \) )
      \( (\text{See talks by V. Vaidya and Esp. I. Scimemi}) \)
  
  – Better oriented for hadron structure studies (e.g. lattice QCD)
  
  – Structurally familiar from phenomenology.
    \[
    H_f \int d^2k_T F_{f/p}(x, k_T - q_T) D_{h/f}(z, zk_T) \]
  
  – Hard parts are fixed by factorization of operator structures.
    \[
    \int d^2k_T F_{f/p}(x, k_T - q_T) D_{h/f}(z, zk_T) = H_f
    \]
New TMD methods

- TMD parton model structure + evolution equations.

**Ex: CSS2**

\[
\frac{d\sigma}{dQ^2 \, dy \, dq_T^2} = \frac{4\pi^2 \alpha^2}{9Q^2 s} \sum_j H_{jj}^{DY}(Q, \mu_Q, a_s(\mu_Q)) \int \frac{d^2b_T}{(2\pi)^2} e^{iq_T \cdot b_T} \tilde{f}_{j/A}(x_A, b_T; Q^2, \mu_Q) \tilde{f}_{j/B}(x_B, b_T; Q^2, \mu_Q)
\]

+ suppressed corrections,

\[
\frac{\partial \ln \tilde{f}(x, b_T; \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu) \quad \tilde{f}_{j/H}(x, b_T; \zeta; \mu) = \sum_k \int_{x \sim 0}^{x \sim 1+} \frac{d\xi}{\xi} \tilde{C}_{j/k}^{PDF}(x/\xi, b_T; \zeta, \mu, a_s(\mu)) f_{k/H}(\xi; \mu) + O((mb_T)^p)
\]

\[
\frac{d\tilde{K}(b_T; \mu)}{d \ln \mu} = -\gamma_K(a_s(\mu))
\]

\[
\frac{d ln \tilde{f}(x, b_T; \mu, \zeta)}{d \ln \mu} = \gamma_j(a_s(\mu)) - \frac{1}{2} \gamma_K(a_s(\mu)) \ln \frac{\zeta}{\mu^2}
\]

\[
\mu_Q \equiv C_2 Q \\
\mu_b \equiv C_1/b_T \\
\mu_{b*} \equiv C_1/b_*
\]
Perturbation-Theory-Optimized Solution

\[
\frac{d\sigma}{dQ^2 \, dy \, dq_T^2} = \frac{4\pi^2 \alpha^2}{9Q^2 s} \sum_{j, JA, JB} H_{jJ}^{DY}(Q, \mu_Q, a_s(\mu_Q)) \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \\
\times e^{-g_{j/A}(x_A, b_T; b_{max})} \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} f_{j/A}(\xi_A; \mu_{b_s}) \tilde{C}_{j/A}^{PDF}(\frac{x_A}{\xi_A}, b_s; \mu_{b_s}^2, \mu_{b_s}, a_s(\mu_{b_s})) \\
\times e^{-g_{\bar{j}/B}(x_B, b_T; b_{max})} \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} f_{\bar{j}/B}(\xi_B; \mu_{b_s}) \tilde{C}_{\bar{j}/B}^{PDF}(\frac{x_B}{\xi_B}, b_s; \mu_{b_s}^2, \mu_{b_s}, a_s(\mu_{b_s})) \\
\times \exp \left\{ -g_K(b_T; b_{max}) \ln \frac{Q^2}{Q_0^2} + \tilde{K}'(b_s; \mu_{b_s}) \ln \frac{Q^2}{\mu_{b_s}^2} + \int_{\mu_{b_s}}^{\mu^{'}} \frac{d\mu'}{\mu'} \left[ 2\gamma_j(a_s(\mu')) - \ln \frac{Q^2}{(\mu^{'})^2} \gamma_K(a_s(\mu')) \right] \right\} \\
+ \text{suppressed corrections.}
\]
Perturbation-Theory-Optimized Solution

\[
\frac{d\sigma}{dQ^2\,dy\,dq_T^2} = \frac{4\pi^2\alpha^2}{9Q^2s} \sum_{j,j_A,j_B} H^{\text{DY}}_{jj}(Q, \mu_Q, a_s(\mu_Q)) \int \frac{d^2b_T}{(2\pi)^2} e^{i \mathbf{q}_T \cdot \mathbf{b}_T} \\
\times e^{-g_{j/A}(x_A,b_T;b_{\text{max}})} \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} f_{j/A}(\xi_A;\mu_{b_*}) \tilde{C}_{V/\pi A}(\frac{x_A}{\xi_A}, b_{*}, \mu_{b_*}, \mu_{b_*}, a_s(\mu_{b_*})) \\
\times e^{-g_{j/B}(x_B,b_T;b_{\text{max}})} \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} f_{j/B}(\xi_B;\mu_{b_*}) \tilde{C}_{V/\pi B}(\frac{x_B}{\xi_B}, b_{*}, \mu_{b_*}, \mu_{b_*}, a_s(\mu_{b_*})) \\
\times \exp \left\{-g_K(b_T;b_{\text{max}}) \ln \frac{Q^2}{Q_0^2} + \tilde{K}(b_{*};\mu_{b_*}) \ln \frac{Q^2}{\mu_{b_*}^2} + \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[ 2\gamma_j(a_s(\mu')) - \ln \frac{Q^2}{(\mu')^2} \gamma_K(a_s(\mu')) \right] \right\}
\]

+ suppressed corrections.
\[
\frac{d\sigma}{dQ^2 \, dy \, dq_T^2} = \frac{4\pi^2 \alpha_s^2}{9Q^2 s} \sum_{j,A,J_B} H_{jj}^{DY}(Q, \mu_Q, a_s(\mu_Q)) \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \\
\times e^{-g_{j/A}(x_A, b_T; b_{\text{max}})} \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} \tilde{f}_{j/A}(\xi_A; \mu_{b_+}) \tilde{C}_{j/A}^{PDF} \left( \frac{x_A}{\xi_A}, b_+; \mu_{b_+}, \mu_{b_*}, a_s(\mu_{b_*}) \right) \\
\times e^{-g_{j/B}(x_B, b_T; b_{\text{max}})} \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} \tilde{f}_{j/B}(\xi_B; \mu_{b_+}) \tilde{C}_{j/B}^{PDF} \left( \frac{x_B}{\xi_B}, b_+; \mu_{b_+}, \mu_{b_*}, a_s(\mu_{b_*}) \right) \\
\times \exp \left\{ -g_K(b_T; b_{\text{max}}) \ln \frac{Q^2}{Q_0^2} + \tilde{K}(b_+; \mu_{b_*}) \ln \frac{Q^2}{\mu_{b_*}^2} + \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[ 2\gamma_j(a_s(\mu')) - \ln \frac{Q^2}{(\mu')^2} \gamma_K(a_s(\mu')) \right] \right\} \\
+ \text{suppressed corrections.}
\]
Perturbation-Theory-Optimized Solution

\[
\frac{d\sigma}{dQ^2 \, dy \, dq_T^2} = \frac{4\pi^2 \alpha^2}{9Q^2 s} \sum_{j, \bar{A}, A, jB} H_{j\bar{j}}^{\text{DY}}(Q, \mu_Q, a_s(\mu_Q)) \int \frac{d^2b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \\
\times e^{-g_{j/A}(x_A, b_T; b_{\text{max}})} \int_{x_A}^1 \frac{d\xi_A}{\xi_A} f_{j/A}(\xi_A; \mu_{b_*}) \tilde{C}_{j/A}^{\text{PDF}} \left( \frac{x_A}{\xi_A}; b_*; \mu_{b_*}^2, \mu_{b_*}, a_s(\mu_{b_*}) \right) \\
\times e^{-g_{j/B}(x_B, b_T; b_{\text{max}})} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{j/B}(\xi_B; \mu_{b_*}) \tilde{C}_{j/B}^{\text{PDF}} \left( \frac{x_B}{\xi_B}; b_*; \mu_{b_*}^2, \mu_{b_*}, a_s(\mu_{b_*}) \right) \\
\times \exp \left\{ -g_K(b_T; b_{\text{max}}) \ln \frac{Q^2}{Q_0^2} + \tilde{K}(b_*; \mu_{b_*}) \ln \frac{Q^2}{\mu_{b_*}^2} + \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[ 2\gamma_j(a_s(\mu')) - \ln \frac{Q^2}{(\mu')^2} \gamma_K(a_s(\mu')) \right] \right\} \\
+ \text{suppressed corrections.}
\]
Compare CSS1 vs. CSS2

\[
\frac{d\sigma}{dQ^2 \, dy \, dq_T^2} = \frac{4\pi^2 \alpha^2}{9Q^2s} \sum_{j_A, j_B} e_j^2 \int \frac{d^2b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \\
\times \int_{x_A}^{1} \frac{d\xi_A}{\xi_A} f_{j_A/A}(\xi_A; \mu_{b*}) \tilde{C}_{j_A/A}^{CSS1, \, DY} \left( \frac{x_A}{\xi_A}, b_\star; \mu_{b_*}^2, \mu_{b_*}, C_2, a_s(\mu_{b_*}) \right) \\
\times \int_{x_B}^{1} \frac{d\xi_B}{\xi_B} f_{j_B/B}(\xi_B; \mu_{b*}) \tilde{C}_{j_B/B}^{CSS1, \, DY} \left( \frac{x_B}{\xi_B}, b_\star; \mu_{b_*}^2, \mu_{b_*}, C_2, a_s(\mu_{b_*}) \right) \\
\times \exp \left\{ - \int_{\mu_{b_*}^2}^{\mu_{b_*}^2} \frac{d\mu_T^2}{\mu_T^2} \left[ A_{CSS1}(a_s(\mu'); C_1) \ln \left( \frac{\mu_{Q}^2}{\mu_T^2} \right) + B_{CSS1, \, DY}(a_s(\mu'); C_1, C_2) \right] \right\} \\
\times \exp \left[ -g_{j_A}^{CSS1}(x_A, b_T; b_{\text{max}}) - g_{j_B}^{CSS1}(x_B, b_T; b_{\text{max}}) - g_K^{CSS1}(b_T; b_{\text{max}}) \ln(Q^2/Q_0^2) \right] \\
\text{+ suppressed corrections.}
\]

\[
\begin{align*}
\mu_Q &\equiv C_2 Q \\
\mu_b &\equiv C_1/b_T \\
\mu_{b*} &\equiv C_1/b_\star
\end{align*}
\]

No explicit hard part
Old Schemes and New Schemes

• Questions:

  – CSS1 involves “A” and “B” functions not explicit in CSS2.
  
  – Non-perturbative parts in CSS1 and in TMD functions?
  
  – Anomalous dimension of PDFs vs. FFs?
  
  – Many high order calculations in old resummation, SCET, etc... how to utilize in, for example, CSS2?
Fast translation to new TMD methods

- CSS1 and CSS2 drop same subleading powers:

\[
\left. \frac{d\sigma}{dQ^2 \, dy \, dq_T^2} \right|_{\text{DY}}^{\text{CSS1}} = \left. \frac{d\sigma}{dQ^2 \, dy \, dq_T^2} \right|_{\text{DY}}^{\text{CSS2}} \; ; \; \left. \frac{d\sigma}{dQ^2 \, dy \, dq_T^2} \right|_{\text{SIDIS}}^{\text{CSS1}} = \left. \frac{d\sigma}{dQ^2 \, dy \, dq_T^2} \right|_{\text{SIDIS}}^{\text{CSS2}}
\]

- Derivatives given by evolution equations. (anomalous dimensions)

- \( b_{\text{max}} \) independence.

- Charge conjugation invariance.
Fast translation to new TMD methods

• First equate \( \ln(Q) \) and \( \ln(b_T) \) derivatives

• Use \( b_{\text{max}} \) independence

\[
A_{\text{CSS1}}(a_s(\mu_{b*}); C_1) = - \frac{d\tilde{K}(b_*; \mu_{b*})}{d \ln b_*^2} + \frac{1}{2} \gamma_K(a_s(\mu_{b*})) = - \frac{\partial \tilde{K}(b_*; \mu)}{\partial \ln b_*^2} \bigg|_{\mu \to \mu_{b*}}
\]

\[
B_{\text{CSS1, DY}}(a_s(\mu_Q); C_1, C_2) = - \tilde{K}(C_1/\mu_Q; \mu_Q) - \frac{\partial \ln H_{jj}^\text{DY}(Q, \mu_Q, a_s(\mu_Q))}{\partial \ln Q^2}
\]

\[
g_K^{\text{CSS1}}(b_T; b_{\text{max}}) = g_K(b_T; b_{\text{max}})
\]
Fast translation to new TMD methods

- Use result and repeat for undifferentiated cross section

\[
e_j^2 \tilde{C}_{\frac{CSS1}{j/A}}^{DY} \left( \frac{x_A}{\xi_A}, b_\ast; \mu_{b_\ast}, \mu_{b_+, C_2, a_s(\mu_{b_+})} \right) \times \tilde{C}_{\frac{CSS1}{j/B}}^{DY} \left( \frac{x_B}{\xi_B}, b_\ast; \mu_{b_\ast}, \mu_{b_+, C_2, a_s(\mu_{b_+})} \right)
\]

\[
= \tilde{C}_{\frac{PDF}{j/A}} \left( \frac{x_A}{\xi_A}, b_\ast; \mu_{b_\ast}, \mu_{b_+, a_s(\mu_{b_+})} \right) \times \tilde{C}_{\frac{PDF}{j/B}} \left( \frac{x_B}{\xi_B}, b_\ast; \mu_{b_\ast}, \mu_{b_+, a_s(\mu_{b_+})} \right) \times H_{\frac{DY}{j j}}^{\mu_{b_+}/C_2, \mu_{b_+}, a_s(\mu_{b_+})} \exp \left[ -2\tilde{K}(b_\ast; \mu_{b_+}) \ln C_2 \right]
\]

\[
g_{j/A}(x_A, b_T; b_{\text{max}}) + g_{j/B}(x_B, b_T; b_{\text{max}}) = g_{\frac{CSS1}{j/A}}(x_A, b_T; b_{\text{max}}) + g_{\frac{CSS1}{j/B}}(x_B, b_T; b_{\text{max}})
\]
Fast translation to new TMD methods

• Charge conjugation invariance:

\[
|e_j|^C_j^{CSS1, DY} \left( \frac{x}{\xi}, b_*; \mu_{b_*}^2, \mu_{b_*}, C_2, a_s(\mu_{b_*}) \right) = C_j^{PDF} \left( \frac{x}{\xi}, b_*; \mu_{b_*}^2, \mu_{b_*}, a_s(\mu_{b_*}) \right) \sqrt{H^{DY}_{jj}(\mu_{b_*}/C_2, \mu_{b_*}, a_s(\mu_{b_*}))} \exp \left[ -\tilde{K}(b_*; \mu_{b_*}) \ln C_2 \right]
\]

\[
g_j/H^{CSS1}(x, b_T; b_{\text{max}}) = g_j/H(x, b_T; b_{\text{max}})
\]

• “Non-perturbative” g-functions are exactly equal in CSS1 and CSS2.
Fast translation to new TMD methods

- Fitted functions are same in new approaches (CSS2 and SCET) as in CSS1.
  - Ex: (Nadolsky, Stump, Yuan (1999), Landry, Brock, Nadolsky, Yuan (2003); Konychev, Nadolsky (2006))

- CSS1 “A,” “B,” “C” fixed by TMD-based expressions.

- TMDs have \( \gamma_j(\mu), \gamma_K(\mu), K(b_T; \mu), H(\alpha_s(\mu); \mu/Q) \)

CSS1 has “A” and “B”
- Need independent information on hard parts and anomalous dimensions
Sudakov Factors and Hard Parts

\[ H_{jj}^{DY}(Q, \mu; a_s(\mu)) = e_j^2 \left| H_{j}^{Sud, \; TL}(Q^2) \right|^2 = e_j^2 \left| H_{j}^{Sud}(-Q^2 - i\epsilon) \right|^2 \]

- TMD factorization hard part determined from Sudakov factor.
- Known for some time:
  (Moch, Vermaseren, Vogt (2005): Three-loop poles)
  (Idilbi, Ji, Yuan (2006))
- Complete three loop result
  (Gehrmann, Glover, Huber, N. Ikizleri, Studerus (2010))
**Sudakov Factors and Hard Parts**

\[ H_{jj}^{DY}(Q, \mu; a_s(\mu)) = e_j^2 \left| H_{j}^{Sud, TL}(Q^2) \right|^2 = e_j^2 \left| H_{j}^{Sud}(-Q^2 - i\epsilon) \right|^2 \]

- Anomalous dimensions of collinear factors: \( \gamma_j \) and \( \gamma_K \)

\[
\frac{d \ln H^{Sud, TL}}{d \ln \mu} = -\gamma_j(a_s(\mu)) - i\frac{\pi}{2} \gamma_K(a_s(\mu)) + \frac{1}{2} \gamma_K(a_s(\mu)) \ln \frac{Q^2}{\mu^2},
\]

\[
\ln F^{Sud, TL}(Q^2) = \ln H^{Sud, TL} + 2 \ln C_j^{bare} + D(a_s, \epsilon) - i\pi E(a_s, \epsilon) + \ln \frac{Q^2}{\mu^2} E(a_s, \epsilon)
\]

- Equality of \( \gamma_j^{PDF} \) and \( \gamma_j^{FF} \) to all orders: TP invariance

*(Collins, TCR (2017), Appendix A)*
Rapidity Evolution Kernels

- Knowledge of B: CSS1 to CSS2, two loops

\[ B_{\text{CSS1, DY}}(a_s(\mu_Q); C_1, C_2) = -\tilde{K}(C_1/\mu_Q; \mu_Q) - \frac{\partial \ln H_{jj}^{\text{DY}}(Q, \mu_Q, a_s(\mu_Q))}{\partial \ln Q^2} \]

*(B calculated in Davies and Stirling (1984))*

- Direct calculation from operators (using SCET): Three loops

*(Li, Neill, Zhu; Li, Zhu (2016, 2017))
(Vladimirov (2017))
Fast translation to new TMD methods

- Charge conjugation invariance:

\[
|e_j| \tilde{C}_{j/k}^{CSS1, DY} \left( \frac{x}{\xi}, b_*; \mu_{b_*}^2, \mu_{b_*}, C_2, a_s(\mu_{b_*}) \right) = \tilde{C}_{j/k}^{PDF} \left( \frac{x}{\xi}, b_*; \mu_{b_*}^2, \mu_{b_*}, a_s(\mu_{b_*}) \right) \sqrt{H_{jj}^{DY}(\mu_{b_*}/C_2, \mu_{b_*}, a_s(\mu_{b_*}))} \exp \left[ -\tilde{K}(b_*; \mu_{b_*}) \ln C_2 \right]
\]

\[
g_{j/H}^{CSS1}(x, b_T; b_{\text{max}}) = g_{j/H}(x, b_T; b_{\text{max}})
\]

- “Non-perturbative” g-functions are exactly equal in CSS1 and CSS2.
Wilson Coefficients

• CSS1 to CSS2: order $\alpha_s^2$

$$\tilde{C}_{j/k}^{\text{PDF}} \left( \frac{x}{\xi}, b_*, \mu^2_{b_*}, \mu_{b_*}, a_s(\mu_{b_*}) \right) = \frac{\tilde{C}_{j/k}^{\text{CSS1, DY}} \left( \frac{x}{\xi}, b_*, \mu^2_{b_*}, \mu_{b_*}, C_2, a_s(\mu_{b_*}) \right)}{\sqrt{(1/e^2_{j})H_{j,j}^{DY}(\mu_{b_*}/C_2, \mu_{b_*}, a_s(\mu_{b_*}))}} \exp \left[ K(b_*; \mu_{b_*}) \ln C_2 \right]$$

$C_{\text{CSS1, DY}}$ taken from (Catani, Cieri, de Florian, Ferrera, Grazzini (2012))

• Directly from operators (using SCET)

(Echevarria, Scimemi, Vladimirov (2016))
The relevant renormalization group scales are

This work was supported by...

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\[ b_{\text{max}} \text{ method} \]

\[
W(q_T, Q) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}(b_T, Q)
\]

\[
= \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{\text{OPE}}(b_*(b_T), Q) \tilde{W}_{\text{NP}}(b_T, Q; b_{\text{max}})
\]
$$W(q_T, Q) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}(b_T, Q)$$

$$= \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{\text{OPE}}(b_*(b_T), Q) \tilde{W}_{\text{NP}}(b_T, Q; b_{\text{max}})$$

**Interesting part.**

*(Esp. for JLab)*

$$\frac{\tilde{W}(b_T, Q)}{\tilde{W}^{\text{OPE}}(b_*(b_T), Q)}$$
**b_{max} method**

\[
W(q_T, Q) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}(b_T, Q) \\
= \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{OPE}(b_*(b_T), Q) \tilde{W}_{NP}(b_T, Q; b_{max})
\]

**Interesting part.**
*(Esp. for JLab)*

\[
\frac{\tilde{W}(b_T, Q)}{\tilde{W}^{OPE}(b_*(b_T), Q)}
\]

\[
\tilde{W}(b_T, Q) = \tilde{W}^{OPE}(b_*(b_T), Q) + O((b_T m)^p)
\]

\[
\bar{f}_{j/H}(x, b_T; \zeta; \mu) = \sum_k \int_{x-}^{1+} \frac{d\xi}{\xi} \bar{C}^{PDF}_{j/k}(x/\xi, b_T; \zeta, \mu, a_s(\mu)) f_{k/H}(\xi; \mu) + O(m b_T)^p
\]
\[ W(q_T, Q) = \int \frac{d^2 b_T}{(2\pi)^2} e^{iq_T \cdot b_T} \tilde{W}(b_T, Q) \]

\[ = \int \frac{d^2 b_T}{(2\pi)^2} e^{iq_T \cdot b_T} \tilde{W}^{\text{OPE}}(b_*(b_T), Q) \tilde{W}_{\text{NP}}(b_T, Q; b_{\text{max}}) \]

- Banish any explicit non-perturbative calculations here.
- Calculated from operator definitions.
- Allow scale dependence.
- Large contribution all the way to \( b_T \approx 1/Q \).
W(q_T, Q) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}(b_T, Q)

= \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{OPE}(b_*(b_T), Q) \tilde{W}_{NP}(b_T, Q; b_{max})

- Banish any explicit non-perturbative calculations here.
- Calculated from operator definitions.
- Allow scale dependence.
- Large contribution all the way to b_T \approx 1/Q.

- Banish any scale dependence here (strictly universal)
- Allow for non-perturbative b_T-dependence (don’t necessarily require)
- Power suppression at small b_T
$b_{\text{max}}$ method

- Combine $b_T$ argument substitution with evolution.

- Maintains exact operator definitions for all factors and for all $b_T$. (Including any possible non-perturbative $b_T$-dependence)
\textbf{b}_{\text{max}} \text{ method}

- Evolution

\[ \tilde{W}(b_T, Q) = \tilde{W}(b_T, Q_0) \exp \left\{ \tilde{K}(b_T; \mu_0) \ln \left( \frac{Q^2}{Q_0^2} \right) + \int_{\mu_0}^{\mu'} \frac{d\mu'}{\mu'} \left[ 2 \gamma_j(\alpha_s(\mu)); 1 - \ln \frac{Q^2}{(\mu')^2} \gamma_K(\alpha_s(\mu')) \right] \right\} \]

\[ \tilde{W}(b_*(b_T), Q) = \tilde{W}(b_*(b_T), Q_0) \exp \left\{ \tilde{K}(b_*(b_T); \mu_0) \ln \left( \frac{Q^2}{Q_0^2} \right) + \int_{\mu_0}^{\mu'} \frac{d\mu'}{\mu'} \left[ 2 \gamma_j(\alpha_s(\mu')); 1 - \ln \frac{Q^2}{(\mu')^2} \gamma_K(\alpha_s(\mu')) \right] \right\} \]

- At a reference scale \( Q_0 \)

\[ \frac{\tilde{W}(b_T, Q_0)}{\tilde{W}(b_*(b_T), Q_0)} = \tilde{W}_{\text{NP}}(b_T, Q_0; b_{\text{max}}) = e^{-g_A(x_A, b_T; b_{\text{max}})} - g_B(z_B, b_T; b_{\text{max}}). \]

- Evolution

\[ \frac{\partial \ln \tilde{W}_j(b_T, Q, x_A, x_B)}{\partial \ln Q^2} = \tilde{K}(b_T; \mu) + b_T \text{ independent parts} \quad \checkmark \text{ Exact for all } b_T. \]
**b_{\text{max}}** method

- Evolution

\[
\frac{\partial \ln \tilde{W}_j(b_T, Q, x_A, x_B)}{\partial \ln Q^2} = \tilde{K}(b_T; \mu) + b_T \text{ independent parts}
\]

- RG for K:

\[
\frac{d\tilde{K}(b_T; \mu)}{d \ln \mu} = -\gamma_K(\alpha_s(\mu))
\]

- At any scale

\[
\tilde{K}(b_T; \mu_0; \alpha_s(\mu_0)) = \tilde{K}(b_T; \mu_b^*; \alpha_s(\mu_b^*)) - \int_{\mu_b^*}^{\mu_0} \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu'))
\]

\[
\frac{\tilde{W}(b_T, Q)}{\tilde{W}(b^*_T, Q)} = \frac{\tilde{W}(b_T, Q_0)}{\tilde{W}(b^*_T, Q_0)} e^{-\left[-\tilde{K}(b_T; \mu_0) + \tilde{K}(b^*_T; \mu_0)\right] \ln \left(\frac{Q^2}{Q_0^2}\right)}
\]

\[
= \frac{\tilde{W}(b_T, Q_0)}{\tilde{W}(b^*_T, Q_0)} e^{-g_K(b_T; b_{\text{max}}) \ln \left(\frac{Q^2}{Q_0^2}\right)}
\]

\[
= e^{-g_A(x_A, b_T; b_{\text{max}}) - g_B(x_B, b_T; b_{\text{max}}) - 2g_K(b_T; b_{\text{max}}) \ln(Q/Q_0)}
\]

*These equations are valid for all b_T*
The relevant renormalization group scales are:

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\[ W(q_T, Q) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}(b_T, Q) \]

\[ = \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{\text{OPE}}(b_\star(b_T), Q) \tilde{W}_{NP}(b_T, Q; b_{\text{max}}) \]

\[ = \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{\text{OPE}}(b_\star(b_T), Q) e^{-g_A(x_A, b_T; b_{\text{max}}) - g_B(x_B, b_T; b_{\text{max}}) - 2g_K(b_T; b_{\text{max}}) \ln(Q/Q_0)} \]

- Note that still:

\[ \frac{d}{db_{\text{max}}} W(q_T, Q) = 0 \]
The large $b_T$ functions are scale independent. $g_K$ is independent of everything except $b_T$.

Everything is written in terms of the $W(b_T, Q)$.

Can directly calculate with the definitions,

$$
\frac{\tilde{W}(b_T, Q)}{\tilde{W}(b_*(b_T), Q)}
$$

$$
g_K(b_T; b_{\text{max}}) \equiv -\tilde{K}(b_T; \mu_0) + \tilde{K}(b_*(b_T); \mu_0)
$$

in perturbation theory, if $b_T$ is small. using non-perturbative theory and operator defs if $b_T$ is large.
The $b_{\text{min}}$ mechanism

$$W(q_T, Q) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}(b_T, Q)$$

$$\int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{\text{OPE}}(b_*(b_T), Q) \tilde{W}_{\text{NP}}(b_T, Q)$$

Ex: ResBos
Generator

$e^{-a b_T^2}$

- Simple ansatz will tend to introduce $b_{\text{max}}$ dependence.
- Other options remove $b_{\text{max}}$ power corrections, e.g.

$$g_K(b_T; b_{\text{max}}) = \frac{C_F}{\pi} b_T^2 b_{\text{max}}^2 \alpha_s(\mu_{b_*}) + O \left( \frac{b_T^4 C_F^2 \alpha_s(\mu_{b_*})^2}{b_{\text{max}}^4 \pi^2 g_0(b_{\text{max}})} \right)$$

Summary

• Many calculations now exist in old CSS, pure collinear factorization, SCET.

• Different methods of calculation produce same results.

• Different approaches to derivation converging on standardized TMD definitions.

• Altogether these give necessary ingredients for new operator-based TMD factorization up to to order $\alpha_s^3$ (except in Wilson coefficient).