MANCHESTER 1824
The University of Manchester

Speaker:
Daniele Paolo Anderle
School of Physics & Astronomy

Recent Progress on FF Analyses

Jefferson Laboratory, Newport News, VA , 05.25.2017

QCD Evolution

FF Panorama

R.D. Field R.P. Feynman, *Phys.Rev.D* 15, 2590 1977		
J.F. Owens E. Reya. M.Gluck, *Phys.Rev.D* 18, 1501 1978	"model estimates consistent with data"	
R. Baier, J. Engels and B. Petersson, *Z.Phys.C 2*, 265 1979		
M. Anselmino, P. Kroll E. Leader, *Z.Phys.C 18*, 307 1983		
...		***LO groundbreaking***

P. Chiappeta et al. , *Nuc.Phys.B 412*, 3 1994	π^0	CGGRW94
J. Binneweis. B. Kniehl, G. Kramer, *Z. Phys. B 65*, 471 1995	"$\pi^\pm, K^\pm$"	BKK95
J. Binneweis. B. Kniehl, G. Kramer, *Phys. Rev. D 52*, 4947 1995	"$\pi^\pm, K^\pm$ LEP"	
J. Binneweis. B. Kniehl, G. Kramer, *Phys. Rev. D 53*, 3553 1996	K^0	
D. de Florian. M.Stratmann, W.Vogelsang, *Phys. Rev. D 57*, 5811 1998	Λ^0	DSV97
L. Bourhis et al. , *Eur. Phys. J.C 19*, 89 2001	$h^\pm$	BFGW00
B. Kniehl G. Kramer, B. Potter, *Nuc. Phys. B 582*, 514 2000	"$\pi^\pm, K^\pm, p/\bar{p}$"	KKP00
S. Kretzer, *Phys. Rev. D 62*, 4001 2000	"flavor tagging"	KRE00
S. Albino, S. Kniehl, G. Kramer, *Nuc. Phys. B 785*, 181 2005	"OPAL tagging"	AKK05
M. Hirai, et al., *Phys. Rev. D 75*, 4009 2007	"uncertainties"	HKNS07
... heavy flavors, hadron mass effects, resummations, **...**		***NLO e+e- paradigm***

D. de Florian, R.S., M. Stratmann , *Phys. Rev. D 75*, 4010 2007	"$e^+e^-, pp, SIDIS$"	DSS07
S. Albino, S. Kniehl, G. Kramer, *Nuc. Phys. B 803*, 42 2008	"e^+e^-, pp"	AKK08
R.S., M. Stratmann, P. Zurita , *Phys. Rev. D 81*, 054001 2010	"nFFs"	SSZ10
C. Aidala, et al., *Phys. Rev. D 83*, 034002 2011	"η"	AESS11
E. Leader, A.V. Sidorov, D. Stamenov, *arXiv:1312.5200*	"$SIDIS$ only"	LSS13
M. Soleymaninia et al., *Phys. Rev. D 88*, 054019 2013	"$e^+e^-, pSIDIS$"	SKMNA13
D. de Florian et al. , *Phys. Rev. D 91*, 4035 2015	"$\pi^\pm$ update"	DSS14
E. Leader, A.V. Sidorov, D. Stamenov, *Phys. Rev. D96*, 074026 2016	"$SIDIS$ only"	LSS15
		Global paradigm

table thanks to R.Sassot

TWO WAY OF ADVANCING IN THIS PANORAMA

Precision extraction of FF

NNLO e+e- with small z resummation

D. Anderle, M. Stratmann, F. Ringer, Phys. Rev. D 92, 114010 2015

D. Anderle, T.Kaufmann, M. Stratmann, F. Ringer, Phys.Rev. D95 (2017) no.5, 054003

Global extraction of FF

Heavy Meson D* Global Fit

preliminary results presented here

Global precision extraction of FF

Outline

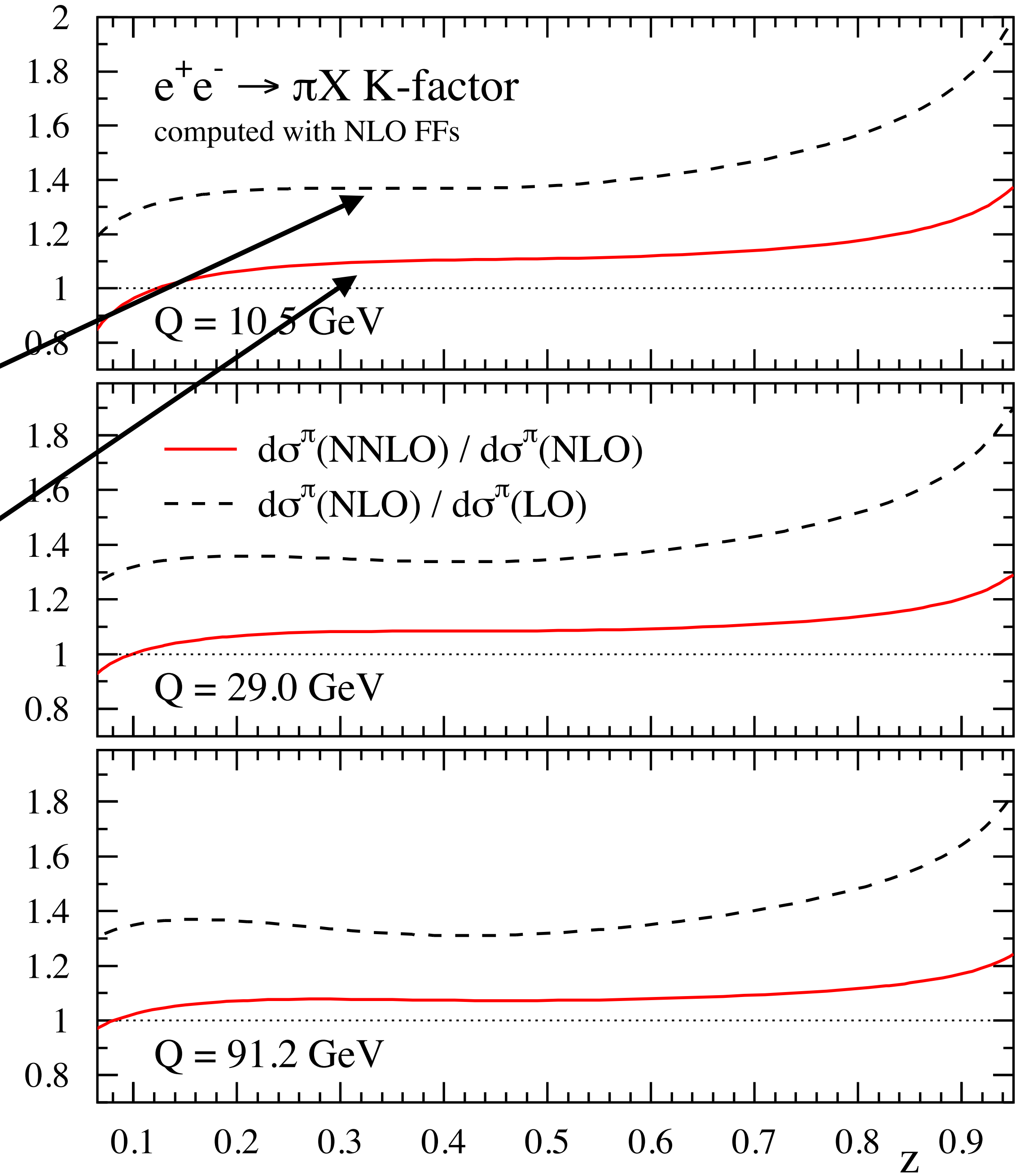
- Our e+e- NNLO Fit
- Improving: Small-z Resummation
- Our D* global fit
- Conclusions & Outlook

NNLO first comfortably precise order

$$K \equiv \frac{d\sigma^{\pi}(\mathrm{N}^{m}\mathrm{LO})}{d\sigma^{\pi}(\mathrm{N}^{m-1}\mathrm{LO})}$$

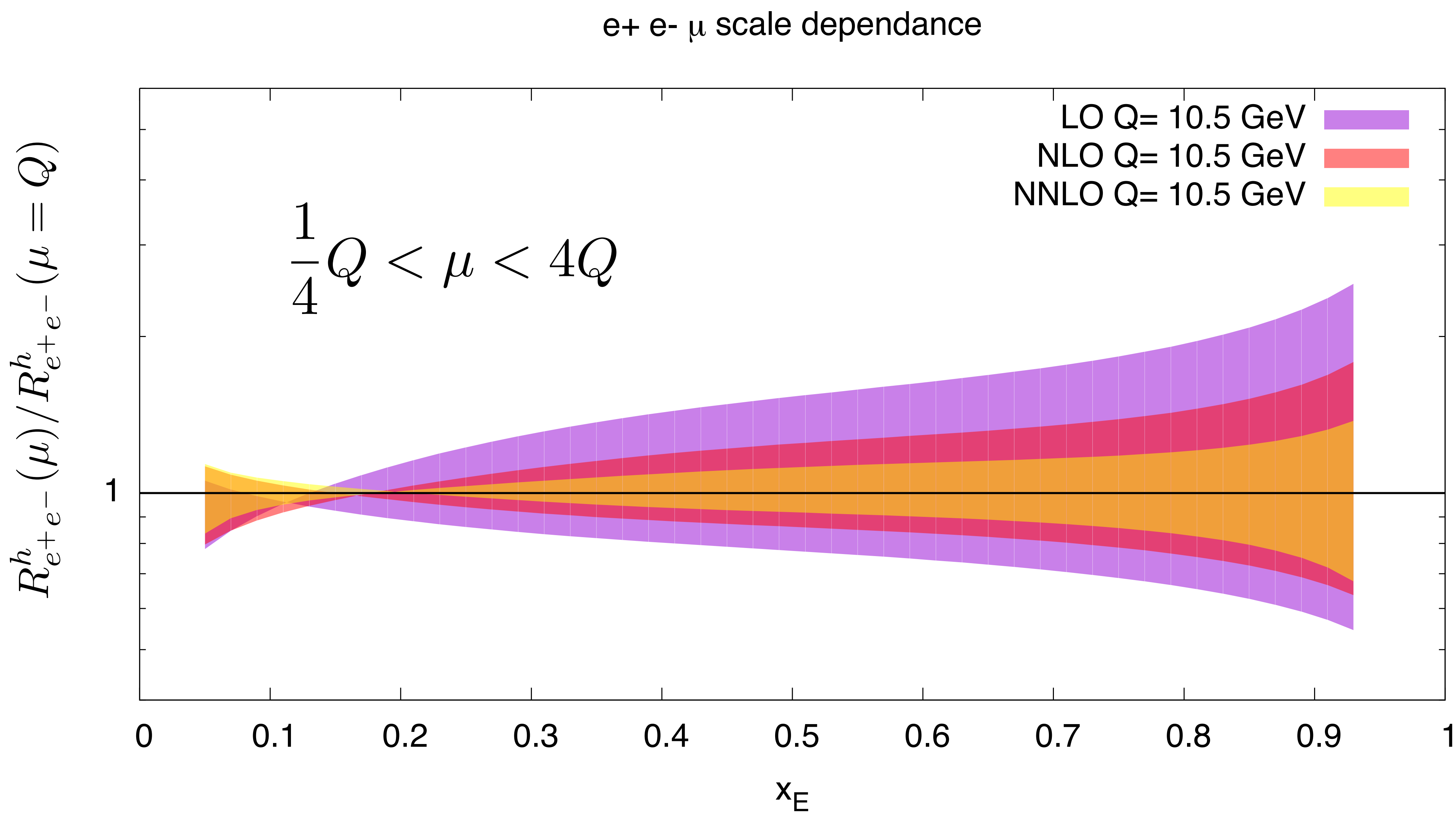
40% correction from LO to NLO

10% correction from NLO to NNLO



D. Anderle, M. Stratmann, F. Ringer, Phys. Rev. D 92, 114010 2015

Scale Dependence



Multiplicity $R^h_{e^+e^-} \equiv \frac{1}{\sigma^{\text{tot}}} \frac{d^2\sigma^h}{dx_E \, d\cos\theta}$

using input parameter for FF of Kretzer (Phys.Rev. D62 (2000) 054001) and truncated-solution

TOWARDS A GLOBAL NNLO FF FIT

Anderle, Ringer, Stratmann

Ingredients needed to achieve the goal:

DATA SETS:

SI-e^+e^- → new: BaBar(Phys. Rev. D 88, 032011 (2013)), Belle(Phys. Rev. Lett. 111, 062002 (2013))

SIDIS → new: HERMES(Ph.D. thesis, Erlangen Univ., Germany, September 2005), Compass(PoS DIS 2013, 202 (2013)), JLAB@12GeV

SI- p(anti-)p → new: Phenix(Phys. Rev. D 76,051106 (2007)), Alice(Phys. Lett. B 717, 162 (2012).), Brahms(Phys. Rev. Lett. 98, 252001 (2007)), Star(Phys. Rev. Lett. 97, 152302 (2006))

pp→ (Jet h)X → future: Star, CMS(JHEP 1210, 087 (2012)), Alice(arXiv:1408.5723), Atlas(Eur. Phys. J. C 71, 1795 (2011))

Towards a Global NNLO FF Fit

Anderle, Ringer, Stratmann

Ingredients needed to achieve the goal:

NNLO COEFFICINT FUNCTIONS:

SI-e^+e^- → **x-Space** Rijken, van Neerven (Phys.Lett.B386(1996)422, Nucl.Phys.B488(1997)233,Phys.Lett.B392(1997)207)

Mellin-Space Mitov, Moch (Nucl.Phys.B751 (2006) 18-52)
Blümlein, Ravindran (Nucl.Phys.B749 (2006) 1-24)

SIDIS → NOT COMPUTED YET only some CL channels

$$\gamma q' \to q\bar{q}q'$$
$$\gamma g \to q\bar{q}q'$$

Anderle, de Florian, Rotstein (Phys.Rev. D95 (2017) no.3, 034027)

SI- p(anti-)p → NOT COMPUTED YET

pp→ (Jet h)X → NOT COMPUTED YET NLO calculation Kaufmann,Asmita Mukherjee,Vogesang (Phys.Rev. D92 (2015) no.5, 054015)

TOWARDS A GLOBAL NNLO FF FIT

Anderle, Ringer, Stratmann

Ingredients needed to achieve the goal:

NNLO COEFFICINT FUNCTIONS:

DIA → To be developed extending to NNLO or approx NNLO results of:

$$\frac{d\sigma}{dz\,du}$$ G. Altarelli, R.K. Ellis, G. Martinelli, So-Young Pi. (Nucl. Phys. B, 160 (1979), p. 301)

$$\frac{d\sigma}{dz_1 dz_2}$$ D. de Florian, L. Vanni (Phys.Lett. B578 (2004) 139-149)

$$\frac{d\sigma}{d\tau}$$ Sterman, Vogelsang (Phys.Rev. D74 (2006) 114002)

where $u = \frac{P_1 \cdot P_2}{P_1 \cdot q}$ $z_2 = \frac{P_2 \cdot q}{Q^2}$ $z \equiv z_1 = \frac{P_1 \cdot q}{Q^2}$ $\tau = z \cdot u = \frac{(P_1 + P_2)^2}{Q^2}$

Towards a Global NNLO FF Fit

Anderle, Ringer, Stratmann

Ingredients needed to achieve the goal:

NNLO EVOLUTION KERNELS:

Splitting functions →

NNLO-Non Singlet: Mitov, Moch, Vogt(Phys.Lett. B638 (2006) 61-67)

NNLO-Singlet: Moch, Vogt(Phys.Lett.B659 (2008) 290-296)

NNLO-Singlet: Almasy, Mitov, Moch, Vogt(Nucl.Phys. B854 (2012)) 133-152)

Both computed in x-Space and in Mellin Space

OUR SIA FIT

Anderle, Ringer, Stratmann Phys.Rev. D92 (2015) no.11, 114017

Based on DSS Mellin Framework

Parametrization of light patrons FF @ μ_0

$$D_i^h(z, Q_0) = \frac{N_i z^{\alpha_i}(1-z)^{\beta_i}[1+\gamma_i(1-z)^{\delta_i}]}{B[2+\alpha_i, \beta_i+1] + \gamma_i B[2+\alpha_i, \beta_i+\delta_i+1]}$$

So that $N_i = \int_0^1 z D_i^h dz$

Heavy Quark Treatment:

NON PERTURBATIVE INPUT: at $\mu > m_q$ the evolution is set to evolve with $n_f + 1$ for flavours and for the q-heavy quark FF the same functional form as for the light quark is set at $\mu = m_q$

Data sets:

15 Data Set: from Sld, Aleph, Delphi, Opal, Tpc, BaBar, Belle either inclusive, uds tagged, b tagged or c tagged (relevant scales 10.5,29,91.2 GeV). We use a GLOBAL CUT 0.075<z<0.95

Our SIA Fit

Anderle, Ringer, Stratmann Phys.Rev. D92 (2015) no.11, 114017

Minimization done with MINUT with norm shift allowed

$$\chi^2 = \sum_{i=1}^{N} \left[\left(\frac{1-\mathcal{N}_i}{\delta \mathcal{N}_i} \right)^2 + \sum_{j=1}^{N_i} \frac{(\mathcal{N}_i T_j - E_j)^2}{\delta E_j^2} \right]$$

with $\mathcal{N}_i = \frac{\sum_{j=1}^{N_i} \frac{\delta \mathcal{N}_i^2}{\delta E_j^2} T_j E_j + 1}{1 + \sum_{j=1}^{N_i} \frac{\delta \mathcal{N}_i^2}{\delta E_j^2} T_j^2}$ and $\left. \frac{\partial \chi^2}{\partial \mathcal{N}} \right|_{\mathcal{N}_i} = 0$

Heavy Quark Treatment:

NON PERTURBATIVE INPUT: at $\mu > m_q$ the evolution is set to evolve with $n_f + 1$ for flavours and for the q-heavy quark FF the same functional form as for the light quark is set at $\mu = m_q$

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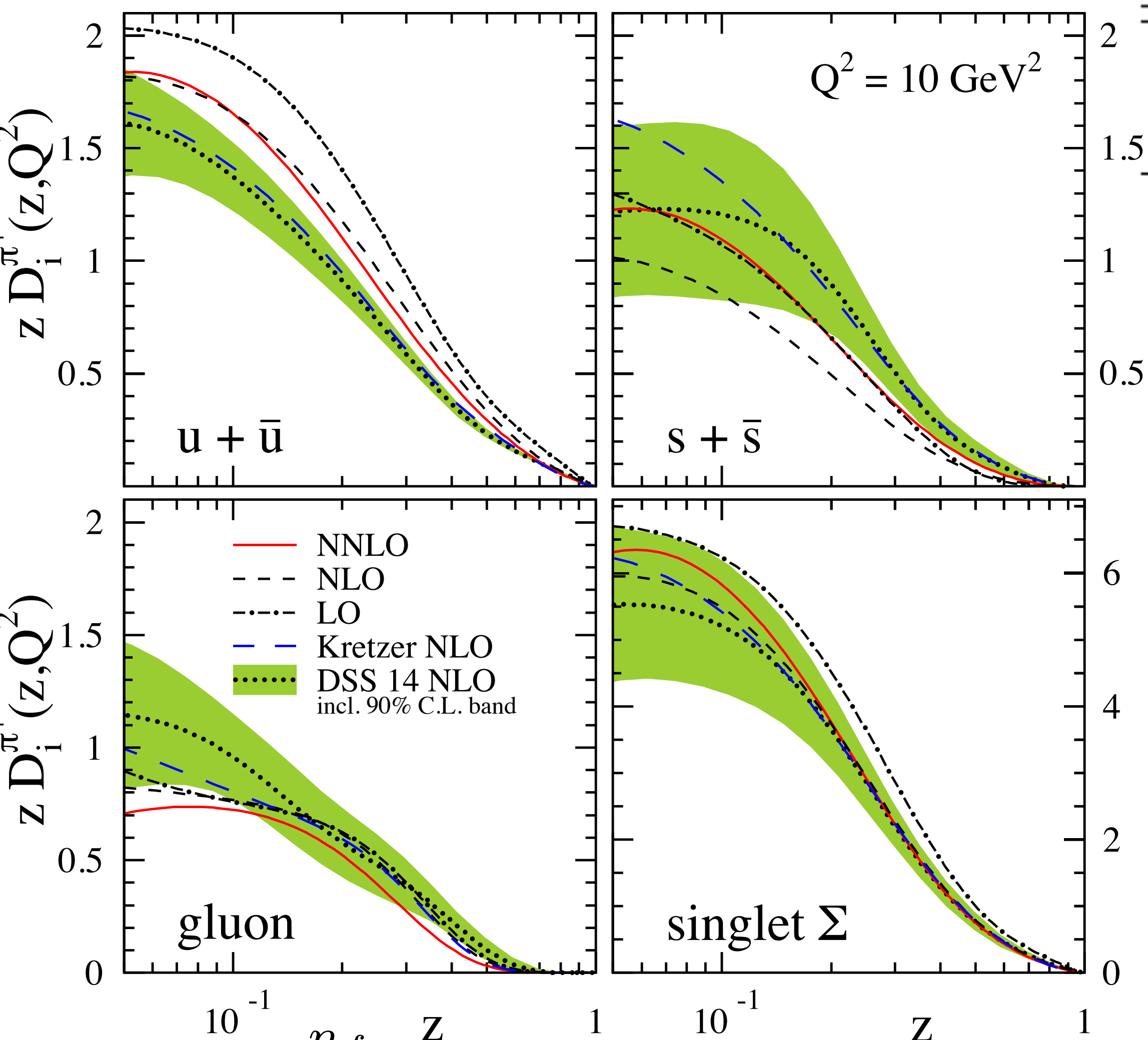
χ^2 Comparison



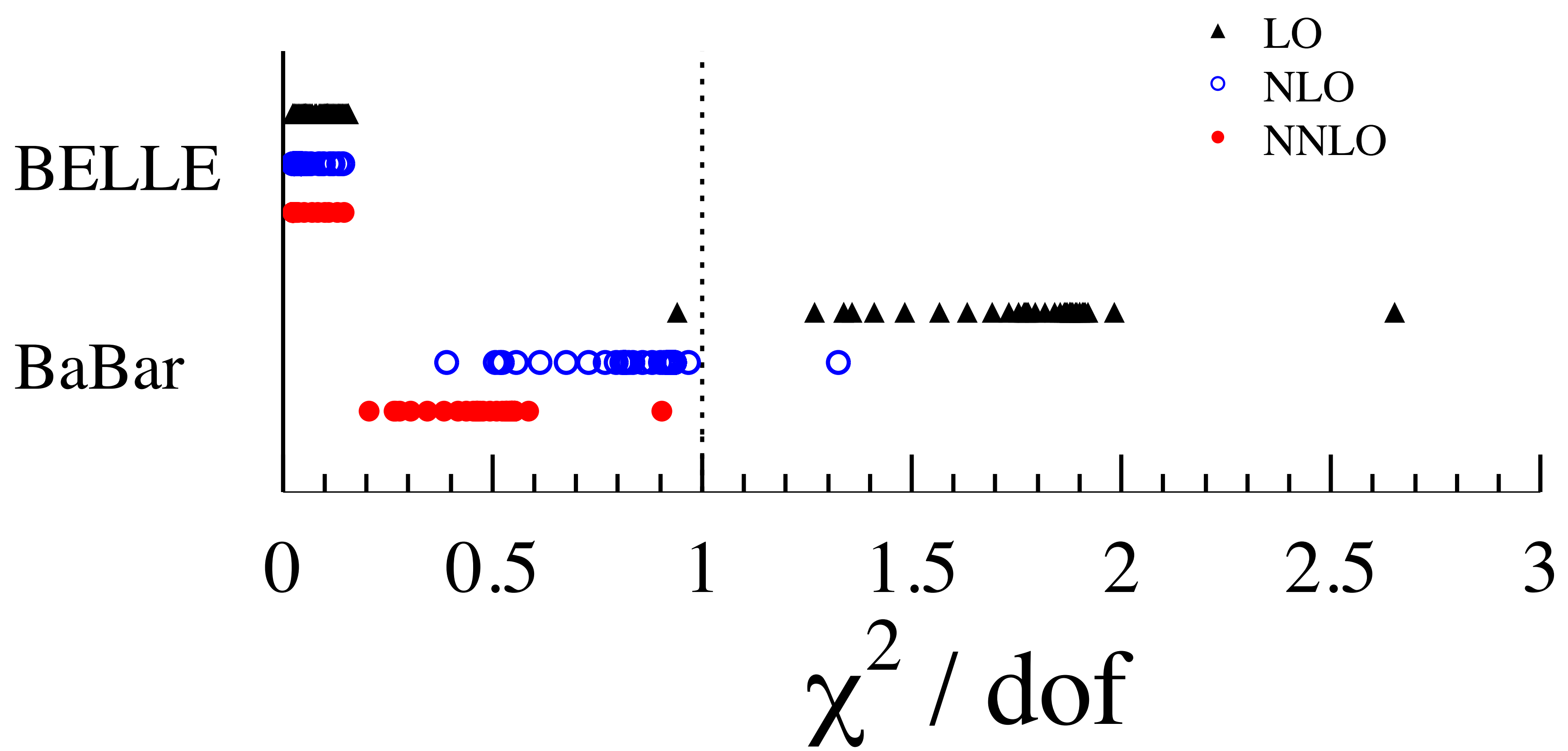
$$D^h_\Sigma = \sum_{i=1}^{n_f} \left(D^h_{q_i} + D^h_{\bar{q}_i} \right)$$

Kretzer FFS (*Phys. Rev. D 62, 054001 (2000)*)

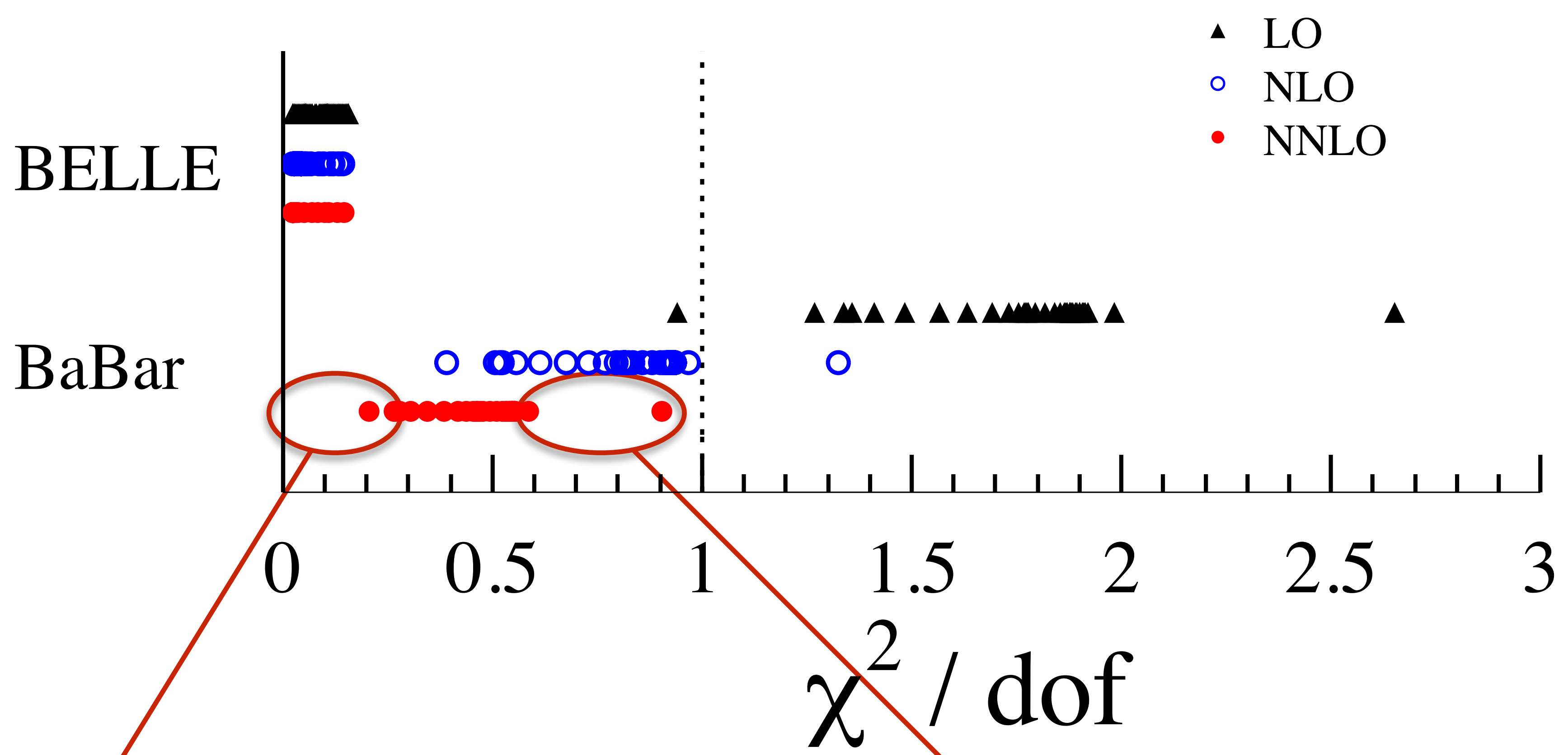
DSS FFS (*Phys. Rev. D 91, 014035 (2015)*)

experiment	data type	# data in fit	χ^2 LO	NLO	NNLO
SLD [40]	incl.	23	15.0	14.8	15.5
	uds tag	14	9.7	18.7	18.8
	c tag	14	10.4	21.0	20.4
	b tag	14	5.9	7.1	8.4
ALEPH [41]	incl.	17	19.2	12.8	12.6
DELPHI [42]	incl.	15	7.4	9.0	9.9
	uds tag	15	8.3	3.8	4.3
	b tag	15	8.5	4.5	4.0
OPAL [43]	incl.	13	8.9	4.9	4.8
TPC [44]	incl.	13	5.3	6.0	6.9
	uds tag	6	1.9	2.1	1.7
	c tag	6	4.0	4.5	4.1
	b tag	6	8.6	8.8	8.6
BABAR [10]	incl.	41	108.7	54.3	37.1
BELLE [9]	incl.	76	11.8	10.9	11.0
NORM. SHIFTS			7.4	6.8	7.1
TOTAL:		288	241.0	190.0	175.2

χ^2 Comparison



χ^2 COMPARISON



Small z Logs

$$\alpha_s^k \frac{\ln^{2k}(z)}{z}$$

Threshold Logs

$$\alpha_s^k \left(\frac{\ln^{2k-1}(1-x)}{1-x}\right)_+$$

Anderle, Ringer, Vogelsang (Phys.Rev. D87 (2013) no.3, 034014)

+Hadron Mass Cor.

Accardi, Anderle, Ringer (Phys.Rev. D91 (2015) 3, 034008)

Outline

- Our E+E- NNLO Fit
- Improving: Small-z Resummation
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Small-z Logarithms (SIA)

N^{k}LO Small-z Logarithms in Splitting Functions and Singlet Coefficient Functions

$$P_{gi}^{(k-1)} \propto a_s^k \frac{\ln^{2k-1-a}(z)}{z}$$

$$C_{T,g}^{(k)} \propto a_s^k \frac{\ln^{2k-1-a}(z)}{z}$$

$$C_{L,g}^{(k)} \propto a_s^k \frac{\ln^{2k-2-a}(z)}{z}$$

$$a = 0, 1, 2 \qquad i \in \{q, g\}$$

Double Log Enhancement

spoils perturbative convergence even for $\alpha_s \ll 1$

In Mellin Space they correspond to N = 1 Poles

$$\mathcal{M}\left[\frac{\ln^{2k-1}(z)}{z}\right] \equiv \int_0^1 dx\, x^{N-1} \frac{\ln^{2k-1}(z)}{z} = \frac{(-1)^k (2k-1)!}{(N-1)^{2k}}$$

RESUMMATION ACCURACY

For example P_{gg} with $N-1=\bar{N}$

Fixed Order →

Resummation ↓

LO	$\alpha_s/\bar{N}$	α_s				
NLO	$\alpha_s/\bar{N}^3$	$\alpha_s/\bar{N}^2$	$\alpha_s/\bar{N}$	α_s		
NNLO	$\alpha_s/\bar{N}^5$	$\alpha_s/\bar{N}^4$	$\alpha_s/\bar{N}^3$	$\alpha_s/\bar{N}^2$	$\alpha_s/\bar{N}$	α_s
...	...	...	...	...	...	
$N^{k-1}LO$	$\alpha_s/\bar{N}^{2k-1}$	$\alpha_s/\bar{N}^{2k-2}$	$\alpha_s/\bar{N}^{2k-3}$	$\alpha_s/\bar{N}^{2k-4}$	$\alpha_s/\bar{N}^{2k-5}$	...

NNLL : *Vogt (2011), Kom ,Vogt, Yeats (2012)*

NLL : *Mueller (83), Albino, Bolzoni, Kniehl, Kotikov (11)*

LL : *Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82).*

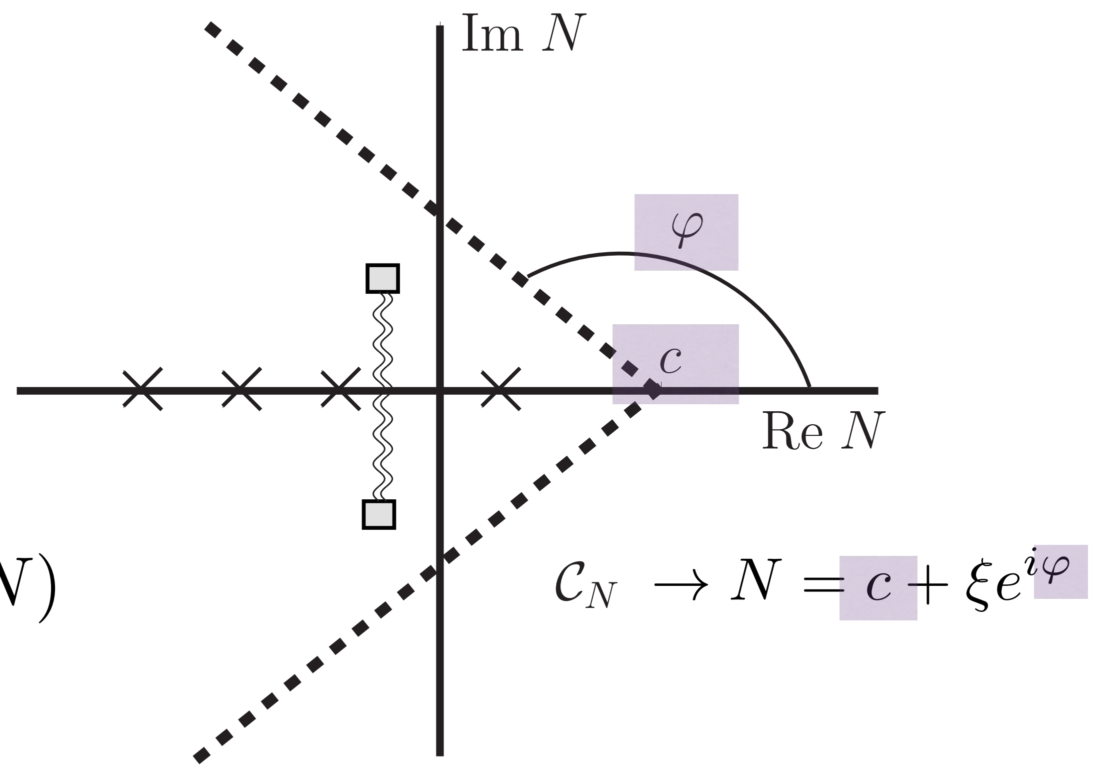
Inversion For Fit

At the end of the day one has to perform the Mellin Inversion (schematically)

Minimal Prescription

Catani, Mangano, Nason, Trentadue

$$D(z) \otimes C(z) = \frac{1}{2\pi i} \int_{\mathcal{C}_N} dN\, z^{-N}\, D(N)\, C(N)$$

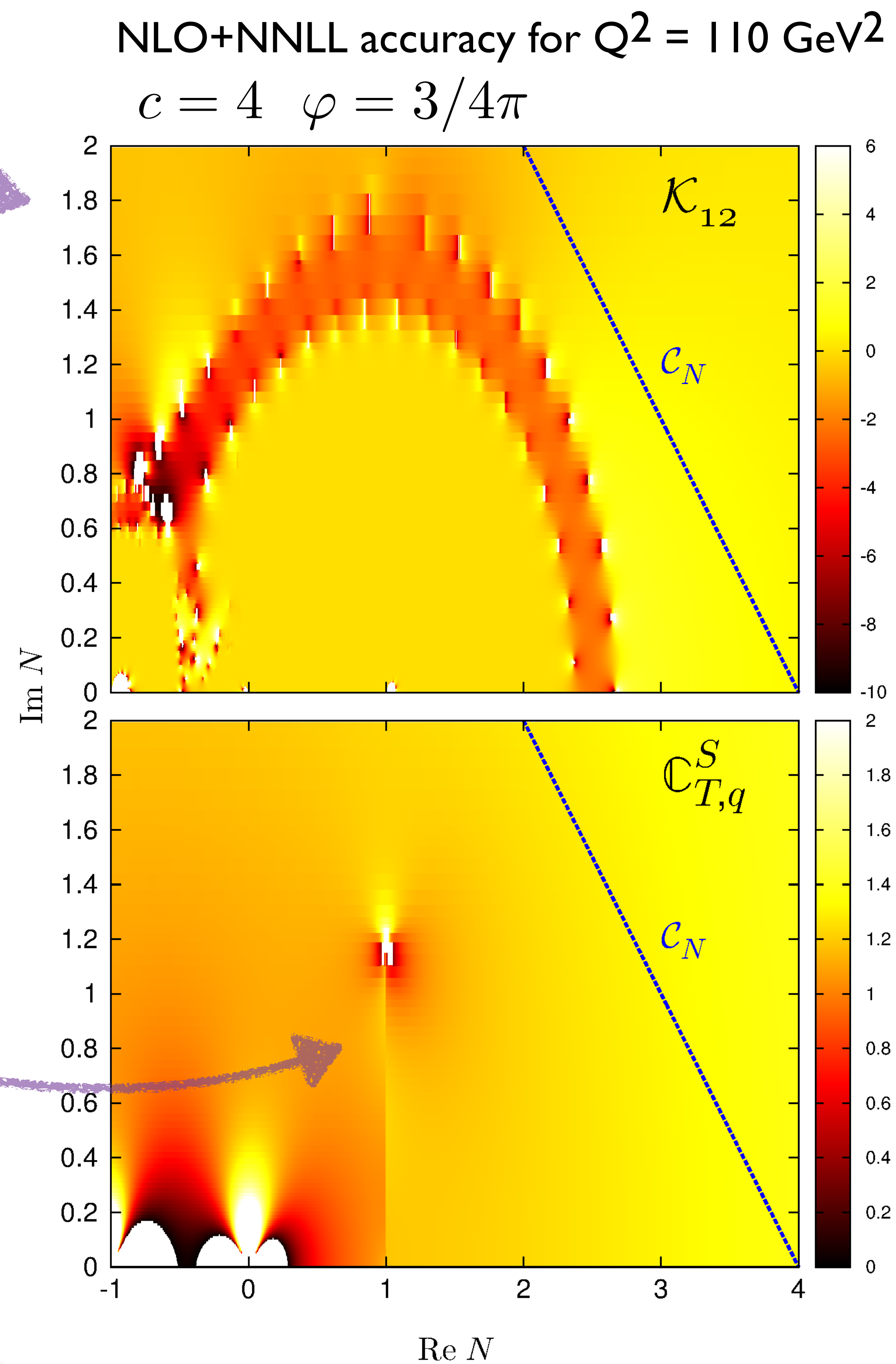


FFs

$$\boldsymbol{D}^h(N, a_s) = \begin{pmatrix} \mathcal{K}^T_{11}(a_s, a_0, N) & \mathcal{K}^T_{12}(a_s, a_0, N) \\ \mathcal{K}^T_{21}(a_s, a_0, N) & \mathcal{K}^T_{22}(a_s, a_0, N) \end{pmatrix} \boldsymbol{D}^h(N, a_0)$$

Coefficient Functions

$$N = 1 \pm i\sqrt{32 C_A a_s(\mu)}$$



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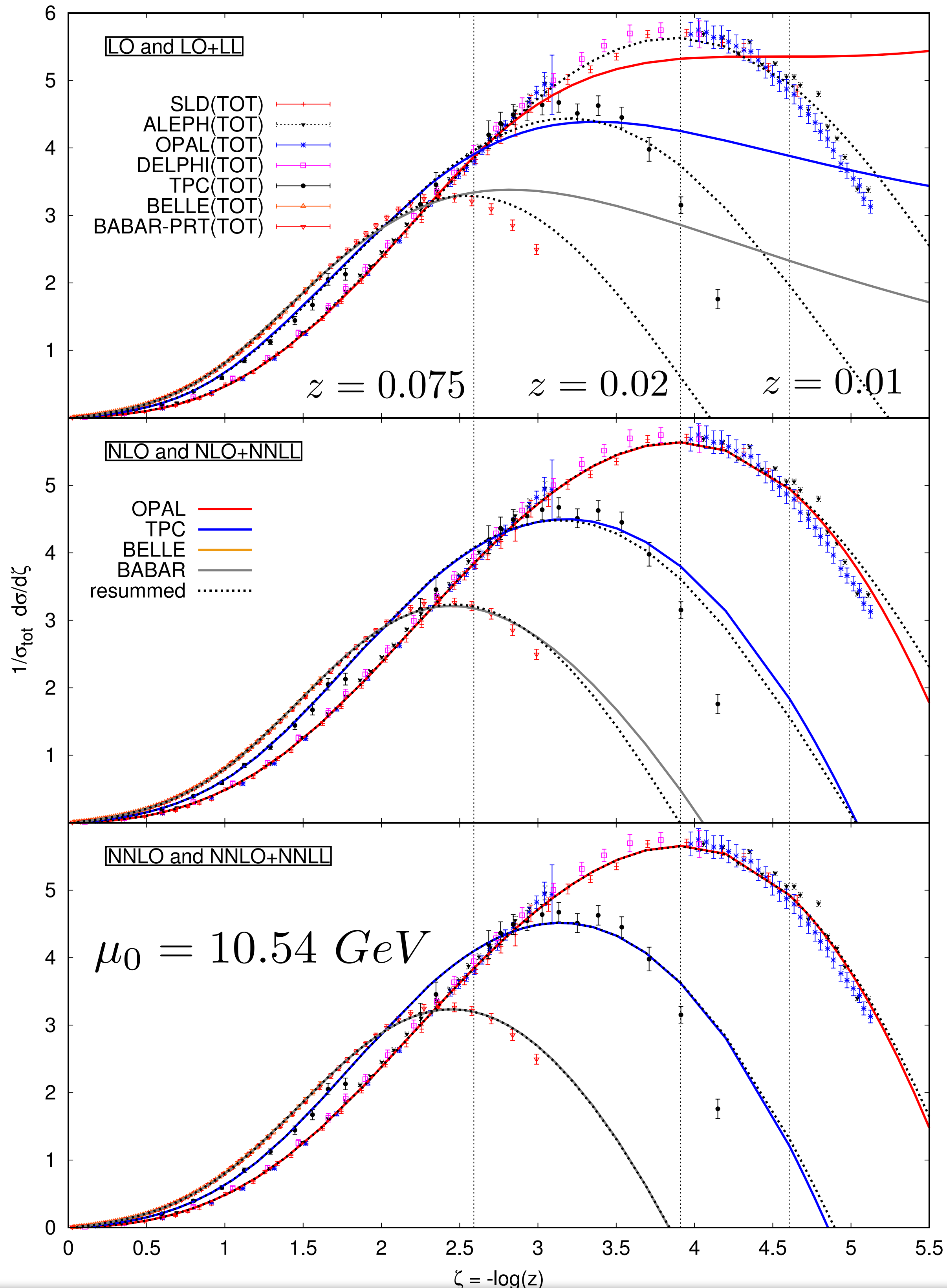
Our Resummed Fit on The Data

436 Total data Points:

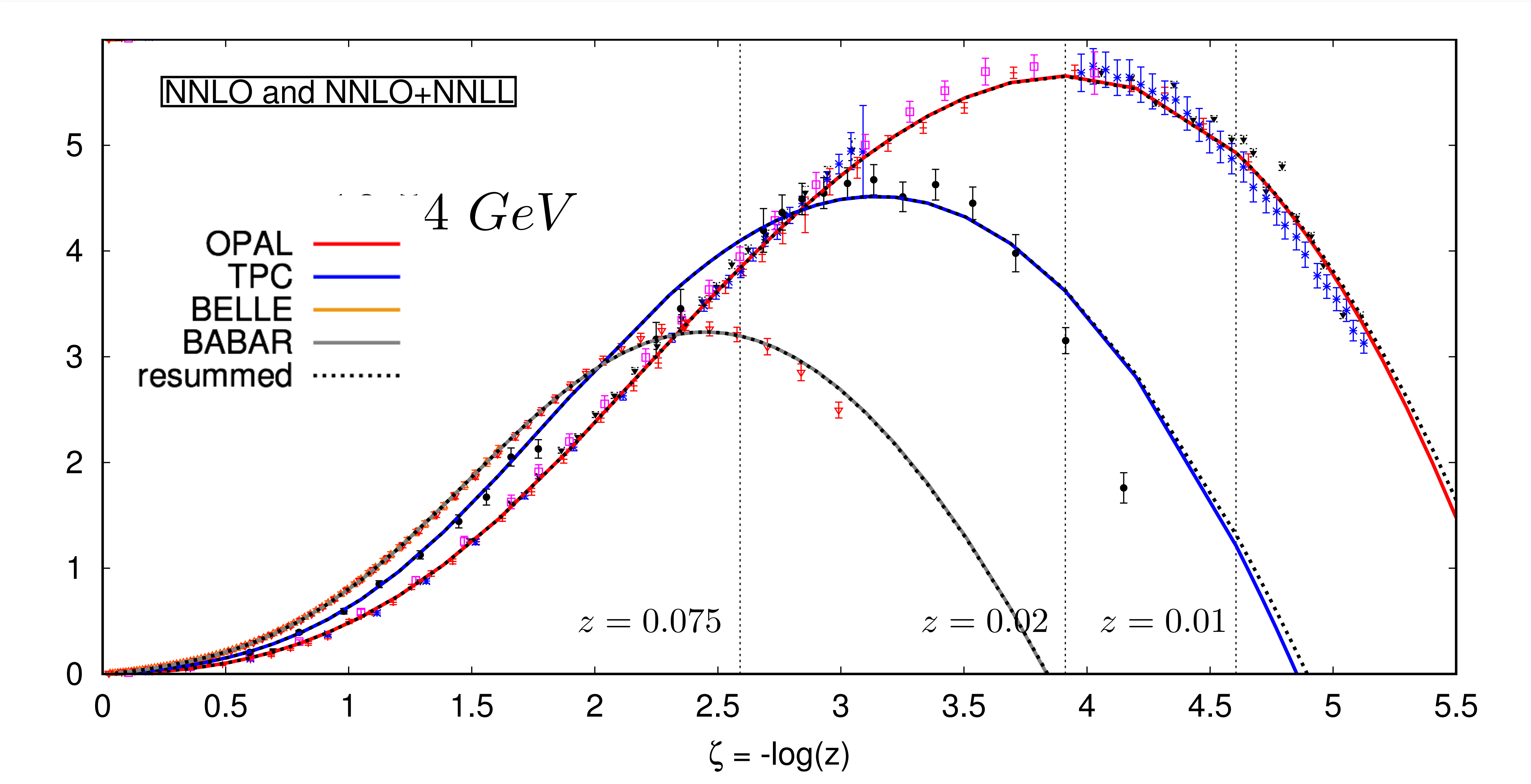
- LEP cut (z = 0.01) due to inconsistency between OPAL and ALEPH

- TPC lower cut (z = 0.02) based on difference of energy fraction $z = 2\,E_h/Q$ and three momentum fraction $x_p = z - 2m_h^2/(zQ^2) + \mathcal{O}(1/Q^4)$ in c.m.s being less than at least 15%

accuracy	χ^2	χ^2/dof
LO	1260.78	2.89
NLO	354.10	0.81
NNLO	330.08	0.76
LO+LL	405.54	0.93
NLO+NNLL	352.28	0.81
NNLO+NNLL	329.96	0.76



Our Resummed Fit on The Data



Outline

- Our E+E- NNLO Fit
- Improving: Small-z Resummation
- OUR D* GLOBAL FIT
- Conclusions & Outlook

A global NLO Analysis for $D^{*\pm}$

Some example of why is it relevant:

- first global fit
- constraint on low-x PDFs
- used in calculation of prompt atmospherical neutrino flux
- to extract information on the medium in heavy ion collisions
- …

Previously only extracted from e+e- data

J. Binnewies, B. A. Kniehl and G. Kramer, Phys. Rev. D 58, 014014 (1998),

B. A. Kniehl, G. Kramer, I. Schienbein and H. Spiesberger, Phys. Rev. D 71, 014018 (2005).

B. A. Kniehl, G. Kramer, I. Schienbein and H. Spiesberger, Eur. Phys. J. C 41, 199 (2005).

B.A. Kniehl and G. Kramer, Phys. Rev. D 74, 037502(2006),

T. Kneesch, B. A. Kniehl, G. Kramer and I. Schienbein, Nucl. Phys. B 799, 34 (2008)

A GLOBAL NLO ANALYSIS FOR $D^{*\pm}$

Some example of why is it relevant:

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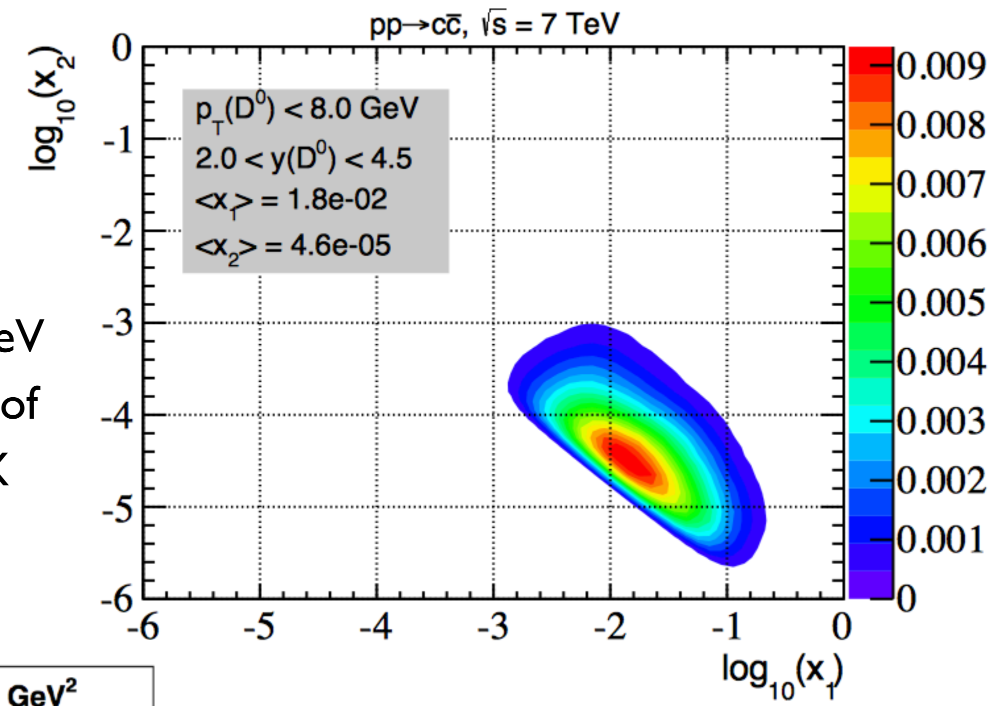
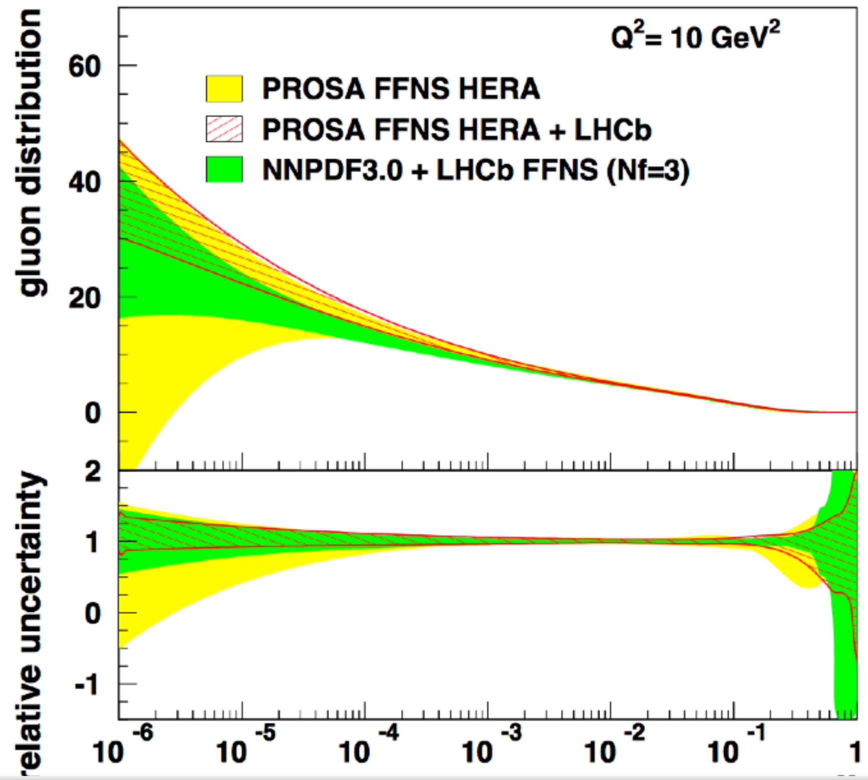
- ...

LHCb D0 with
2.0 ≤ y ≤ 4.5 and pT < 8 GeV
x1 and x2 Bjorken Variables of
respective PDF in pp~>DX





PROSA Collaboration, Eur.Phys.J. C75 (2015) no.8, 396

Gauld, R., Rojo, J., Rottoli, L. et al. J. High Energ. Phys (2015) 2015

A GLOBAL NLO ANALYSIS FOR $D^{*\pm}$

Some example of why is it relevant:

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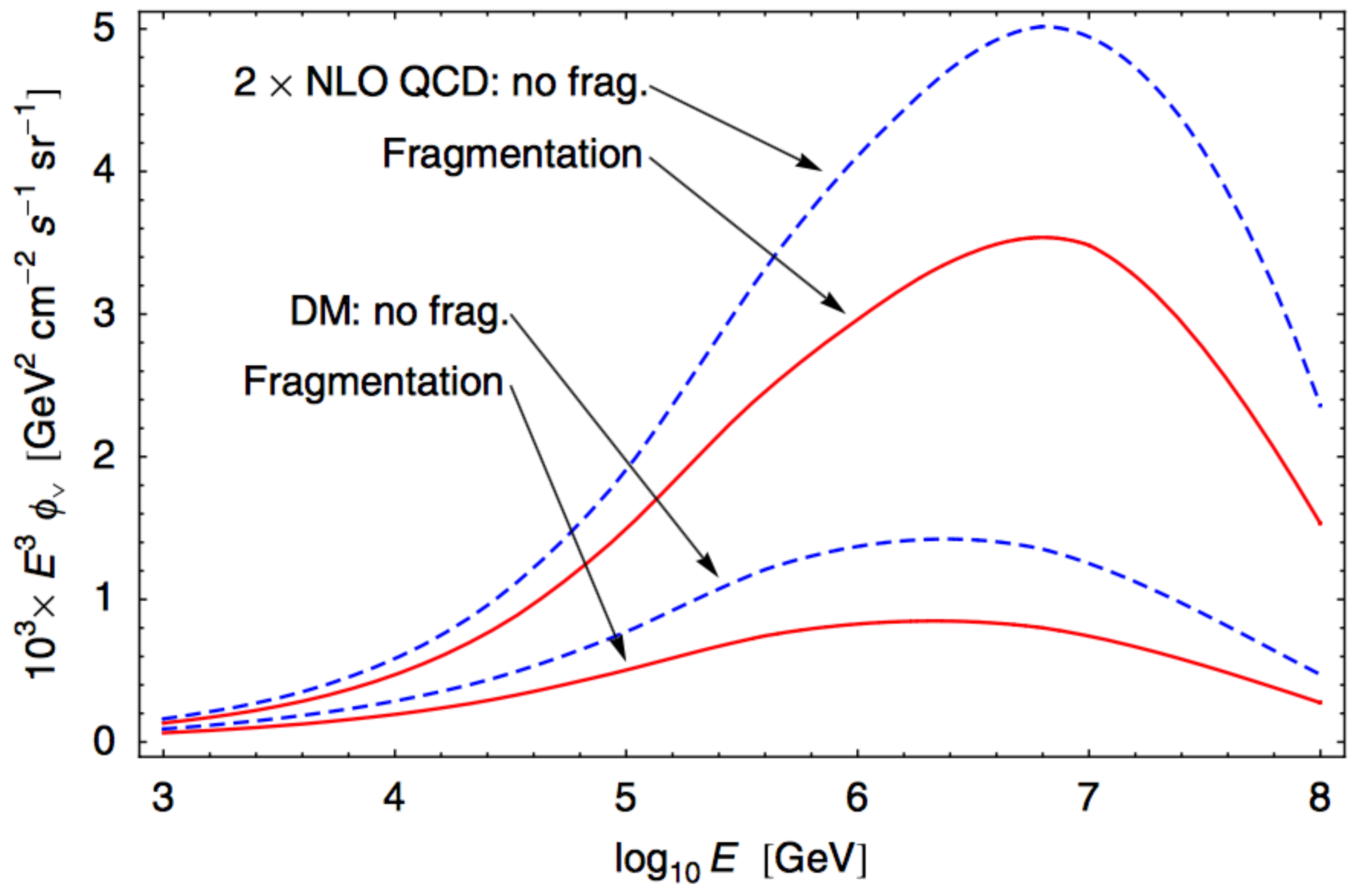
- …

w or w/o considering $\frac{d\sigma(pp \to hX)}{dE_h} = \int_{E_h}^{\infty} \frac{dE_c}{E_c} \frac{d\sigma(pp \to cX)}{dE_c} D_c^h(E_h/E_c)$



R.Enberg, M.H.Reno, I.Sarcevic PRD 78, 043005 (2008)

Bhattacharya, R. Enberg, M. H. Reno, I. Sarcevic , A. Stasto, JHEP 1506, 110 (2015)

A GLOBAL NLO ANALYSIS FOR $D^{*\pm}$

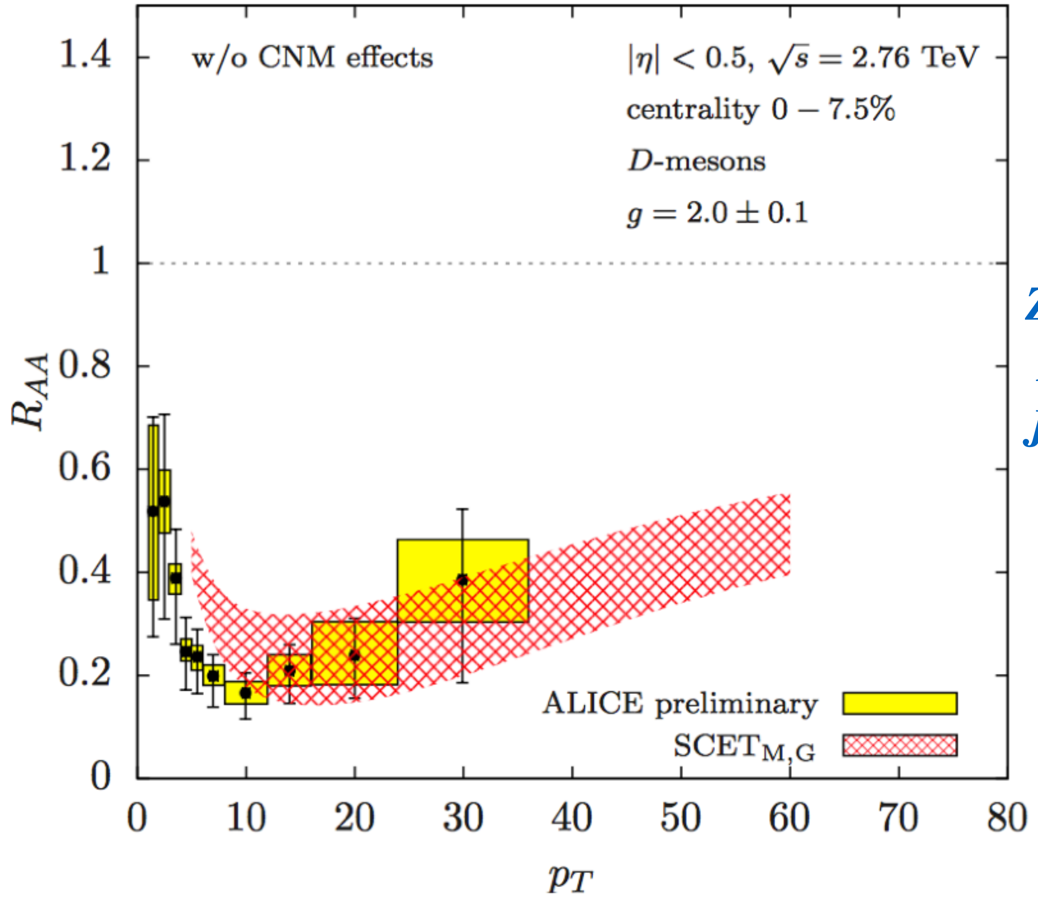
Some example of why is it relevant:

$$R_{AA} = \frac{d\sigma^H_{\rm PbPb}/d\eta dp_T}{\langle N_{\rm bin}\rangle\, d\sigma^H_{pp}/d\eta dp_T}$$

$$d\sigma^H_{\rm PbPb} = d\sigma^{H,{\rm NLO}}_{pp} + d\sigma^{H,{\rm med}}_{\rm PbPb}$$

$$d\sigma^{H,{\rm med}}_{\rm PbPb} = \sum_j \hat{\sigma}^{(0)}_i \otimes \mathcal{P}^{\rm med}_{i\to jk} \otimes D^H_j \equiv \hat{\sigma}^{(0)}_i \otimes D^{H,{\rm med}}_i$$

- first global fit
- constraint on low-x PDFs
- used in calculation of prompt atmospherical neutrino flux
- to extract information on the medium in heavy ion collisions
- …



See I.Vitev Talk

Zhong-Bo Kang, Felix Ringer, Ivan Vitev
JHEP 1703 (2017) 146

First Global Fit Including pp→(Jet h) X → Direct Gluon Distribution Scan

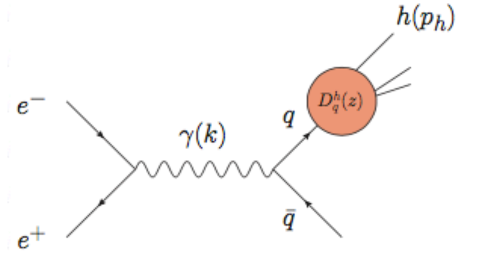
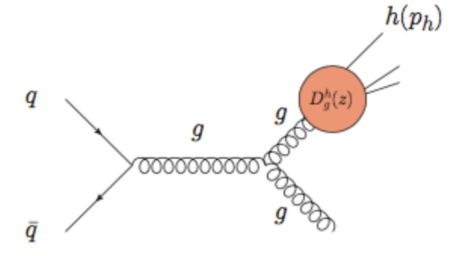
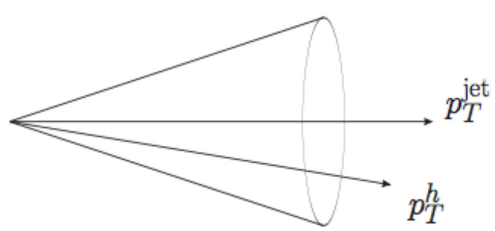
Process	D_g^h ?	Direct scan?
$e^+e^- \to hX$ $ep \to ehX$		  $z = \frac{2p_h \cdot k}{k^2}$
$pp \to hX$		  $\hat{p}_T = \frac{p_T^h}{z_c}$ $\quad z_c^{\min} = \frac{2p_T^h}{\sqrt{S}} \cosh\eta$ $d\sigma \propto \int_{z_c^{\min}}^{1} \frac{dz_c}{z_c^2} \left[f_a \otimes f_b \otimes d\hat{\sigma}_{ab}^c(\hat{p}_T, \ldots)\right] D_c^h(z_c)$
$pp \to (\text{jet } h)\, X$		  $z = z_h \equiv \frac{p_T^h}{p_T^{\text{jet}}}$

table thanks to T. Kaufmann

First Global Fit Including pp→(Jet h) X → Direct Gluon Distribution Scan

At LO direct probe of FF at z_h

$$\left.\frac{d\sigma^{H_1H_2\to(\text{jet }h)X}}{dp_T^{\text{jet}}\,d\eta^{\text{jet}}\,dz_h}\right|_{\text{LO}} = \frac{2p_T^{\text{jet}}}{S}\sum_{a,b,c}\int_{x_a^{\min}}^{1}\frac{dx_a}{x_a}f_a^{H_1}(x_a,\mu_F)\int_{x_b^{\min}}^{1}\frac{dx_b}{x_b}f_b^{H_2}(x_b,\mu_F)$$

$$\times\frac{d\hat{\sigma}_{ab}^{c,\text{Born}}(\hat{s},\hat{p}_T,\hat{\eta},\mu_F,\mu'_F,\mu_R)}{v\,dv\,dw}\times D_c^h(z_h,\mu''_F)$$

$$\propto\quad D_c^h(z_h,\mu''_F)\,.$$

pp→(Jet h) X puzzle for Heavy H

$$\frac{d\sigma^{pp\to(\text{jet }h)X}}{dp_T^{\text{jet}}d\eta^{\text{jet}}dz_h} = \frac{2p_T^{\text{jet}}}{S}\sum_{a,b,c}\int_{x_a^{\min}}^1 \frac{dx_a}{x_a} f_a(x_a,\mu)\int_{x_b^{\min}}^1 \frac{dx_b}{x_b} f_b(x_b,\mu)\int_{z_c^{\min}}^1 \frac{dz_c}{z_c^2}\frac{d\hat{\sigma}_{ab}^c(\hat{s},\hat{p}_T,\hat{\eta},\mu)}{vdvdw}\mathcal{G}_c^h(z_c,z_h,\mu,R),$$

QCD: NLO

T. Kaufmann, A. Mukherjee, W. Vogelsang, Phys. Rev. D92 (2015), no. 5 054015

$$\mathcal{G}_c^{h,\text{QCD}} = \sum_e j_{c\to e}(z_c,R,\mu)\sum_{c'}\int_{z_h}^1 \frac{dz_p}{z_p}\tilde{j}_{e\to c'}(z_p,R,\mu)\, D_{c'}^h\left(\frac{z_h}{z_p},\mu\right)$$

SCET: NLO+NLL_R

Zhong-Bo Kang, Felix Ringer, Ivan Vitev JHEP 1611 (2016) 155

See F.Ringer Talk

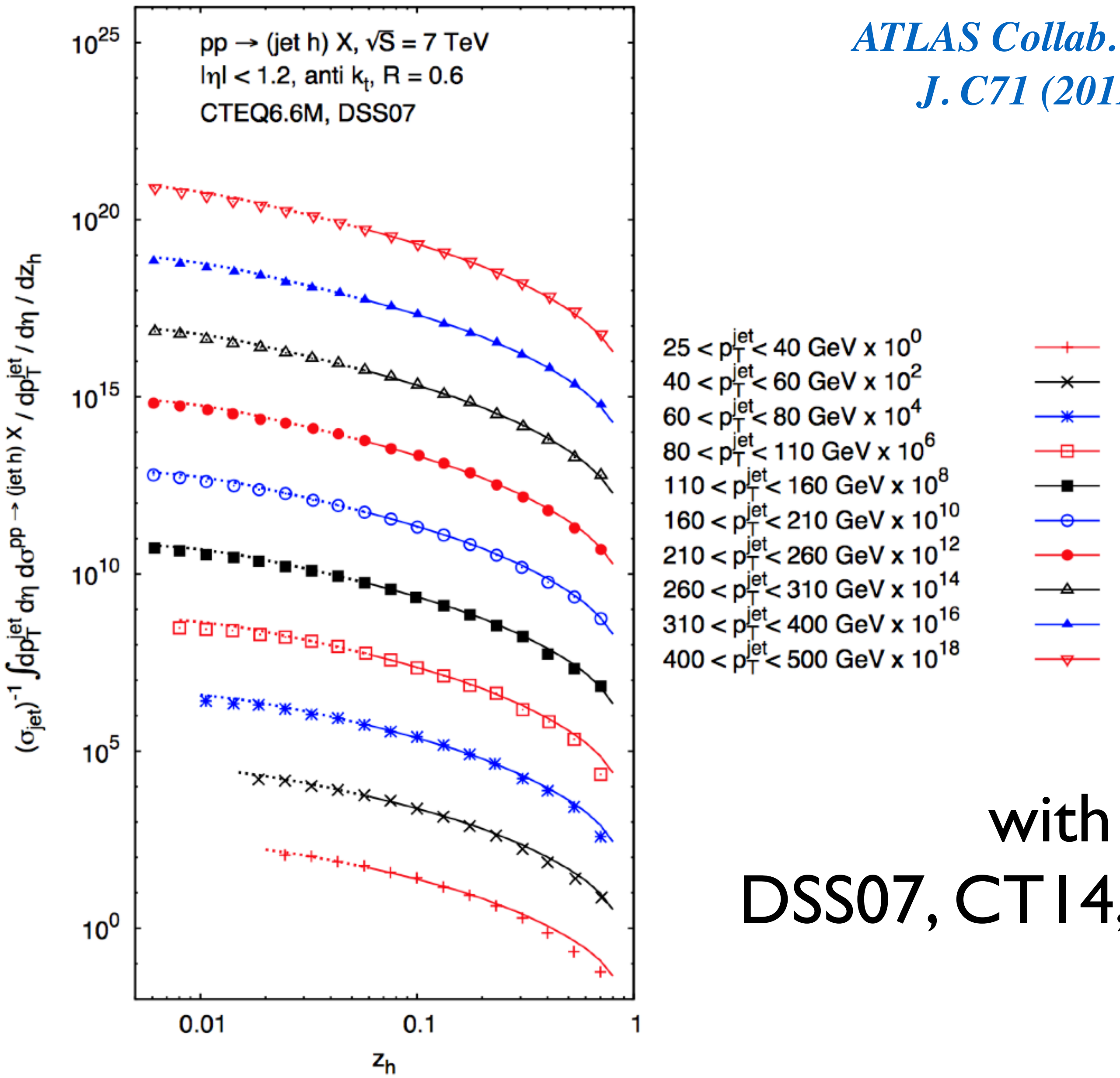
$$\mathcal{G}_c^{h,\text{SCET}}(z,z_h,\omega_J R,\mu) = \sum_j\int_{z_h}^1 \frac{dz_h'}{z_h'}\mathcal{J}_{ij}(z,z_h',\omega_J R,\mu)D_j^h\left(\frac{z_h}{z_h'},\mu\right)$$

$\alpha_s^n \ln^n R$ resummed through DGLAP →

$$\frac{d}{d\ln\mu^2}\mathcal{G}_i^h(z,z_h,\omega_J R,\mu) = \sum_j\int_z^1 \frac{dz'}{z'}P_{ji}^T\left(\frac{z}{z'},\mu\right)\mathcal{G}_j^h(z',z_h,\omega_J R,\mu)$$

LIGHT CHARGED HADRONS IN JET

QCD

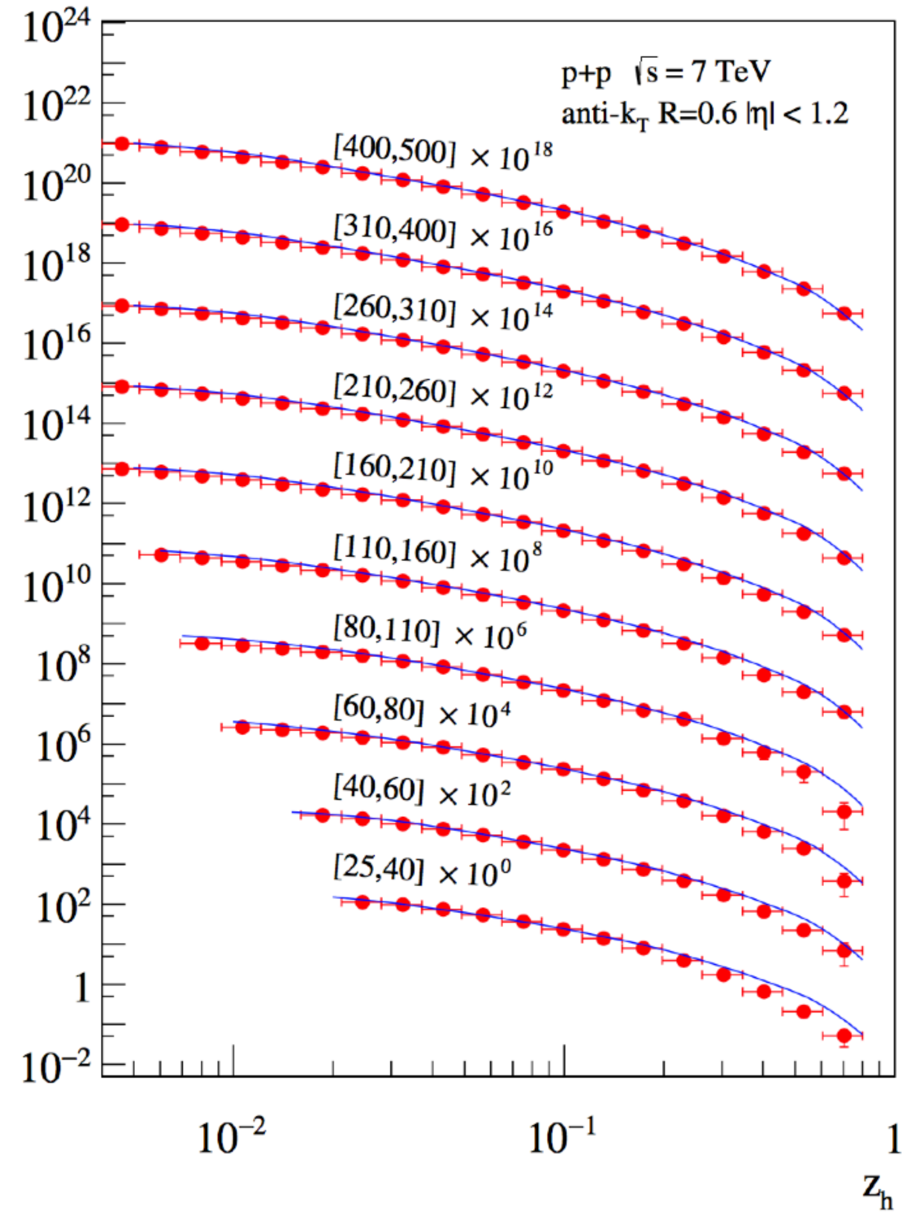


T. Kaufmann, A. Mukherjee, W. Vogelsang, Phys. Rev. D92 (2015), no. 5 054015

ATLAS Collab. Eur. Phys. J. C71 (2011) 1795

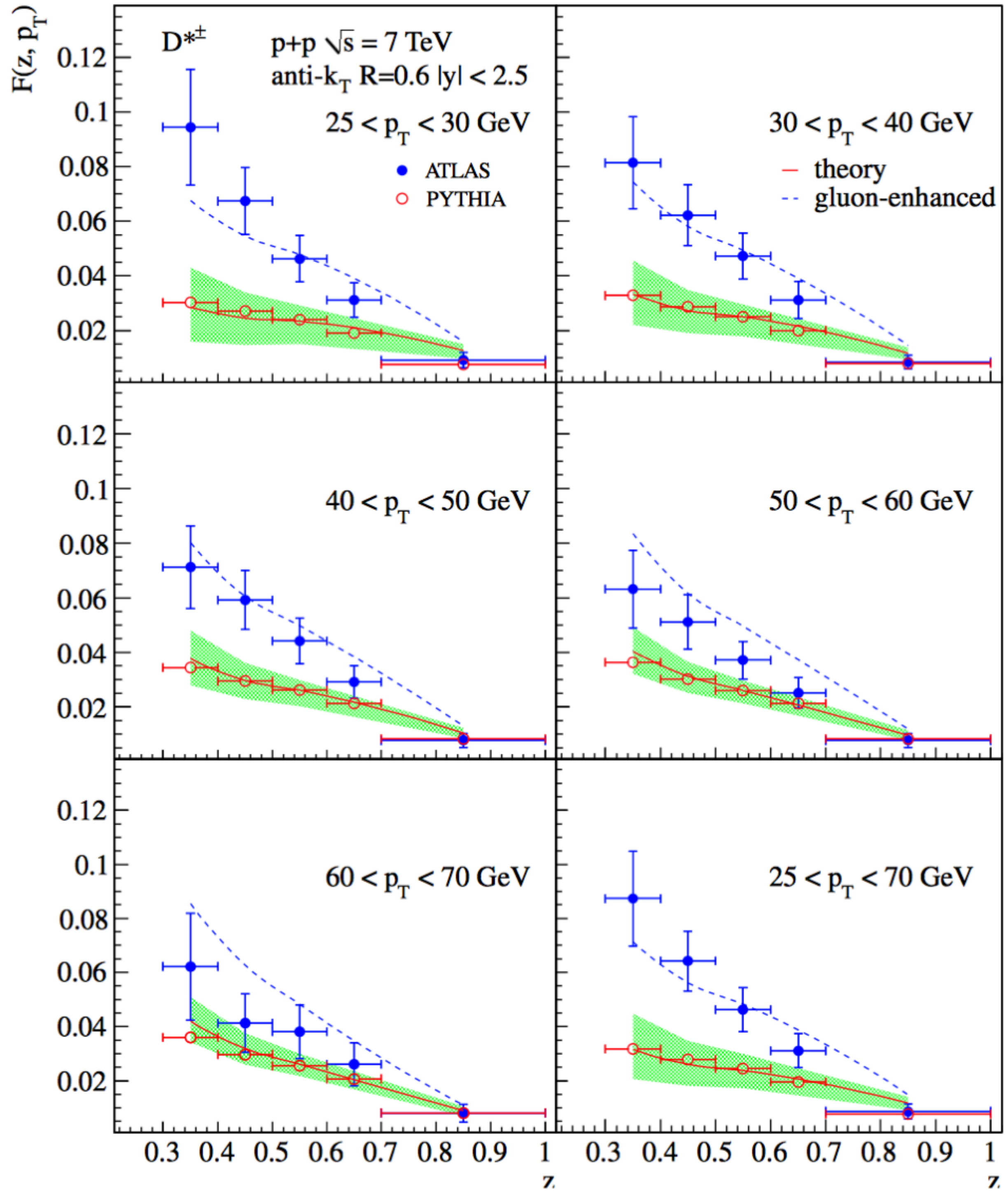
with
DSS07, CT14,CT6.6M

SCET



Zhong-Bo Kang, Felix Ringer, Ivan Vitev JHEP 1611 (2016) 155

$D^{*\pm}$ HADRONS IN JET



ATLAS Collab. Phys. Rev. D85 (2012) 052005

with CT14 and KKK08 FF

T. Kneesch, B. A. Kniehl, G. Kramer, and I. Schienbein, Nucl. Phys. B799 (2008) 34–59

KKK08 FF gluon poorly constrained only from e+e- data

with $D_g^D(z,\mu) \to 2\, D_g^D(z,\mu)$ guess
data well described

CALL FOR A GLOBAL FIT

Yang-Ting Chien, Zhong-Bo Kang, Felix Ringer, Ivan Vitev, Hongxi Xing JHEP 1605 (2016) 125

OUR $D^{*\pm}$ FIT

DISCLAIMER: last week erratum on LHCb JHEP 1603,159(2016)
published on arXiv:1510.01707v6

g & Heavy Flavour parametrisation

$$D_i^{D^{*+}}(z,\mu_0^2) = \frac{N_i\, z^{\alpha_i}(1-z)^{\beta_i}}{B[2+\alpha_i, \beta_i+1]}$$

g & charm input $\mu_0 = m_c = 1.3$ Ge

bottom input $\mu_0^b = m_b =$

Conside

0GeV

L parametrisation

$$D_{i_{\text{light}}}^{D^{*+}}(z,\mu_0^2) = 0$$

experiment				
ALEPH [80]				38
OPAL [81]				6.999
				8.388
			9	5.342
		$D^{*\pm}$	5	3.598
		D^{*+}	3	0.126
	6 TeV	D^{*+}	1	0.007
.91, 241804 (2003)		D^{*+}	2	1.289
	$2 \leq \eta \leq 2.5$	$D^{*\pm}$	5	10.984
013 (2016)	$2.5 \leq \eta \leq 3$	$D^{*\pm}$	5	2.607
	$3 \leq \eta \leq 3.5$	$D^{*\pm}$	5	8.229
	$3.5 \leq \eta \leq 4$	$D^{*\pm}$	2	10.411
ATLAS [68]	$25 \leq \frac{p_T^{\text{jet}}}{\text{GeV}} \leq 30$	(jet $D^{*\pm}$)	5	4.146
Phys. Rev. D 85, 052005 (2012)	$30 \leq \frac{p_T^{\text{jet}}}{\text{GeV}} \leq 40$	(jet $D^{*\pm}$)	5	1.977
	$40 \leq \frac{p_T^{\text{jet}}}{\text{GeV}} \leq 50$	(jet $D^{*\pm}$)	5	0.659
	$50 \leq \frac{p_T^{\text{jet}}}{\text{GeV}} \leq 60$	(jet $D^{*\pm}$)	5	0.791
	$60 \leq \frac{p_T^{\text{jet}}}{\text{GeV}} \leq 70$	(jet $D^{*\pm}$)	5	1.333
TOTAL:			97	100.980

OUR $D^{*\pm}$ FIT

g & Heavy Flavour parametrisation

$$D_i^{D^{*+}}(z,\mu_0^2) = \frac{N_i\, z^{\alpha_i}(1-z)^{\beta_i}}{B[2+\alpha_i,\beta_i+1]}$$

g & charm input $\mu_0 = m_c = 1.3$ GeV

bottom input $\mu_0^b = m_b = 4.75$ GeV

Considered massless (ZMVFNS)
pT > pTcut=5 or 10GeV

Light Flavours parametrisation

$$D_{i_{\text{light}}}^{D^{*+}}(z,\mu_0^2) = 0$$

experiment		data type	#data in fit	χ^2
ALEPH [80] *Eur. Phys. J. C 16, 597 (2000)*		incl.	17	33.738
OPAL [81] *Z. Phys. C 67, 27 (1995)*		incl.	9	6.999
		c tag	9	8.388
		b tag	9	5.342
ATLAS [94] *Nucl. Phys. B 907, 717 (2016).*		$D^{*\pm}$	5	3.598
ALICE [60, 61] *JHEP 1201, 128 (2012)*	$\sqrt{S} = 7$ TeV	D^{*+}	3	0.126
JHEP 1207, 191 (2012)	$\sqrt{S} = 2.76$ TeV	D^{*+}	1	0.007
CDF [62] *Phys. Rev. Lett.91, 241804 (2003)*		D^{*+}	2	1.289
LHCb [64]	$2 \leq \eta \leq 2.5$	$D^{*\pm}$	5	10.984
JHEP 1609, 013 (2016)	$2.5 \leq \eta \leq 3$	$D^{*\pm}$	5	2.607
	$3 \leq \eta \leq 3.5$	$D^{*\pm}$	5	8.229
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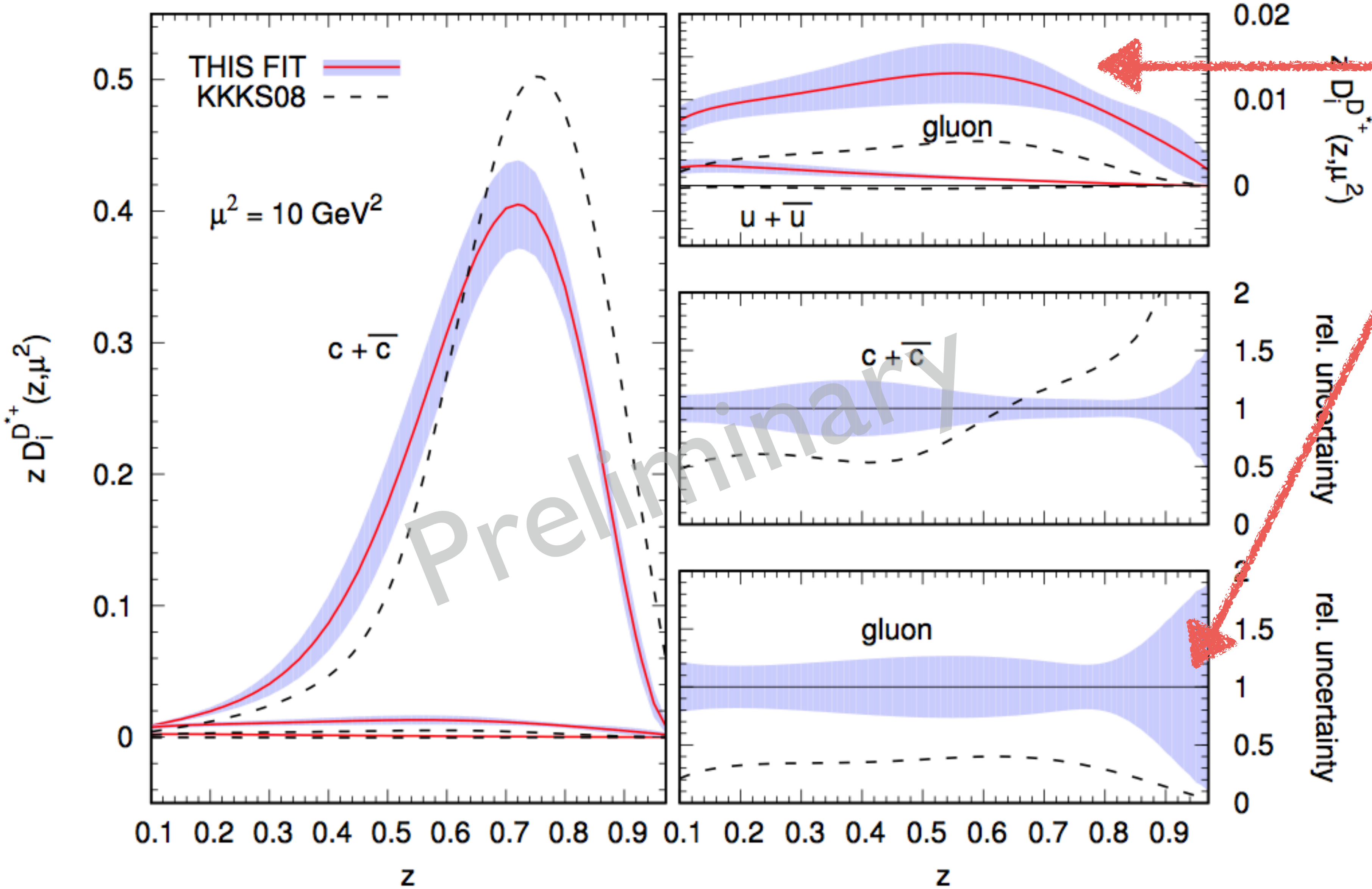
	$40 \leq \frac{p_T^{\text{jet}}}{\text{GeV}} \leq 50$	(jet $D^{*\pm}$)	5	0.659
	$50 \leq \frac{p_T^{\text{jet}}}{\text{GeV}} \leq 60$	(jet $D^{*\pm}$)	5	0.791
	$60 \leq \frac{p_T^{\text{jet}}}{\text{GeV}} \leq 70$	(jet $D^{*\pm}$)	5	1.333
TOTAL:			97	100.980

OUR $D^{*\pm}$ FIT

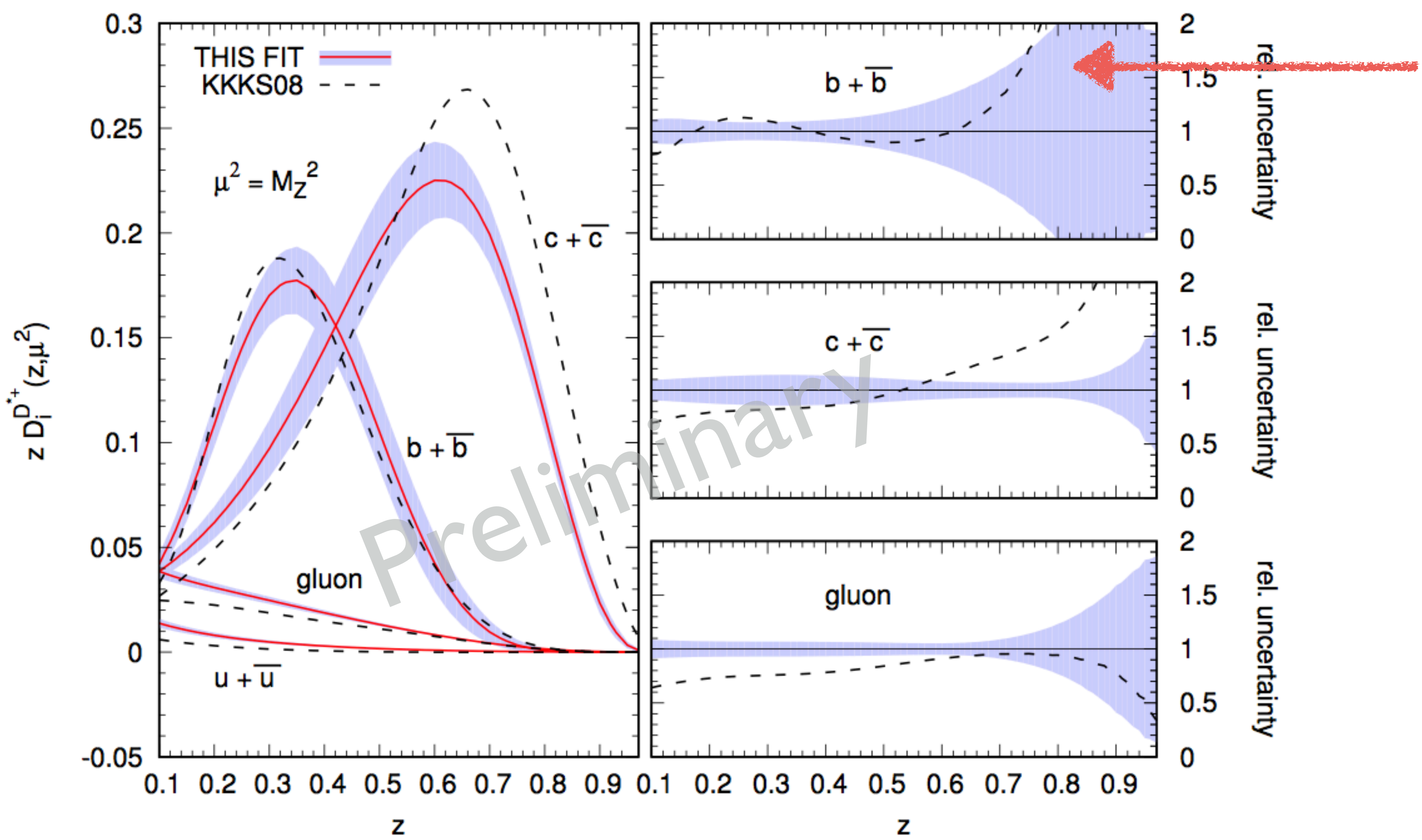


Confirms

JHEP 1605 (2016) 125

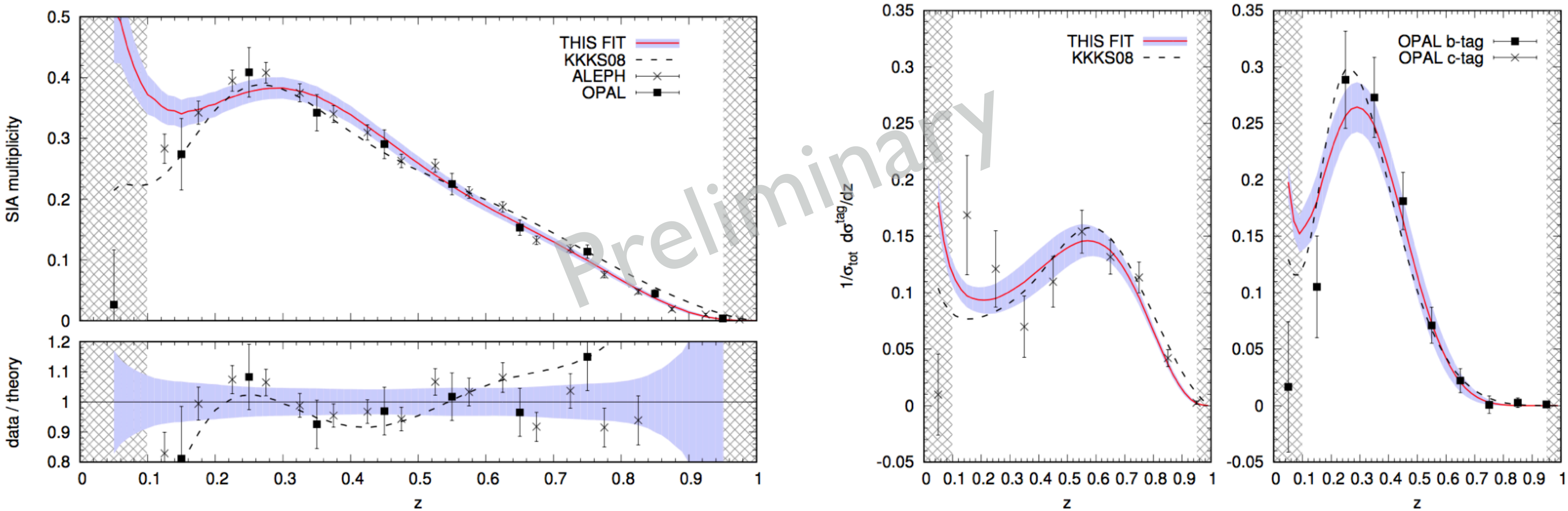
Guess

OUR $D^{*\pm}$ FIT

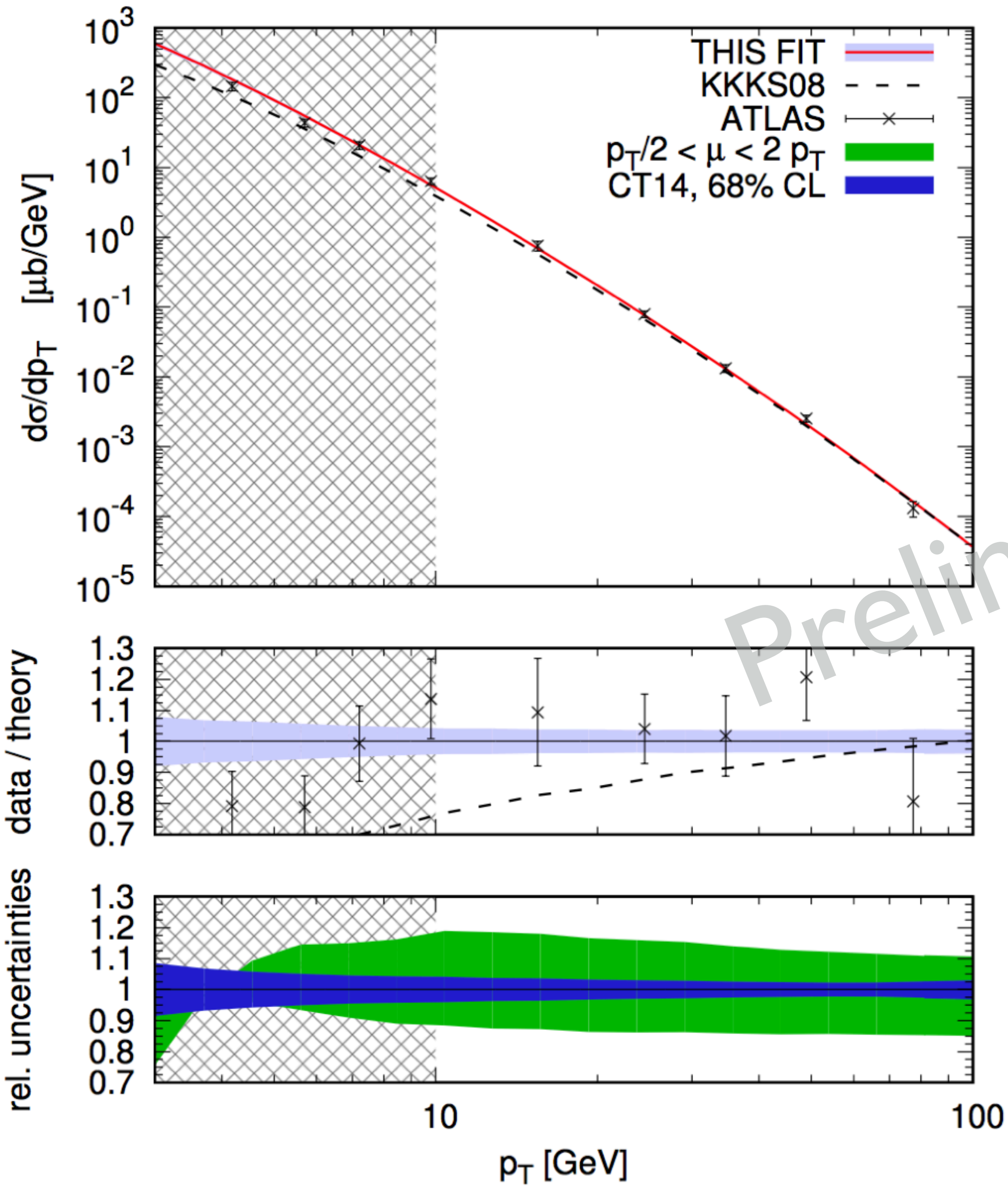


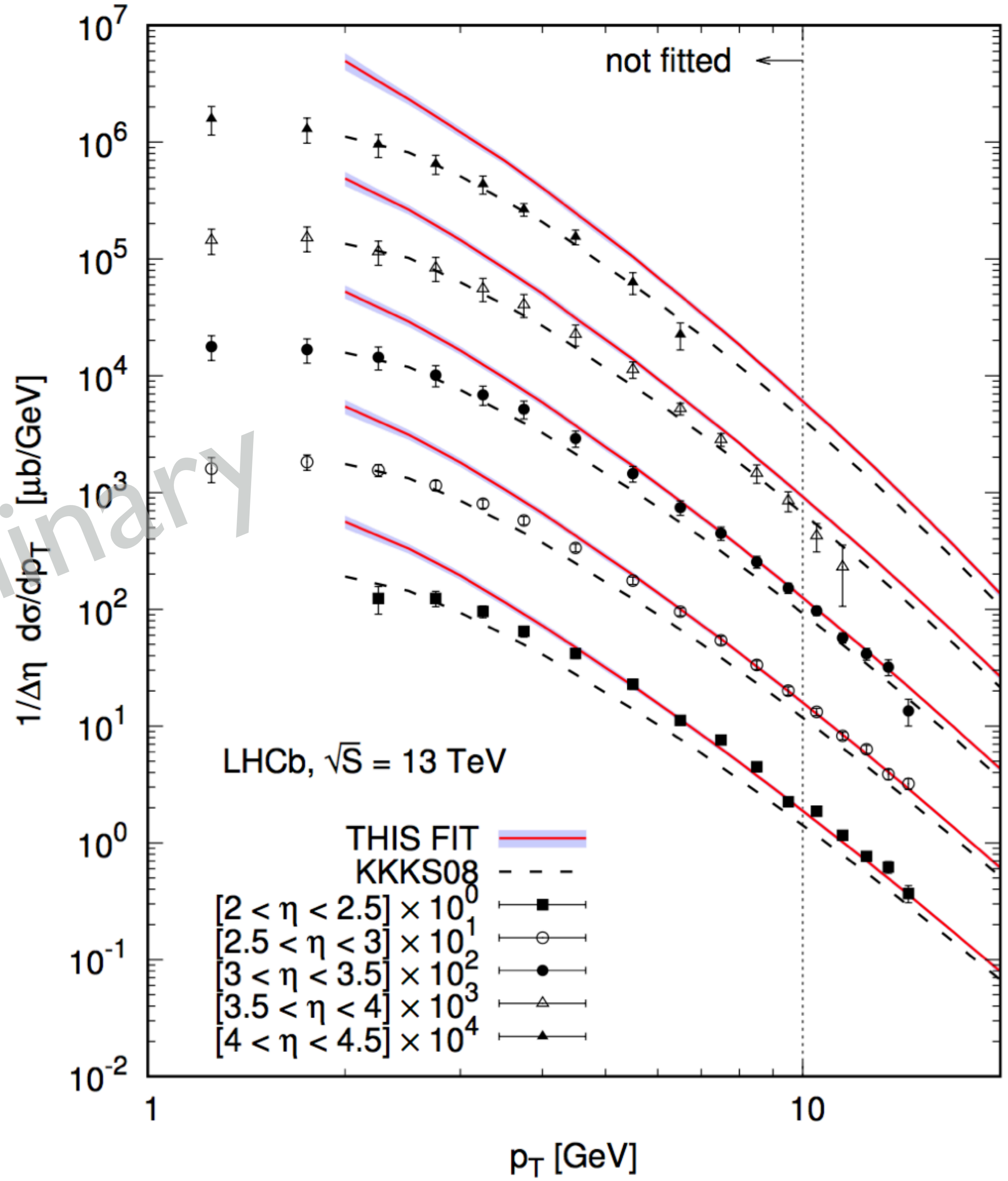
KKK08
b FF within
our errors

Theory vs data: SIA

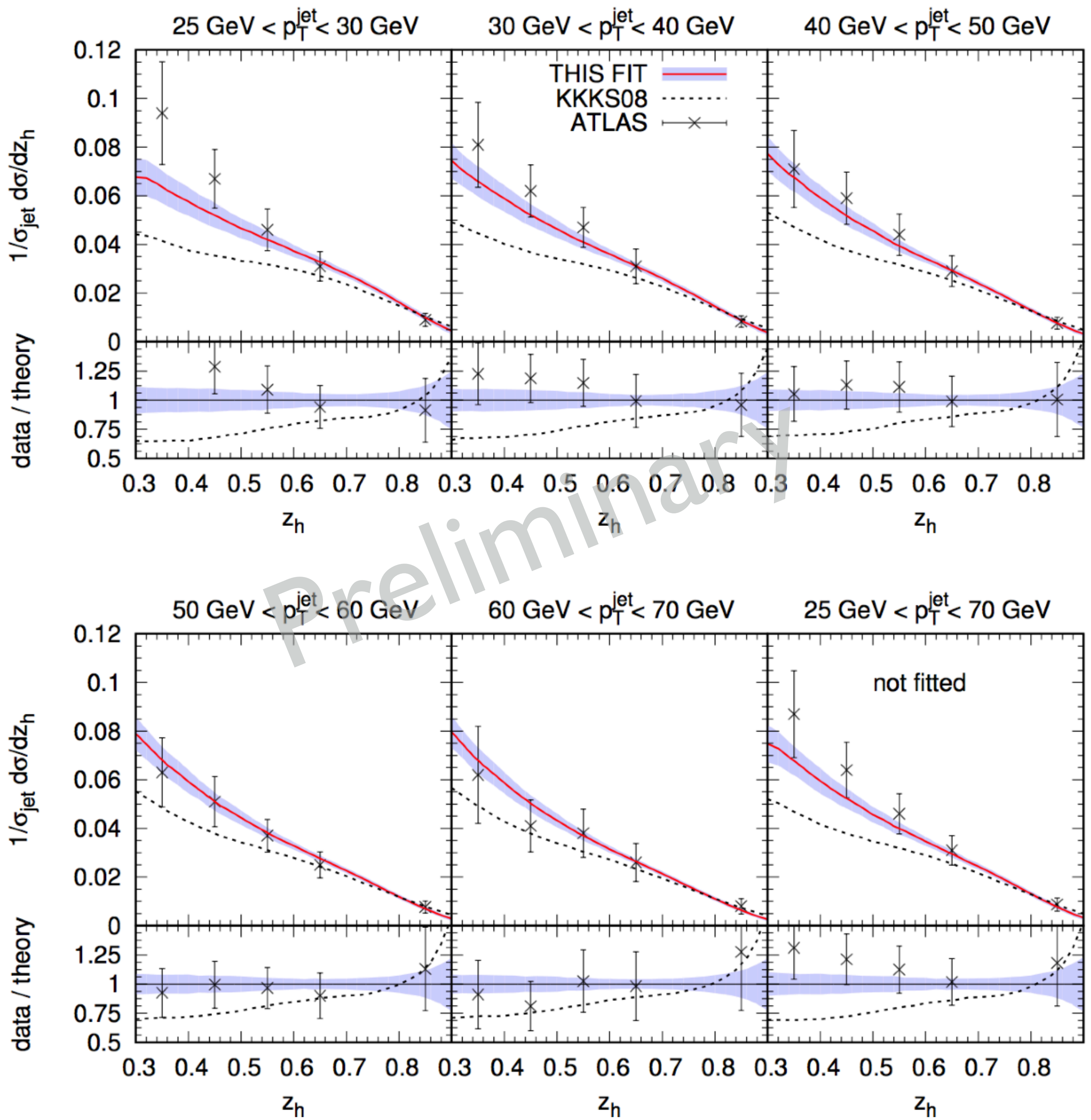


Theory vs data: pp→hX





THEORY VS DATA: PP→(JET H)X



Confirms what seen in *JHEP 1605 (2016) 125* :
with KKK08
bad agreement

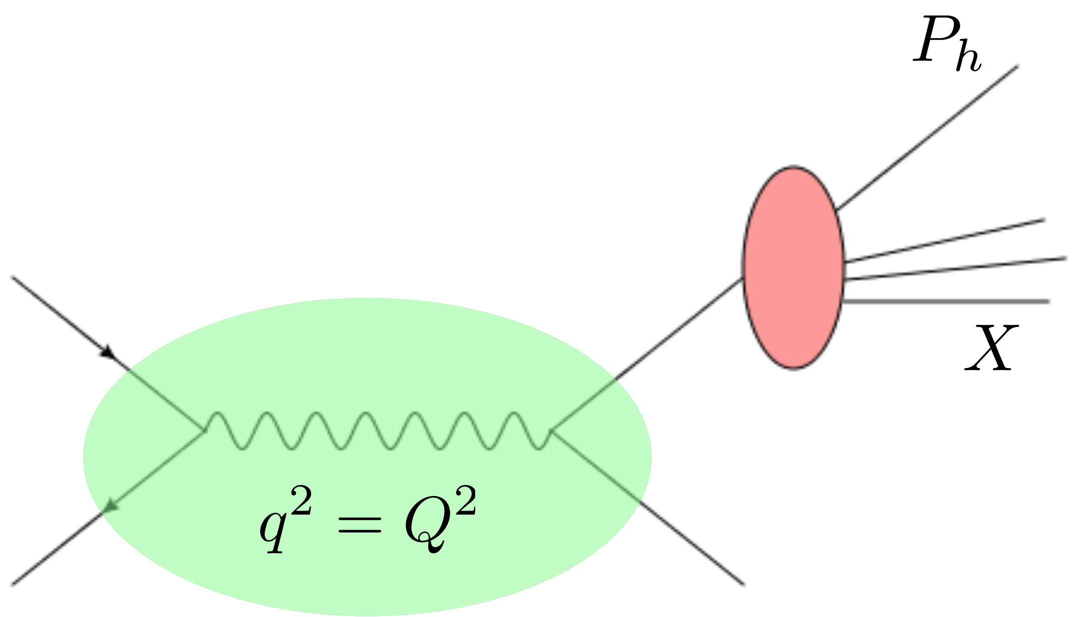
Conclusions

- We have performed a NNLO fit for SIA exploring the new features appearing @ NNLO. At NNLO it is clear that the extreme phase space regions are better described and that more of the small z and large z logs are taken into account.
- We have presented our results when including small z effects. For all practical uses, we conclude that NNLO results already capture most of the small-z dynamics relevant for phenomenology.
- We have presented a first preliminary global fit of D* FF using for the first time hadron in jet observable to constrain the gluon FF
- The gluon distribution seams to agree with *Kang et al.(JHEP 1605 (2016) 125)* guess and at the same time consistent with pp->HX data

Speaker:
Daniele Paolo Anderle
School of Physics & Astronomy

Thanks for Your Attention

Single-inclusive e^+e^- Annihilation



Hadron multiplicities

$$R^h_{e^+e^-} \equiv \frac{1}{\sigma^{\rm tot}} \frac{d^2\sigma^h}{dx_E\, d\cos\theta}$$

where

$$x_E \equiv \frac{2P_h \cdot q}{Q^2}$$

$$\sigma^{\rm tot} = \frac{4\pi\alpha^2}{3Q^2} N_C \sum_q e_q^2 \left(1 + \frac{\alpha_s}{\pi}\right)$$

$$\frac{d^2\sigma^h}{dx_E d\cos\theta} = \frac{\pi\alpha^2}{Q^2} N_C \left[\frac{1+\cos^2\theta}{2}\mathcal{F}^h_T(x_E, Q^2) + \sin^2\theta\, \mathcal{F}^h_L(x_E, Q^2)\right]$$

Nason, Webber; Furmanski, Petronzio

In Collinear Factorization

Nason, Webber; Furmanski, Petronzio

$$\mathcal{F}^h_i(x_E, Q^2) = \sum_f \int_{x_E}^1 \frac{d\hat{z}}{\hat{z}} D^h_f\left(\frac{x_E}{\hat{z}}, \mu^2\right) \mathcal{C}^i_f\left(\hat{z}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$$

Resummed Matched solution for evolution

...doing the Matching as

$$\begin{cases} \boldsymbol{P}^{T,(k)} \equiv \boldsymbol{P}^{T\ \mathrm{FO},(k)} & k \le m \\ \boldsymbol{P}^{T\ \mathrm{N}^{\kappa}\mathrm{LL},(k)} & k > m \end{cases}$$

This preserves the total momentum conservation sum rule

$$P^T_{qq}(N=2) + P^T_{gq}(N=2) = 0$$
$$P^T_{gg}(N=2) + P^T_{qg}(N=2) = 0$$

up to errors of few per-mille

The NNLO Evolution Code "Pegasus_FF"

Existing NNLO Evolution CODES:

X-SPACE APFEL(time-like version C/C++, Fortran77, Python)
Bertone1,Carrazza, Rojo (CERN-PH-TH/2013-209)

Mellin SPACE MELA(Fortran77)
Bertone1,Carrazza, Nocera (CERN-PH-TH-2014-265)

Benchmarked but NNLO matching for VFNS still missing

Our Mellin Evolution Program:

Mellin SPACE Pegasus_FF (Fortran77) → based on Pegasus(Fortran77)
Anderle, Ringer, Stratmann — Vogt (Comput.Phys.Commun.170:65-92,2005)

TOWARDS A GLOBAL NNLO FF FIT

Anderle, Ringer, Stratmann

Ingredients needed to achieve the goal:

NNLO COEFFICINT FUNCTIONS:

SIDIS → Soft gluon Resummed results (can be expanded @ NNLO)

Anderle, Ringer, Vogelsang (Phys.Rev. D87 (2013) 094021,
Phys.Rev. D87 (2013) 3, 034014)

SI- p(anti-)p → Soft gluon Resummed results (can be expanded @ NNLO)

Work in progress for $\frac{d\sigma}{dp_T d\eta}^{(NNLL)}$ Hinderer, Ringer, Sterman, Vogelsang
(continuing from di-hadron production Phys.Rev. D91 (2015) no.1, 014016)

pp→(Jet h)X → Resummed results (can be expanded @ NNLO)

Work in progress from T. Kaufmann, Vogelsang
SCET approach see Kang, Ringer, Vitev arXiv:1610.02043

Towards a Global NNLO FF Fit

Anderle, Ringer, Stratmann

Ingredients needed to achieve the goal:

NNLO COEFFICINT FUNCTIONS:

attempt to do everything Analytically calculated

To include the last processes we need a

NNLO Mellin Space Fitting Program

Resummation via Unfactorized SIA

van Neerven, Rijken (1996)
Vogt (2011), Kom, Vogt, Yeats(2012)

One can proceed by using "all-order" mass factorization: e.g.

A) starting from the *unfactorized gluon singlet transversal parton structure function in dimensional regularisation (IR-singularities not yet factorized out and "re-absorbed" in FF)*

$$\hat{\mathcal{F}}_g^T(N, a_s, \epsilon) = \sum_{i=q,g} \bar{C}_i^T(N, a_s, \epsilon)\, \Gamma_{ig}^N(N, a_s, \epsilon)$$

D-Dimensional coef. function:
only positive powers of ϵ

$$\bar{C}_i^T(N, a_s, \epsilon) = \delta_{iq} + \sum_{l=1}^{\infty} a_s^l \sum_{k=0}^{\infty} \epsilon^k \bar{c}_{T,i}^{(l,k)}(N)$$

Transition function:
incorporates all IR $1/\epsilon$ poles, calculable order by order as a combination of splitting functions

$$\beta_D(a_s)\, \frac{\partial \Gamma_{ik}}{\partial a_s}\, \Gamma_{kj}^{-1} = P_{ij}$$

B) "Plug-in" the small $\bar{N} = N - 1$ limit for known fixed order coefficient functions and splitting functions (e.g. NNLO)

C) Impose equality order by order in a_s with the small $\bar{N} = N - 1$ limit for the unfactorized structure function which reads

A.Vogt JHEP10 (2011) 025

$$\hat{\mathcal{F}}_g^{T,(n)}(N,\epsilon) = a_s^n \frac{1}{\epsilon^{2n-1}} \sum_{l=0}^{n-1} \frac{1}{N-1-2(n-l)\epsilon}\left(A_{T,g}^{(l,n)} + \epsilon B_{T,g}^{(l,n)} + \epsilon^2 C_{T,g}^{(l,n)} + \dots\right)$$

LL NLL NNLL

D) solve recursively order by order for $\bar{c}_{T,i}^{(n,k)}$ $P_{ij}^{(n-1)}$ $A_{T,g}^{(l,n)}$ $B_{T,g}^{(l,n)}$ $C_{T,g}^{(l,n)}$:

- **KLN - Cancellations**
- **fixed order calculation constrains**

→ System of equation solvable

E) From the coefficient of the small N expansion deduct closed form

THE TIME-LIKE EVOLUTION

In the factorisation procedure, the absorption of *collinear singularities* by fragmentation functions (FF)(in case of massless partons) leads to scaling violation and the appearance of a factorisation scale μ_F

The scale dependance of FF is governed by the Time-Like DGLAP

$$\frac{\partial}{\partial \ln \mu_F^2} D_i^h(x, \mu_F^2) = \sum_j \int_x^1 \frac{dy}{y} P_{ji}\left(y, \alpha_s(\mu_F^2)\right) D_j^h\left(\frac{x}{y}, \mu_F^2\right)$$

Time-Like Splitting function perturbatively calculable $P_{ji}(y, \alpha_s) = \sum_{k=0} a_s^{k+1} P_{ji}^{(k)}(y)$

Usually rewritten into $2n_f - 1$ equations (charge conjugation and flavour symmetry)

NON-SINGLET

$$D^h_{\mathrm{NS};v} = \sum_{i=1}^{n_f} (D^h_{q_i} - D^h_{\bar{q}_i})$$

$$D^h_{\mathrm{NS};\pm} = (D^h_{q_i} \pm D^h_{\bar{q}_i}) - (D^h_{q_j} \pm D^h_{\bar{q}_j})$$

$$\frac{\partial}{\partial \ln \mu_F^2} D^h_{\mathrm{NS};\pm,v}(x, \mu_F^2) = P^{\pm,\mathrm{v}}(x, \mu_F^2) \otimes D^h_{\mathrm{NS};\pm,v}(x, \mu_F^2)$$

and two coupled

SINGLET

$$D^h_{\Sigma} = \sum_{i=1}^{n_f} (D^h_{q_i} + D^h_{\bar{q}_i})$$

$$D^h_g$$

$$\frac{\partial}{\partial \ln \mu_F^2} \begin{pmatrix} D^h_{\Sigma}(x, \mu_F^2) \\ D^h_g(x, \mu_F^2) \end{pmatrix} = \begin{pmatrix} P^{\mathrm{qq}} & 2n_f P^{\mathrm{gq}} \\ \frac{1}{2n_f} P^{\mathrm{qg}} & P^{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} D^h_{\Sigma}(x, \mu_F^2) \\ D^h_g(x, \mu_F^2) \end{pmatrix}$$

THE SOLUTION

We can solve the integro-differential DGLAP equation analytically in Mellin space at each fixed order since it becomes an Ordinary Differential Equation

$$\frac{\partial \boldsymbol{q}(N,a_s)}{\partial a_s} = \{\beta_{\mathrm{N^mLO}}(a_s)\}^{-1} \boldsymbol{P}_{\mathrm{N^mLO}}(N,a_s)\,\boldsymbol{q}(N,a_s)$$

$$= -\frac{1}{\beta_0 a_s}\Big[\boldsymbol{P}^{(0)}(N) + a_s\big(\boldsymbol{P}^{(1)}(N) \ldots b_1 \boldsymbol{P}^{(0)}(N)\big)$$

$$+ a_s^2\big(\boldsymbol{P}^{(} \ldots (b_1^2 - b_2)\boldsymbol{P}^{(0)}(N)\big) + \ldots\Big]\,\boldsymbol{q}(N,a_s)$$

$$f(N,\alpha_s) = \int_0^1 dy\, y^{N-1} f(y,\alpha_s) \qquad N \in \mathbb{C}$$

where here $\mathbf{P}(N,\alpha_S)$ and $\mathbf{q}(N,\alpha_S)$ are the Mellin-Transform of either singlet or non-singlet splitting function and FF respectively the general solution can be expressed in terms of the evolution matrices ***U*** (constructed from the splitting functions) as a simple multiplication

$$\begin{aligned}\boldsymbol{q}(N,a_s) &= \boldsymbol{U}(N,a_s)\,\boldsymbol{L}(N,a_s,a_0)\,\boldsymbol{U}^{-1}(N,a_0)\,\boldsymbol{q}(N,a_0)\\ &= \left[1+\sum_{k=1}^{\infty} a_s^k\,\boldsymbol{U}_k(N)\right]\boldsymbol{L}(a_s,a_0,N)\left[1+\sum_{k=1}^{\infty} a_0^k\,\boldsymbol{U}_k(N)\right]^{-1}\boldsymbol{q}(a_0,N)\end{aligned}$$

where ***L*** is defined by the LO solution

$$\boldsymbol{q}_{\rm LO}(N,a_s,N) = \left(\frac{a_s}{a_0}\right)^{-\boldsymbol{R}_0(N)}\boldsymbol{q}(N,a_0) \equiv \boldsymbol{L}(N,a_s,a_0)\,\boldsymbol{q}(N,a_0)$$

$$\boldsymbol{R}_0 \equiv \frac{1}{\beta_0}\boldsymbol{P}^{(0)}$$

TRUNCATED AND ITERATED SOLUTION

Since both $\beta_{\mathrm{N^mLO}}$ and $\boldsymbol{P}_{\mathrm{N^mLO}}$ have an expansion in powers of α_s

there are different ways of defining the N^m^LO solution

$$\begin{aligned}\boldsymbol{q}_{\mathrm{N^3LO}}(a_\mathrm{s}) \;=\; \Big[& \boldsymbol{L} + a_\mathrm{s}\,\boldsymbol{U}_1\,\boldsymbol{L} - a_0\,\boldsymbol{L}\,\boldsymbol{U}_1 \\ & + a_\mathrm{s}^2\,\boldsymbol{U}_2\,\boldsymbol{L} - a_\mathrm{s}a_0\,\boldsymbol{U}_1\,\boldsymbol{L}\,\boldsymbol{U}_1 + a_0^2\,\boldsymbol{L}\left(\boldsymbol{U}_1^2 - \boldsymbol{U}_2\right) \\ & + a_\mathrm{s}^3\,\boldsymbol{U}_3\,\boldsymbol{L} - a_\mathrm{s}^2 a_0\,\boldsymbol{U}_2\,\boldsymbol{L}\,\boldsymbol{U}_1 + a_\mathrm{s}a_0^2\,\boldsymbol{U}_1\,\boldsymbol{L}\left(\boldsymbol{U}_1^2 - \boldsymbol{U}_2\right) \\ & - a_0^3\,\boldsymbol{L}\left(\boldsymbol{U}_1^3 - \boldsymbol{U}_1\boldsymbol{U}_2 - \boldsymbol{U}_1\boldsymbol{U}_2 + \boldsymbol{U}_3\right)\Big]\,\boldsymbol{q}(a_0)\end{aligned}$$

$$\boldsymbol{R}_0 \equiv \frac{1}{\beta_0}\boldsymbol{P}^{T,(0)}\,, \quad \boldsymbol{R}_k \equiv \frac{1}{\beta_0}\boldsymbol{P}^{T,(k)} - \sum_{i=1}^{k} b_i \boldsymbol{R}_{k-i}\,,$$

$$[\boldsymbol{U}_k, \boldsymbol{R}_0] = \boldsymbol{R}_k + \sum_{i=1}^{k-1} \boldsymbol{R}_{k-1}\boldsymbol{U}_i + k\boldsymbol{U}_k\,.$$

TRUNCATED AND ITERATED SOLUTION

TRUNCATED: Keep only terms up to $\alpha_s^{\,m}$ in the solution

$$\boldsymbol{q}_{\mathrm{N^3LO}}(a_s) = \Big[\boldsymbol{L} + a_s\,\boldsymbol{U}_1\,\boldsymbol{L} - a_0\,\boldsymbol{L}\,\boldsymbol{U}_1 + a_s^2\,\boldsymbol{U}_2\,\boldsymbol{L} - a_s a_0\,\boldsymbol{U}_1\,\boldsymbol{L}\,\boldsymbol{U}_1 + a_0^2\,\boldsymbol{L}\left(\boldsymbol{U}_1^2 - \boldsymbol{U}_2\right) + a_s^3\,\boldsymbol{U}_3\,\boldsymbol{L} - a_s^2 a_0\,\boldsymbol{U}_2\,\boldsymbol{L}\,\boldsymbol{U}_1 + a_s a_0^2\,\boldsymbol{U}_1\,\boldsymbol{L}\left(\boldsymbol{U}_1^2 - \boldsymbol{U}_2\right) - a_0^3\,\boldsymbol{L}\left(\boldsymbol{U}_1^3 - \boldsymbol{U}_1\boldsymbol{U}_2 - \boldsymbol{U}_1\boldsymbol{U}_2 + \boldsymbol{U}_3\right)\Big]\boldsymbol{q}(a_0)$$

$$\boldsymbol{R}_0 \equiv \frac{1}{\beta_0}\boldsymbol{P}^{T,(0)}\,, \quad \boldsymbol{R}_k \equiv \frac{1}{\beta_0}\boldsymbol{P}^{T,(k)} - \sum_{i=1}^{k} b_i \boldsymbol{R}_{k-i}\,, \qquad [\boldsymbol{U}_k, \boldsymbol{R}_0] = \boldsymbol{R}_k + \sum_{i=1}^{k-1} \boldsymbol{R}_{k-1}\boldsymbol{U}_i + k\boldsymbol{U}_k\,.$$

- It solves the equation exactly only up to terms of order n > m

Truncated and Iterated Solution

ITERATED: Keep the all the m-terms generated from $\beta_{\mathrm{N^mLO}}$ and $\boldsymbol{P}_{\mathrm{N^mLO}}$

$$\boldsymbol{q}_{\mathrm{N^3LO}}(a_s) = \Big[\boldsymbol{L} + a_s\,\boldsymbol{U}_1\,\boldsymbol{L} - a_0\,\boldsymbol{L}\,\boldsymbol{U}_1 + a_s^2\,\boldsymbol{U}_2\,\boldsymbol{L} - a_s a_0\,\boldsymbol{U}_1\,\boldsymbol{L}\,\boldsymbol{U}_1 + a_0^2\,\boldsymbol{L}\left(\boldsymbol{U}_1^2 - \boldsymbol{U}_2\right) + a_s^3\,\boldsymbol{U}_3\,\boldsymbol{L} - a_s^2 a_0\,\boldsymbol{U}_2\,\boldsymbol{L}\,\boldsymbol{U}_1 + a_s a_0^2\,\boldsymbol{U}_1\,\boldsymbol{L}\left(\boldsymbol{U}_1^2 - \boldsymbol{U}_2\right) - a_0^3\,\boldsymbol{L}\left(\boldsymbol{U}_1^3 - \boldsymbol{U}_1\boldsymbol{U}_2 - \boldsymbol{U}_1\boldsymbol{U}_2 + \boldsymbol{U}_3\right)\Big]\boldsymbol{q}(a_0)$$

$$\boldsymbol{R}_0 \equiv \frac{1}{\beta_0}\boldsymbol{P}^{T,(0)}\,, \quad \boldsymbol{R}_k \equiv \frac{1}{\beta_0}\boldsymbol{P}^{T,(k)} - \sum_{i=1}^{k} b_i \boldsymbol{R}_{k-i}\,, \qquad [\boldsymbol{U}_k, \boldsymbol{R}_0] = \boldsymbol{R}_k + \sum_{i=1}^{k-1}\boldsymbol{R}_{k-1}\boldsymbol{U}_i + k\boldsymbol{U}_k\,.$$

- It corresponds to the solution done in x-Space

- It introduces more higher order scheme-dependent terms

TRUNCATED AND ITERATED SOLUTION

ITERATED-TRUNCATED = theoretical uncertainty of order $\mathcal{O}(\alpha_s^{m+1})$

RESUMMED MATCHED SOLUTION FOR EVOLUTION

The $\mathrm{N}^m\mathrm{LO} + \mathrm{N}^\kappa\mathrm{LL}$ matched solution is defined by taking the *iterated solution* and…

$$\boldsymbol{q}(N, a_s) = \boldsymbol{U}(N, a_s)\, \boldsymbol{L}(N, a_s, a_0)\, \boldsymbol{U}^{-1}(N, a_0)\, \boldsymbol{q}(N, a_0)$$

$$= \left[1 + \sum_{k=1}^{\infty} a_s^k\, \boldsymbol{U}_k(N)\right] \boldsymbol{L}(a_s, a_0, N \ldots \Big]^{-1} \boldsymbol{q}(a_0, N)$$

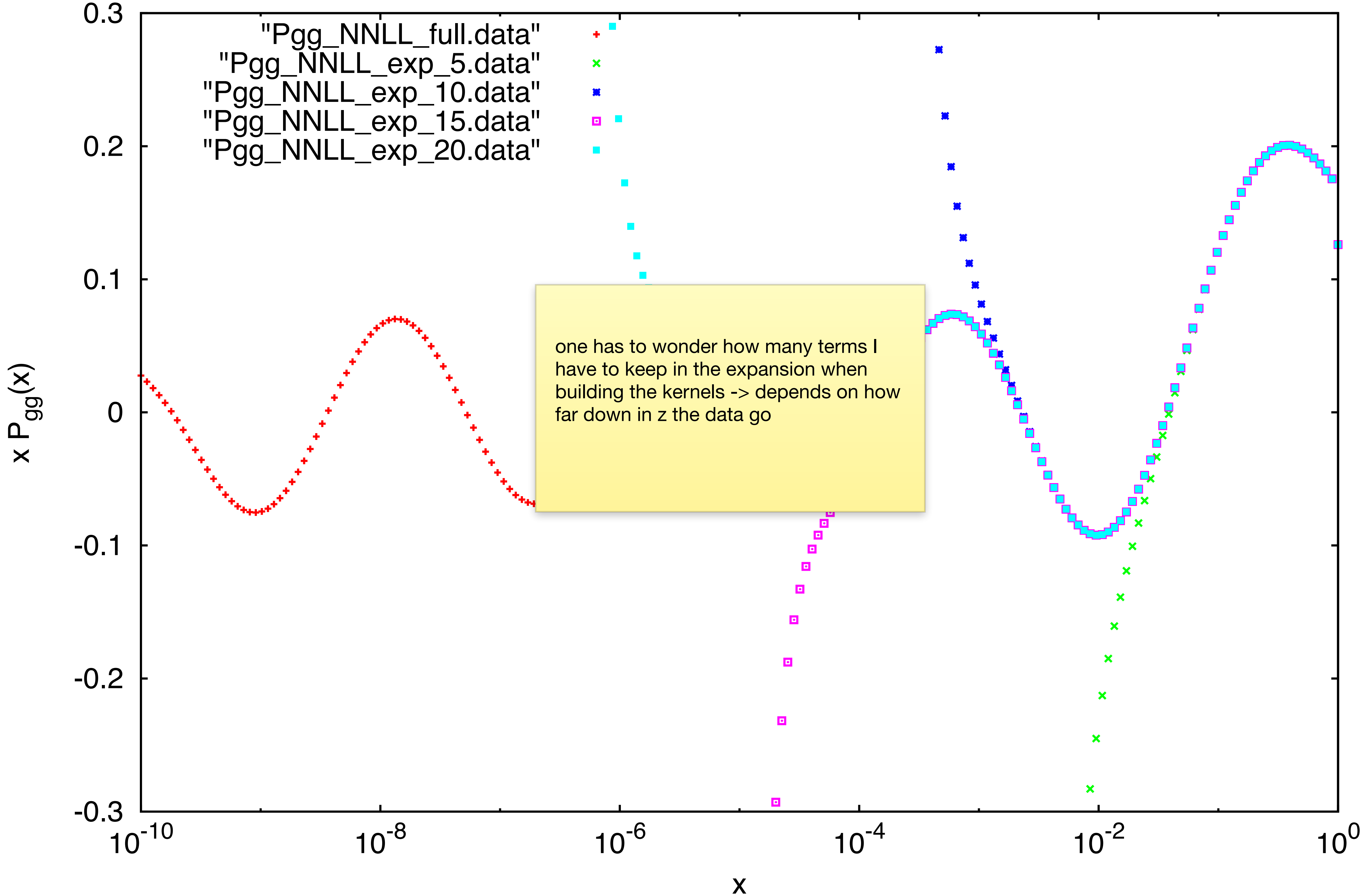
$$\boldsymbol{R}_0 \equiv \frac{1}{\beta_0}\boldsymbol{P}^{T,(0)}\,, \quad \boldsymbol{R}_k \equiv \frac{1}{\beta_0}\boldsymbol{P}^{T,(k)} - \sum \ldots$$

$$[\boldsymbol{U}_k, \boldsymbol{R}_0] = \boldsymbol{R}_k + \sum_{i=1}^{k-1} \boldsymbol{R}_{k-1}\boldsymbol{U}_i + k \ldots$$

Technical issues: evolution-> how many terms

and

inversion



one has to wonder how many terms I have to keep in the expansion when building the kernels -> depends on how far down in z the data go

RESUMMED SCALE DEPENDANCE

In SIA the dependance of the coefficient functions on the factorization scale μ_F can be expressed through the coefficients $c_{k,i}^{(l,m)}$

$$C_{k,i}(N,\alpha_s,\log(Q^2/\mu_F^2)) = c_{k,i}^{(0)}(N) + \sum_{l=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^l \left(c_{k,i}^{(l)}(N) + \sum_{m=1}^{l} c_{k,i}^{(l,m)}(N)\log^m\left(\frac{Q^2}{\mu_F^2}\right)\right)$$

which can be calculated order by order solving the renormalization group equation:

$$\left[\left\{\frac{\partial}{\partial \log \mu_F^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right\}\delta_{ij} - P_{ij}^T\right] C_{k,i}(N,\alpha_s,\log(Q^2/\mu_F^2)) = 0$$

$$k = L,T \qquad i = q,g$$

This leads to the following recursive formula for the coefficients $c_{k,i}^{(l,m)}$

$$c_{k,j}^{(l,m)} = \frac{1}{m}\sum_{w=m-1}^{l-1} c_{k,i}^{(w,m-1)}\left(P_{ij}^{T\ (l-w-1)} - w\beta_{l-w-1}\delta_{ij}\right)$$

RESUMMED SCALE DEPENDANCE

Taking the small $\bar{N} = N - 1$ limit, one can write the $L_M^m = \log^m \left(\frac{Q^2}{\mu_F^2}\right)$ dependance up to NNNLL

LL
$$C_{k,g}^{\mathrm{S,LL},(n)} = c_{k,g}^{\mathrm{LL},(n)}$$

NLL
$$C_{T,g}^{\mathrm{S,NLL},(n)} = c_{T,g}^{\mathrm{NLL},(n)} + L_M \left\{ P_{gq}^{T\,\mathrm{LL},(n-1)} + \sum_{j=0}^{n-2} c_{T,g}^{\mathrm{LL},(n-1-j)} P_{gg}^{T\,\mathrm{LL},(j)} \right\}$$

NNLL
$$C_{T,g}^{\mathrm{S,NNLL},(n)} = c_{T,g}^{\mathrm{NNLL},(n)} + L_M \left\{ P_{gq}^{T\,\mathrm{NLL},(n-1)} - (n-1)\beta_0 c_{T,g}^{\mathrm{LL},(n-1)} + \sum_{j=0}^{n-3} c_{T,q}^{\mathrm{NLL},(n-1-j)} P_{gq}^{T\,\mathrm{LL},(j)} + \sum_{j=0}^{n-2} \left(c_{T,g}^{\mathrm{LL},(n-1-j)} P_{gg}^{T\,\mathrm{NLL},(j)} + c_{T,g}^{\mathrm{NLL},(n-1-j)} P_{gg}^{T\,\mathrm{LL},(j)} \right) \right\}$$
$$+ \frac{L_M^2}{2} \left[\sum_{j=0}^{n-2} P_{gq}^{T\,\mathrm{LL},(n-2-j)} P_{gg}^{T\,\mathrm{LL},(j)} + \sum_{i=0}^{n-3} \sum_{j=0}^{n-2-i} c_{T,g}^{\mathrm{LL},(n-2-i-j)} P_{gg}^{T\,\mathrm{LL},(i)} P_{gg}^{T\,\mathrm{LL},(j)} \right]$$

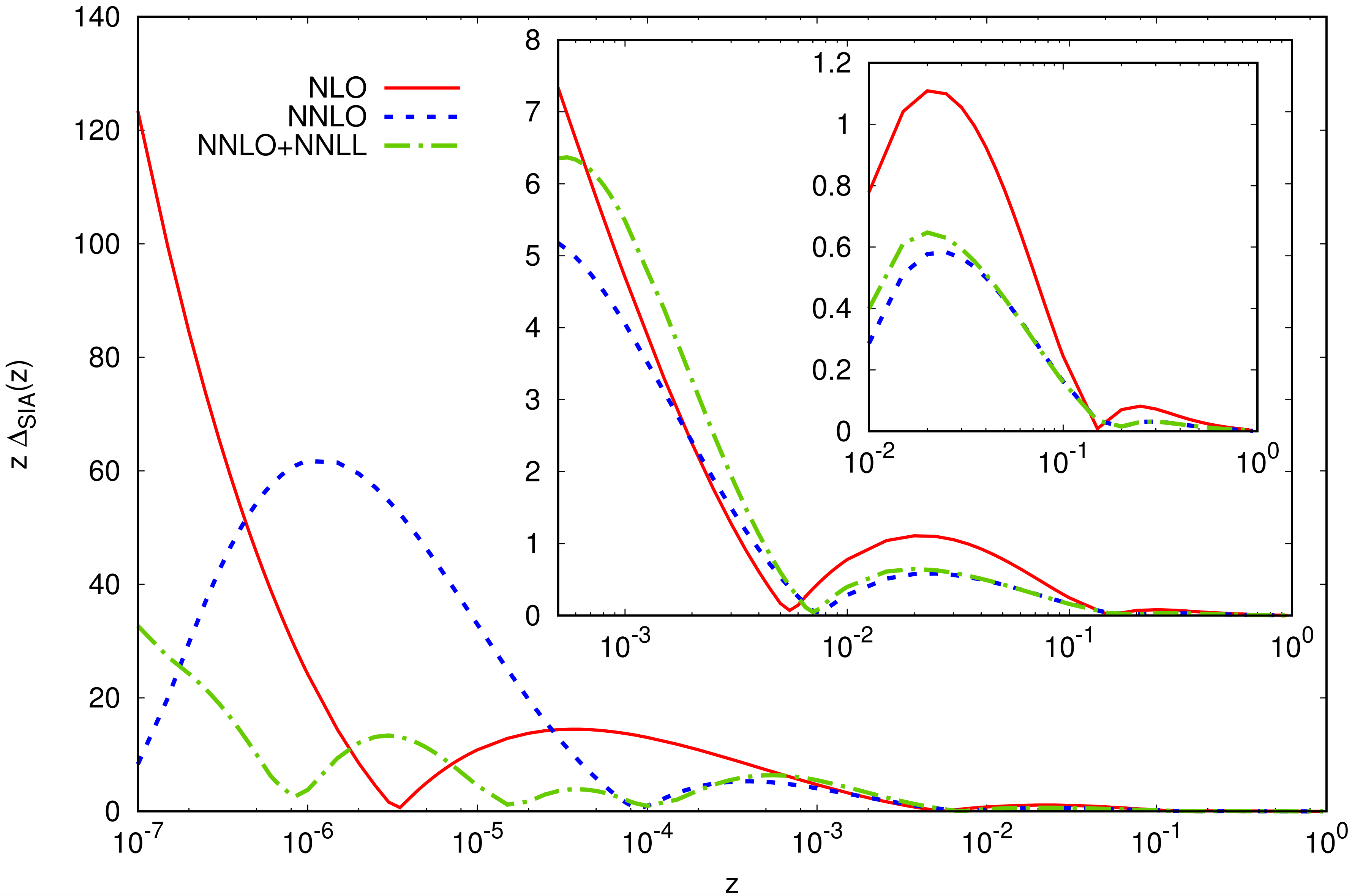
NNNLL → Scale dependance given by NNLL quantities and 3 powers of $\log\left(\frac{Q^2}{\mu_F^2}\right)$

Resummed Scale Dependance

Max Error Band in Pion Multiplicities

$$\Delta_T(z) \equiv \max[T_{\xi=1}(z), T_{\xi=2}(z), T_{\xi=0.5}(z)] - \min[T_{\xi=1}(z), T_{\xi=2}(z), T_{\xi=0.5}(z)]$$

where $\xi \equiv \mu_F^2/Q^2$



$$Q = 10.54\ GeV$$