Three-loop evolution equation for non-singlet leading-twist operator
based on:

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Motivation & Introduction

- Remarkable achievements in experimental accuracy
- intended to increase even more in near future (e.g. JLAB 12 GeV update or EIC)
- needs to be matched by theoretical accuracy
  → NNLO is becoming the state-of-the-art in most fields
- NNLO evolution for off-forwards distributions needs to be investigated
Operators under consideration

Non-local “light-ray” twist-two operator

\[ n^2 = 0 \quad D_+ := n_\mu D^\mu \]

\[ \mathcal{O}^{(n)}(x; z_1, z_2) = Z\bar{q}(x + n z_1)\gamma q(x + n z_2) \]

Generating object for local operators

\[ \mathcal{O}^{(n)}(x; z_1, z_2) = \sum_{l,m=0}^{\infty} \frac{z_1^l z_2^m}{l! m!} [Z \mathcal{O}]_{lm}(x), \quad \mathcal{O}_{lm}(x) = \bar{q}(x)(\overrightarrow{D}_+)^l \gamma (\overleftarrow{D}_+)^m q(x). \]

We tacitly assume different flavors.

Renormalization factor in $\overline{MS}$ - scheme

\[ Z = 1 + \sum_{k=0}^{\infty} \frac{1}{\epsilon^k} \sum_{\ell=k}^{\infty} a^\ell Z^{(\ell)}_k, \quad a = \frac{\alpha_s}{4\pi}. \]

Relation to evolution kernel

\[ \mathbb{H} = -\frac{d}{d \ln \mu} \ln(Z), \quad \gamma^{l'm'}_{lm} = -\frac{d}{d \ln \mu} \ln(Z^{l'm'}_{lm}) \]
Evolution equation

RGE

\[ [\mu \partial_\mu + \beta(a) \partial_a + \mathbb{H}] O(x; z_1, z_2) = 0, \quad \text{Balitsky, Braun '89} \]

\[ [\mu \partial_\mu + \beta(a) \partial_a + \hat{\gamma}] O_{lm}(x) = 0. \quad \text{Radyushkin, etc. '84} \]

QCD $\beta$-function

\[ \beta(a) = \mu \frac{\partial a}{\partial \mu} = -2a(\epsilon + a\beta_0 + a^2 \beta_1 + \ldots). \quad \beta_{\text{QCD}}(a) \]

General form

\[ [\mathbb{H} O](z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta h(\alpha, \beta) O(z_{12}^\alpha, z_{21}^\beta), \quad z_{12}^\alpha = (1 - \alpha)z_1 + \alpha z_2. \]
Conformal symmetry at the classical level

Three symmetry generators on the light-cone

- Translations: \( S_{(0)}^{-} = -\partial z_1 - \partial z_2 \)
- Dilatations: \( S_{(0)}^{0} = z_1 \partial z_1 + z_1 \partial z_1 + 2(j_1 + j_2) \)
- Spec. conf. transf.: \( S_{(0)}^{+} = z_1^2 \partial z_1 + z_2^2 \partial z_2 + 2j_1 z_1 + 2j_2 z_2 \)

All three commute with the LO evolution kernel

\[
[S_{(0)}^{(0)}, \mathbb{H}^{(1)}] = 0
\]
Evolution equations at the LO

Shape of the evolution kernel fixed by \([S^{(0)}_+, \mathbb{H}^{(1)}] = 0\)

\[
[\mathbb{H}^{(1)} \mathcal{O}] (z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta h^{(1)}_{\text{inv}} (\tau) \mathcal{O}(z_{12}^\alpha, z_{21}^\beta), \quad \tau = \frac{\alpha \beta}{\bar{\alpha} \bar{\beta}}.
\]

Anomalous dimension \(\gamma_N\) of local operators with spin \(N\) are eigenvalues

\[
\mathbb{H} (z_1 - z_2)^N = (z_1 - z_2)^N \int_0^1 d\alpha \int_0^1 d\beta h(\alpha, \beta)(1 - \alpha - \beta)^N = (z_1 - z_2)^N \gamma_N
\]

Knowledge of anomalous dimension \(\gamma_N\) of local operators with spin \(N\) allows to reconstruct kernel \(h(\alpha, \beta)\), if \(h(\alpha, \beta) = h_{\text{inv}}(\tau)\)

\[
h_{\text{inv}}(\tau) = \int_{-i\infty}^{+i\infty} dN (2N + 1) \gamma_N P_N \left( \frac{1 + \tau}{1 - \tau} \right)
\]

\(P_N(x)\) : Legendre polynomial
Conformal symmetry breaks down!

- canonical conformal symmetry broken
- Find critical point $\beta(a_* ) = 0 \rightarrow a_* = a_*(\epsilon)$.
- exact symmetry can be reconstructed order by order (in modified theory in $d = 4 - 2\epsilon$)
- done by conformal Ward identities $\delta_C \langle [O(x; z_1, z_2)] [O(y; w_1, w_2)] \rangle = 0$

Corrections added to canonical generators

\[
\begin{align*}
S_- &= S_-^{(0)} \\
S_0 &= S_0^{(0)} + \Delta S_0(a_*) \\
S_+ &= S_+^{(0)} + \Delta S_+(a_*)
\end{align*}
\]


$O(a_*^2 )$: V.M. Braun, A.N. Manashov, S. Moch, M. S. , JHEP 03 (2016) 142
Constraints on evolution equations from conformal symmetry

The three generators must commute with evolution kernel

\[ [S_\alpha, H] = 0 \]

At fixed perturbative order

\[
[S_+^{(0)}, H^{(\ell)}] = - \sum_{j=1}^{\ell-1} [\Delta S_+^{(j)}, H^{(\ell-j)}] \quad (2)
\]

- lhs: differential equation for evolution kernel at \( \mathcal{O}(\alpha_s^\ell) \)
- rhs: only operators to at most \( \mathcal{O}(\alpha_s^{\ell-1}) \) enter.

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Three-loop evolution equations
Solution for evolution kernel

Eq. (2) fixes $\mathbb{H}^{(\ell)}$ only up to solutions of homogeneous equation $[S^{(0)}_+, \mathbb{H}_{\text{inv}}^{(\ell)}] = 0$

- Split $\mathbb{H}^{(\ell)} = \mathbb{H}_{\text{inv}}^{(\ell)} + \Delta \mathbb{H}^{(\ell)}$
- Determine $\Delta \mathbb{H}^{(\ell)}$ as solution of

$$[S^{(0)}_+, \Delta \mathbb{H}^{(\ell)}] = - \sum_{j=1}^{\ell-1} [\Delta S^{(j)}_+, \mathbb{H}^{(\ell-j)}]$$

- Fix $\mathbb{H}_{\text{inv}}^{(\ell)}$, i.e. canonically invariant part, with the knowledge of $\ell$-loop anomalous dimensions $\gamma^{(\ell)}_N$ by

$$h_{\text{inv}}(\tau) = \int_{-i\infty}^{+i\infty} dN (2N + 1) [\gamma_N - \Delta \gamma_N] P_N \left( \frac{1 + \tau}{1 - \tau} \right)$$
Non-invariant part

Result for evolution kernel in $\overline{MS}$-scheme

\[
\mathbb{H}^{(3)} = \mathbb{H}^{(3)}_{\text{inv}} \\
+ T^{(1)} \left( \beta_1 + \frac{1}{2} \mathbb{H}^{(2)}_{\text{inv}} \right) + \frac{1}{2} T^{(1)} T^{(1)} \left( \beta_0 + \frac{1}{2} \mathbb{H}^{(1)} \right) \\
+ T_2^{(1)} \left( \beta_0 + \frac{1}{2} \mathbb{H}^{(1)} \right)^2 + T^{(2)} \left( \beta_0 + \frac{1}{2} \mathbb{H}^{(1)} \right) \\
+ [\mathbb{H}^{(2)}, X^{(1)}] + \frac{1}{2} [T^{(1)} \mathbb{H}^{(1)}, X^{(1)}] + \frac{1}{2} [\mathbb{H}^{(1)}, X^{(1)}_2] \mathbb{H}^{(1)} \\
+ [\mathbb{H}^{(1)}, X^{(2)}] + \beta_0 \left( [T^{(1)}, X^{(1)}] + [\mathbb{H}^{(1)}, X^{(1)}_2] \right) \\
+ \frac{1}{2} \left[ [\mathbb{H}^{(1)}, X^{(1)}], X^{(1)} \right] - \frac{1}{2} \left[ \mathbb{H}^{(1)}, X^{(2)}_2 \right]
\]

All of the new operators are defined by eq’s a la

\[
[S^{(0)}_+, T] = \mathcal{F}(\Delta S_+, \mathbb{H}), \quad [S^{(0)}_+, X] = \tilde{\mathcal{F}}(\Delta S_+, \mathbb{H}).
\]
Non-invariant part: Side remark

Finite scheme-transformation $U(a) = e^{\mathbf{X}(a)}$

$$\left[U \mathcal{H} U^{-1}\right]^{(3)} = \mathcal{H}^{(3)}_{\text{inv}}$$

$$+ T^{(1)} \left( \beta_1 + \frac{1}{2} \mathcal{H}^{(2)}_{\text{inv}} \right) + \frac{1}{2} T^{(1)} T^{(1)} \left( \beta_0 + \frac{1}{2} \mathcal{H}^{(1)} \right)$$

$$+ T^{(1)}_{2} \left( \beta_0 + \frac{1}{2} \mathcal{H}^{(1)} \right)^2 + T^{(2)} \left( \beta_0 + \frac{1}{2} \mathcal{H}^{(1)} \right)$$

Satisfies new evolution equation

$$[\mu \partial_\mu + \beta(a) \partial_a (1 + \ln U) + U \mathcal{H} U^{-1}] [U \mathcal{O}(z_1, z_2)] = 0.$$
Invariant part I

Large-N asymptotics

\[
\gamma_N = f(j_N + \beta_{QCD}^2(a) + \frac{1}{2} \gamma_N) \quad \quad j_N = N + 2,
\]

\[
f(j_N) = f(1 - j_N) \rightarrow f(j_N) = \bar{f}(\mathcal{J}^2) \quad \quad \mathcal{J}^2 = j_N(j_N - 1),
\]

Large-spin expansion:

\[
f(j_N) = f^{(0)} \ln \mathcal{J}^2 + f^{(\text{const})} + \sum_n \frac{f^{(n)}(\ln \mathcal{J}^2)}{\mathcal{J}^{2n}}
\]

In all known cases the function \( f(N) \) turns out to be simpler than \( \gamma_N \).

Our solution for the non-invariant part turns out to be a lucky one:

\[
\gamma_{\text{inv}}(N) = f(j_N).
\]
Reminder:

We need to solve Eq. (1):

\[ h_{\text{inv}}(\tau) = \int_{-i\infty}^{+i\infty} dN(2N + 1)f(j_N)P_N \left( \frac{1 + \tau}{1 - \tau} \right) \]
Reminder:

We need to solve Eq. (1):

$$h_{\text{inv}}(\tau) = \int_{-i\infty}^{+i\infty} dN(2N + 1)f(j_N)P_N \left( \frac{1 + \tau}{1 - \tau} \right)$$

Did not find a smart way to solve this problem analytically! 😞
Invariant part III

Schedule

- fix leading asymptotic terms to some order
- parametrize the remainder by some simple function with only few fit parameters

\[
\mathcal{H}^{(3)}_{\text{inv}} \mathcal{O}(z_1, z_2) = \Gamma^{(3)}_{\text{cusp}} \int_0^1 d\alpha \frac{\tilde{\alpha}}{\alpha} \left( 2 \mathcal{O}(z_1, z_2) - \mathcal{O}(z_{12}^\alpha, z_2) - \mathcal{O}(z_1, z_{21}^\alpha) \right) \\
+ \chi_0^{(3)} \mathcal{O}(z_1, z_2) + \int_0^1 d\alpha \int_0^{\tilde{\alpha}} d\beta \left( \chi_{\text{inv}}^{(3)}(\tau) + \chi_{\text{inv}}^{P(3)}(\tau)P_{12} \right) \mathcal{O}(z_{12}^\alpha, z_{21}^\beta).
\]

Leading two terms \((\Gamma_{\text{cusp}}, \chi_0)\) can simple be taken from literature:
Invariant part IV

\[ \chi_{\text{inv}}(\tau) = \frac{C_F^2}{N_c} \left\{ \left( -176\zeta_3 + \frac{1886}{3} - \frac{52\pi^2}{9} \right) \varphi_1(\tau) + \left( \frac{3632}{9} - \frac{16\pi^2}{3} \right) \varphi_2(\tau) \right. \\
\left. - \frac{520}{3} \varphi_3(\tau) - 64\varphi_4(\tau) - \left( \frac{352}{3} \zeta_3 + \frac{81196}{27} - \frac{3200\pi^2}{27} + \frac{176\pi^4}{45} \right) \right. \\
\left. + \left( \frac{16\pi^2}{3} - \frac{376}{3} \right) \ln(\tau/\bar{\tau}) + \delta\chi_{\text{inv}}(\tau) \right\} + \text{five more color structures} \]

with \( \varphi_i(\tau) \simeq \text{harmonic polylogarithm with at most weight } i. \)

Convers the asymptotic expansion

\[ \int d\alpha d\beta \chi_{\text{inv}}(\tau)(1 - \alpha - \beta)^N \sim 1/J^2, \ldots, 1/J^{10}, \ln J^2/J^2, \text{ whatever remains} \]
Invariant part V

Restrictions on $\delta \chi(\tau)$:

- As simple as possible (concerning number of parameters)
- Reciprocity respecting: moments are functions of $J^2 = j_N(j_N - 1)$.

We make the Ansatz:

$$\delta \chi^{(3)}_{\text{inv}}(\tau) = \frac{H_0}{(1 + 4a\tau/\bar{\tau})^{5/2}} \left[ 1 + a\frac{\tau}{\bar{\tau}}(4 - 6b) \right] - H_0,$$

with e.g. for structure $\frac{C_F^2}{N_c}$:

$$H_0 = - \frac{368\zeta_3}{3} - \frac{992}{9} + \frac{176\pi^2}{9} + \frac{4\pi^4}{9},$$

$$a = 0.05174 \quad b = 4.116.$$
Accuracy of approximation (for $n_f = 4$)

Compare $\chi_{\text{inv}}(\tau)$ in our parametrization with what one gets by numerical solution of $\chi_{\text{inv}}^{\text{exact}}(\tau) = \int_C dN (2N + 1) \Delta \gamma_{\text{inv}}(N) P_N \left( \frac{1+\tau}{1-\tau} \right)$ with $\Delta \gamma_{\text{inv}}(N) = f(j_N) - 2\Gamma_{\text{cusp}}(S_1(N) - 1) - \chi_0$. 

\[
1 - \frac{\chi_{\text{inv}}}{\chi_{\text{inv}}^{\text{exact}}}
\]

\[
1 - \frac{\chi_{\text{even}}}{\chi_{\text{even,exact}}}
\]

\[
1 - \frac{\chi_{\text{odd}}}{\chi_{\text{odd,exact}}}
\]
How big is the NNLO correction? \( \alpha_s / \pi = 0.1, \ n_f = 4. \)

\[
\chi_{\text{inv}}(\tau) = a \Gamma^{(1)}_cusp(1 - 0.7935a - 141.3a^2 + \ldots) = a \chi^{(1)}_0(1 - 0.0198 - 0.0883 + \ldots).
\]
OPE of light-ray operator

In general

\[ \mathcal{O}(z_1, z_2) = \sum_{n,m} \varphi_{nm}(z_1, z_2) \mathcal{O}_{nm}, \quad \mathcal{O}_{nm}(x) = P_{nm}(\partial_{z_1}, \partial_{z_2}) \bar{q}(z_1) q(z_2) \bigg|_{z_1=z_2=0} \]

Particular choice:

\[ \varphi_{nm}(z_1, z_2) = \omega_{nm} (S_+^{(0)})^{m-n} (z_1 - z_2)^n, \]

\[ \mathcal{O}_{nm} = (\partial_{z_1} + \partial_{z_2})^m C_n^{3/2} \left( \frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}} \right) \mathcal{O}(z_1, z_2) \bigg|_{z_1=z_2=0}, \quad m \geq n. \]

There exists a scalar product:

\[ P_{n'm'}(\partial_{z_1}, \partial_{z_2}) \varphi_{nm}(z_1, z_2) \bigg|_{z_1=z_2=0} = \delta_{n'n} \delta_{m'm} \]
Relation between mixing matrix and evolution kernel

Compare evolution equation for non-local and local operators:

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} \right) [\mathcal{O}(z_1, z_2)] = - \mathbb{H} [\mathcal{O}(z_1, z_2)],$$

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} \right) [\mathcal{O}_{nk}] = - \sum_{n' = 0}^{n} \gamma_{nn'} [\mathcal{O}_{n' k}].$$

Local matrix is given by “matrix elements” of scalar product:

$$\langle nm | \mathbb{H} | n'm \rangle \equiv P_{nm}(\partial_1, \partial_2) \mathbb{H} \varphi_{n'm}(z_1, z_2)|_{z=0} = \gamma_{nn'}$$
Three-loop results for mixing matrix

We have found for the first few elements (taking \( n_f = 4 \))

\[
\gamma^{(3)} = \begin{pmatrix}
0 & 0 & 0 & \cdots \\
0 & 0 & \frac{64(105587-35640\zeta_3)}{6561} & 0 \\
0 & \frac{43600}{243} & 0 & \frac{39786575}{26244} - \frac{45800\zeta_3}{81} \\
0 & \frac{400717}{30375} & \frac{634360}{2187} & 0 \\
& \vdots & & \ddots \\
\end{pmatrix}
\]

Comparing different orders in perturbative expansion

\[
\hat{\gamma} = a\hat{\gamma}^{(1)} [\text{Id} + a\Delta\hat{\gamma}]
\]

with

\[
\Delta\hat{\gamma} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 + 86a & 0 & 0 & 0 & 0 \\
1 + 16a & 0 & 9.1 + 78a & 0 & 0 & 0 \\
0 & 1.6 + 21a & 0 & 8.6 + 72a & 0 & 0 \\
-0.21 - 0.82a & 0 & 1.5 + 18a & 0 & 8.3 + 71a & \cdots
\end{pmatrix}.
\]
What can be done next?

- Exact solution for $\mathbb{H}$ most likely irrelevant
- Evolution kernel in momentum space
- Sample application for physical process, e.g. pion DA in $\gamma \to \gamma^* \pi$
- What other sets of local operators might be of interest for physical applications?
- Big goal: extend analysis for singlet operators!!!
Thank you for your attention

Any questions?