# Three-loop evolution equation for non-singlet leading-twist operator

based on:
V. M. Braun, A. N. Manashov, S. Moch, M. S., not yet published, arXiv:1703.09532

Matthias Strohmaier

University of Regensburg

given at QCD Evolution 05.25.2017, JLAB

# Motivation & Introduction

- Remarkable archievements in experimental accuracy
- intended to increase even more in near future (e.g. JLAB 12 GeV update or EIC)
- needs to be matched by theoretical accuracy
  $\longrightarrow$ NNLO is becoming the state-of-the-art in most fields
- NNLO evolution for off-forwards distributions needs to be investigated

# Operators under consideration

Non-local "light-ray" twist-two operator $n^2 = 0$ $D_+ := n_\mu D^\mu$

$$[\mathcal{O}^{(n)}(x; z_1, z_2)] = \mathbb{Z}\bar{q}(x + nz_1)\not{n}q(x + nz_2)$$

Generating object for local operators

$$[\mathcal{O}^{(n)}(x; z_1, z_2)] = \sum_{l,m=0}^{\infty} \frac{z_1^l z_2^m}{l!m!}[\mathcal{Z}\mathcal{O}]_{lm}(x)\,, \quad \mathcal{O}_{lm}(x) = \bar{q}(x)(\overleftarrow{D}_+)^l \not{n}(\overrightarrow{D}_+)^m q(x)\,.$$

We tacitely assume different flavors.

Renormalization factor in $\overline{MS}$ - scheme

$$Z = 1 + \sum_{k=0}^{\infty} \frac{1}{\epsilon^k} \sum_{\ell=k}^{\infty} a^\ell Z_k^{(\ell)}\,, \qquad a = \frac{\alpha_s}{4\pi}\,.$$

Relation to evolution kernel

$$\mathbb{H} = -\frac{d}{d\ln\mu}\ln(\mathbb{Z})\,, \qquad \gamma_{lm}^{l'm'} = -\frac{d}{d\ln\mu}\ln(\mathcal{Z}_{lm}^{l'm'})$$

# Evolution equation

## RGE

$$[\mu\partial_\mu + \beta(a)\partial_a + \mathbb{H}]O(x;z_1,z_2) = 0\,,$$

Balitsky, Braun '89

$$[\mu\partial_\mu + \beta(a)\partial_a + \hat{\gamma}]O_{lm}(x) = 0\,.$$

Radyushkin, etc. '84

## QCD $\beta$-function

$$\beta(a) = \mu\frac{\partial a}{\partial\mu} = -2a(\epsilon + \underbrace{a\beta_0 + a^2\beta_1 + \ldots}_{\beta^{\mathrm{QCD}}(a)})\,.$$

## General form

$$[\mathbb{H}\mathcal{O}](z_1,z_2) = \int_0^1 d\alpha \int_0^1 d\beta h(\alpha,\beta)\mathcal{O}(z_{12}^\alpha, z_{21}^\beta)\,, \quad z_{12}^\alpha = (1-\alpha)z_1 + \alpha z_2\,.$$

# Conformal symmetry at the classical level

Three symmetry generators on the light-cone

- Translations: $S_-^{(0)} = -\partial_{z_1} - \partial_{z_2}$
- Dilatations: $S_0^{(0)} = z_1\partial_{z_1} + z_1\partial_{z_1} + 2(j_1 + j_2)$
- Spec. conf. transf.: $S_+^{(0)} = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2j_1 z_1 + 2j_2 z_2$

All three commute with the LO evolution kernel

$$[S_\alpha^{(0)}, \mathbb{H}^{(1)}] = 0$$

# Evolution equations at the LO

Shape of the evolution kernel fixed by $[S_+^{(0)}, \mathbb{H}^{(1)}] = 0$

$$[\mathbb{H}^{(1)}\mathcal{O}](z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta h_{\text{inv}}^{(1)}(\tau)\, O(z_{12}^{\alpha}, z_{21}^{\beta})\,, \qquad \tau = \frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}\,.$$

Anomalous dimension $\gamma_N$ of local operators with spin $N$ are eigenvalues

$$\mathbb{H}(z_1 - z_2)^N = (z_1 - z_2)^N \int_0^1 d\alpha \int_0^1 d\beta h(\alpha, \beta)(1 - \alpha - \beta)^N = (z_1 - z_2)^N \gamma_N$$

Knowledge of anomalous dimension $\gamma_N$ of local operators with spin $N$ allows to reconstruct kernel $h(\alpha, \beta)$, if $h(\alpha, \beta) = h_{\text{inv}}(\tau)$

$$h_{\text{inv}}(\tau) = \int_{-i\infty}^{+i\infty} dN(2N+1)\gamma_N P_N\left(\frac{1+\tau}{1-\tau}\right) \qquad (1)$$

$P_N(x)$ : Legendre polynomial

# Conformal symmetry beyond LO

**Conformal symmetry breaks down!**

- canonical conformal symmetry broken
- Find critical point $\beta(a_*) = 0 \to a_* = a_*(\epsilon)$.
- exact symmetry can be reconstructed order by order (in modified theory in $d = 4 - 2\epsilon$)
- done by conformal Ward identities $\delta_C \langle [O(x; z_1, z_2)][O(y; w_1, w_2)] \rangle = 0$

Corrections added to canonical generators

$$
\begin{aligned}
S_- &= S_-^{(0)} \\
S_0 &= S_0^{(0)} + \Delta S_0(a_*) \\
S_+ &= S_+^{(0)} + \Delta S_+(a_*)
\end{aligned}
$$

$\mathcal{O}(a_*)$: V.M. Braun, A.N. Manashov, Eur.Phys.J. C73 (2013) 2544
$\mathcal{O}(a_*^2)$: V.M. Braun, A.N. Manashov, S. Moch, M. S. , JHEP 03 (2016) 142

# Constraints on evolution equations from conformal symmetry

The three generators must commute with evolution kernel

$$[S_\alpha, \mathbb{H}] = 0$$

At fixed perturbative order

$$\boxed{[S_+^{(0)}, \mathbb{H}^{(\ell)}] = -\sum_{j=1}^{\ell-1} [\Delta S_+^{(j)}, \mathbb{H}^{(\ell-j)}]} \qquad (2)$$

- lhs: differential equation for evolution kernel at $\mathcal{O}(\alpha_s^\ell)$
- rhs: only operators to at most $\mathcal{O}(\alpha_s^{\ell-1})$ enter.

# Solution for evolution kernel

Eq. (2) fixes $\mathbb{H}^{(\ell)}$ only up to solutions of homogeneous equation $[S_+^{(0)}, \mathbb{H}_{\text{inv}}^{(\ell)}] = 0$

- Split $\mathbb{H}^{(\ell)} = \mathbb{H}_{\text{inv}}^{(\ell)} + \Delta\mathbb{H}^{(\ell)}$
- Determine $\Delta\mathbb{H}^{(\ell)}$ as solution of

$$[S_+^{(0)}, \Delta\mathbb{H}^{(\ell)}] = -\sum_{j=1}^{\ell-1} [\Delta S_+^{(j)}, \mathbb{H}^{(\ell-j)}]$$

- Fix $\mathbb{H}_{\text{inv}}^{(\ell)}$, i.e. canonically invariant part, with the knowledge of $\ell$-loop anomalous dimensions $\gamma_N^{(\ell)}$ by

$$h_{\text{inv}}(\tau) = \int_{-i\infty}^{+i\infty} dN(2N+1)\left[\gamma_N - \Delta\gamma_N\right] P_N\left(\frac{1+\tau}{1-\tau}\right)$$

# Non-invariant part

Result for evolution kernel in $\overline{MS}$-scheme

$$\begin{aligned}\mathbb{H}^{(3)} = \mathbb{H}^{(3)}_{\text{inv}} &+ \mathbb{T}^{(1)}\big(\beta_1 + \frac{1}{2}\mathbb{H}^{(2)}_{\text{inv}}\big) + \frac{1}{2}\mathbb{T}^{(1)}\mathbb{T}^{(1)}\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) \\ &+ \mathbb{T}^{(1)}_2\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right)^2 + \mathbb{T}^{(2)}\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) \\ &+ [\mathbb{H}^{(2)}_{\text{inv}}, \mathbb{X}^{(1)}] + \frac{1}{2}\big[\mathbb{T}^{(1)}\mathbb{H}^{(1)}, \mathbb{X}^{(1)}\big] + \frac{1}{2}\big[\mathbb{H}^{(1)}, \mathbb{X}^{(1)}_2\big]\mathbb{H}^{(1)} \\ &+ [\mathbb{H}^{(1)}, \mathbb{X}^{(2)}] + \beta_0\Big(\big[\mathbb{T}^{(1)}, \mathbb{X}^{(1)}\big] + \big[\mathbb{H}^{(1)}, \mathbb{X}^{(1)}_2\big]\Big) \\ &+ \frac{1}{2}\big[\big[\mathbb{H}^{(1)}, \mathbb{X}^{(1)}\big], \mathbb{X}^{(1)}\big] - \frac{1}{2}\big[\mathbb{H}^{(1)}, \mathbb{X}^{(2)}_2\big]\end{aligned}$$

All of the new operators are defined by eq's a la

$$[S^{(0)}_+, \mathbb{T}] = \mathcal{F}(\Delta S_+, \mathbb{H}), \qquad [S^{(0)}_+, \mathbb{X}] = \widetilde{\mathcal{F}}(\Delta S_+, \mathbb{H}).$$

# Non-invariant part: Side remark

Finite scheme-transformation $U(a) = e^{\mathbb{X}(a)}$

$$\begin{aligned}\left[U\mathbb{H}U^{-1}\right]^{(3)} = \mathbb{H}_{\text{inv}}^{(3)} \\ &+ \mathbb{T}^{(1)}\big(\beta_1 + \frac{1}{2}\mathbb{H}_{\text{inv}}^{(2)}\big) + \frac{1}{2}\mathbb{T}^{(1)}\mathbb{T}^{(1)}\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) \\ &+ \mathbb{T}_2^{(1)}\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right)^2 + \mathbb{T}^{(2)}\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right)\end{aligned}$$

Satisfies new evolution equation

$$[\mu\partial_\mu + \beta(a)\partial_a\big(1 + \ln U\big) + U\mathbb{H}U^{-1}][U\mathcal{O}(z_1, z_2)] = 0.$$

# Invariant part I

Large-N asymptotics

$$\gamma_N = f(j_N + \beta^{\mathrm{QCD}}(a) + \frac{1}{2}\gamma_N) \qquad j_N = N + 2\,,$$

$$f(j_N) = f(1 - j_N) \longrightarrow f(j_N) = \bar{f}(\mathbb{J}^2) \qquad \mathbb{J}^2 = j_N(j_N - 1)\,,$$

Large-spin expansion:

$$f(j_N) = f^{(0)} \ln \mathbb{J}^2 + f^{(\mathrm{const})} + \sum_n \frac{f^{(n)}(\ln \mathbb{J}^2)}{\mathbb{J}^{2n}}$$

In all known cases the function $f(N)$ turns out to be simpler than $\gamma_N$.

Our solution for the non-invariant part turns out to be a lucky one:

$$\boxed{\gamma_{\mathrm{inv}}(N) = f(j_N).}$$

# Invariant part II

Reminder:

We need to solve Eq. (1):

$$h_{\text{inv}}(\tau) = \int_{-i\infty}^{+i\infty} dN(2N+1)f(j_N)P_N\left(\frac{1+\tau}{1-\tau}\right)$$

# Invariant part II

Reminder:

We need to solve Eq. (1):

$$h_{\text{inv}}(\tau) = \int_{-i\infty}^{+i\infty} dN(2N+1)f(j_N)P_N\left(\frac{1+\tau}{1-\tau}\right)$$

Did not find a smart way to solve this problem analytically!

☹

# Invariant part III

Schedule

- fix leading asymptotic terms to some order
- parametrize the remainder by some simple function with only few fit parameters

$$\mathbb{H}^{(3)}_{\text{inv}}\mathcal{O}(z_1,z_2) = \Gamma^{(3)}_{\text{cusp}}\int_0^1 d\alpha\,\frac{\bar\alpha}{\alpha}\Big(2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^{\alpha},z_2)-\mathcal{O}(z_1,z_{21}^{\alpha})\Big)$$
$$+\chi_0^{(3)}\mathcal{O}(z_1,z_2)+\int_0^1 d\alpha\int_0^{\bar\alpha}d\beta\Big(\chi^{(3)}_{\text{inv}}(\tau)+\chi^{\mathbb{P}(3)}_{\text{inv}}(\tau)\mathbb{P}_{12}\Big)\mathcal{O}(z_{12}^{\alpha},z_{21}^{\beta})\,.$$

Leading two terms ($\Gamma_{\text{cusp}},\chi_0$) can simple be taken from literature:
S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B 688 (2004) 101.

# Invariant part IV

$$\chi_{\text{inv}}(\tau) = \frac{C_F^2}{N_c}\Bigg\{ \left(-176\zeta_3 + \frac{1886}{3} - \frac{52\pi^2}{9}\right)\varphi_1(\tau) + \left(\frac{3632}{9} - \frac{16\pi^2}{3}\right)\varphi_2(\tau)$$
$$- \frac{520}{3}\varphi_3(\tau) - 64\varphi_4(\tau) - \left(\frac{352}{3}\zeta_3 + \frac{81196}{27} - \frac{3200\pi^2}{27} + \frac{176\pi^4}{45}\right)$$
$$+ \left(\frac{16\pi^2}{3} - \frac{376}{3}\right)\ln(\tau/\bar{\tau}) + \delta\chi_{\text{inv}}(\tau)\Bigg\} + \text{five more color structures}$$

with $\varphi_i(\tau) \simeq$ harmonic polylogarithm with at most weight $i$.
Convers the asymptotic expansion

$$\int d\alpha d\beta \chi_{\text{inv}}(\tau)(1-\alpha-\beta)^N \sim 1/\mathbb{J}^2, \ldots, 1/\mathbb{J}^{10}, \ln \mathbb{J}^2/\mathbb{J}^2, \text{ whatever remains}$$

# Invariant part V

Restrictions on $\delta\chi(\tau)$:

- As simple as possible (concerning number of parameters)
- Reciprocity respecting: moments are functions of $\mathbb{J}^2 = j_N(j_N - 1)$.

We make the Ansatz:

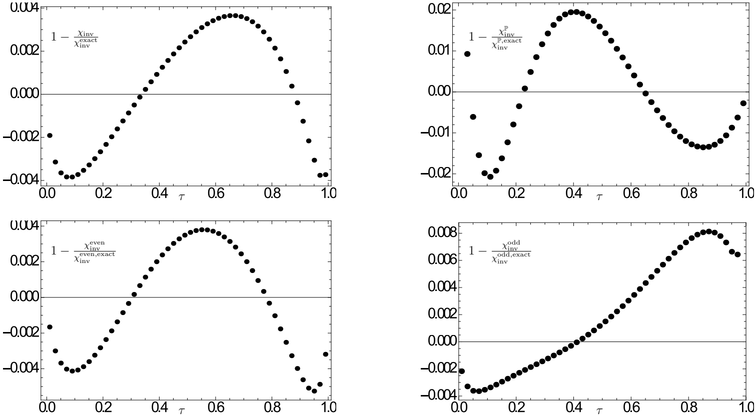
$$\delta\chi^{(3)}_{\text{inv}}(\tau) = \frac{H_0}{(1 + 4a\tau/\bar{\tau})^{5/2}} \left[1 + a\frac{\tau}{\bar{\tau}}(4 - 6\,b)\right] - H_0\,,$$

with e.g. for structure $\dfrac{C_F^2}{N_c}$:

$$H_0 = -\frac{368\zeta_3}{3} - \frac{992}{9} + \frac{176\pi^2}{9} + \frac{4\pi^4}{9}$$
$$a = 0.05174 \qquad b = 4.116\,.$$

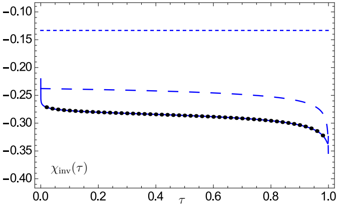
# Accuracy of approximation (for $n_f = 4$)

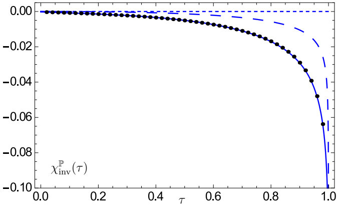
Compare $\chi_{\text{inv}}(\tau)$ in our parametrization with what one gets by numerical solution of $\chi_{\text{inv}}^{\text{exact}}(\tau) = \int_C dN(2N+1)\Delta\gamma_{\text{inv}}(N)P_N\left(\frac{1+\tau}{1-\tau}\right)$ with $\Delta\gamma_{\text{inv}}(N) = f(j_N) - 2\Gamma_{\text{cusp}}(S_1(N)-1) - \chi_0$.



# Numerical impact of NNLO

How big is the NNLO correction? $\alpha_s/\pi = 0.1,\ n_f = 4.$




Seems to be not too small, however: the leading contributions are

$$\Gamma_{\rm cusp} = a\Gamma^{(1)}(1 + 8.019a + 80.53a^2 + \ldots) = a\Gamma^{(1)}(1 + 0.2005 + 0.0503 + \ldots)\,,$$
$$\chi_0 = a\chi_0^{(1)}(1 - 0.7935a - 141.3a^2 + \ldots) = a\chi_0^{(1)}(1 - 0.0198 - 0.0883 + \ldots)\,.$$

# OPE of light-ray operator

In general

$$\mathcal{O}(z_1, z_2) = \sum_{n,m} \varphi_{nm}(z_1, z_2)\mathcal{O}_{nm}\,, \quad \mathcal{O}_{nm}(x) = P_{nm}(\partial_{z_1}, \partial_{z_2})\bar{q}(z_1)q(z_2)\Big|_{z_1=z_2=0}.$$

Particular choice:

$$\varphi_{nm}(z_1, z_2) = \omega_{nm}(S_+^{(0)})^{m-n}(z_1 - z_2)^n\,,$$
$$\mathcal{O}_{nm} = (\partial_{z_1} + \partial_{z_2})^m C_n^{3/2}\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right)\mathcal{O}(z_1, z_2)\Big|_{z_1=z_2=0}\,, \qquad m \geq n\,.$$

There exists a scalar product:

$$P_{n'm'}(\partial_{z_1}, \partial_{z_2})\varphi_{nm}(z_1, z_2)\Big|_{z_1=z_2=0} = \delta_{n'n}\delta_{m'm}$$

# Relation between mixing matrix and evolution kernel

Compare evolution equation for non-local and local operators:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(a)\frac{\partial}{\partial a}\right)[\mathcal{O}(z_1,z_2)]=-\,\mathbb{H}[\mathcal{O}(z_1,z_2)]\,,$$

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(a)\frac{\partial}{\partial a}\right)[\mathcal{O}_{nk}]=-\sum_{n'=0}^{n}\gamma_{nn'}\,[\mathcal{O}_{n'k}]\,.$$

Local matrix is given by “matrix elements” of scalar product:

$$\boxed{\langle nm|\mathbb{H}|n'm\rangle\equiv P_{nm}(\partial_1,\partial_2)\mathbb{H}\varphi_{n'm}(z_1,z_2)\Big|_{z=0}=\gamma_{nn'}}$$

# Three-loop results for mixing matrix

We have found for the first few elements (taking $n_f = 4$)

$$\gamma^{(3)} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & \frac{64(105587-35640\zeta_3)}{6561} & 0 & \\ \frac{43600}{243} & 0 & \frac{39786575}{26244} - \frac{45800\zeta_3}{81} & \\ 0 & \frac{634360}{2187} & 0 & \\ -\frac{400717}{30375} & 0 & \frac{162370621}{546750} & \\ \vdots & & & \ddots \end{pmatrix}$$

Comparing different orders in perturbative expansion

$$\widehat{\boldsymbol{\gamma}} = a\widehat{\boldsymbol{\gamma}}^{(1)}\left[\mathrm{Id} + a\Delta\widehat{\boldsymbol{\gamma}}\right]$$

with

$$\Delta\widehat{\boldsymbol{\gamma}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 10+86a & 0 & 0 & 0 \\ 1+16a & 0 & 9.1+78a & 0 & 0 \\ 0 & 1.6+21a & 0 & 8.6+72a & 0 \\ -0.21-0.82a & 0 & 1.5+18a & 0 & 8.3+71a \end{pmatrix}.$$

# Conclusion and Outlook

What can be done next?

- Exact solution for $\mathbb{H}$ most likely irrelevant
- Evolution kernel in momentum space
- Sample application for physical process, e.g. pion DA in $\gamma \to \gamma^* \pi$
- What other sets of local operators might be of interest for physical applications?
- Big goal: extend analysis for singlet operators!!!

# Thank you for your attention

# Any questions?