Small x Asymptotics of the Gluon Helicity Distribution

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work with Dan Pitonyak and Matt Sievert,

in preparation

Outline

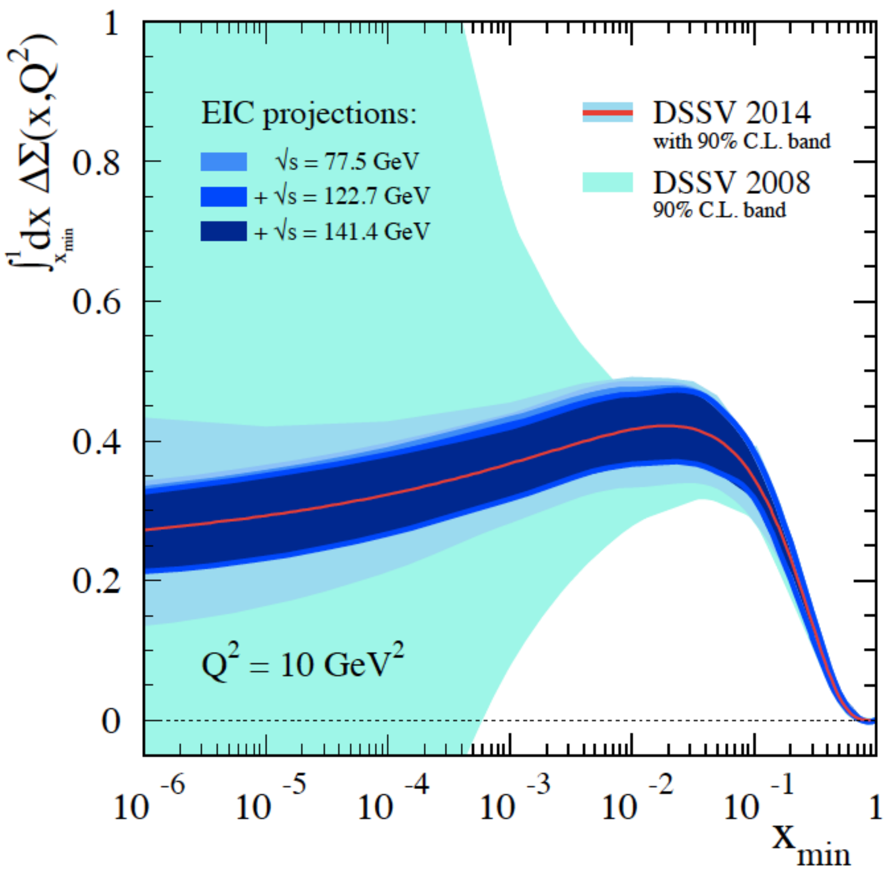
- Quark helicity evolution: re-derivation and review in the operator form.
- Gluon helicity TMDs at small x.
- Small-x evolution equations for the gluon helicity TMDs at large N_c.
- Solution of the evolution equations and the small-x asymptotics of the gluon helicity distribution:

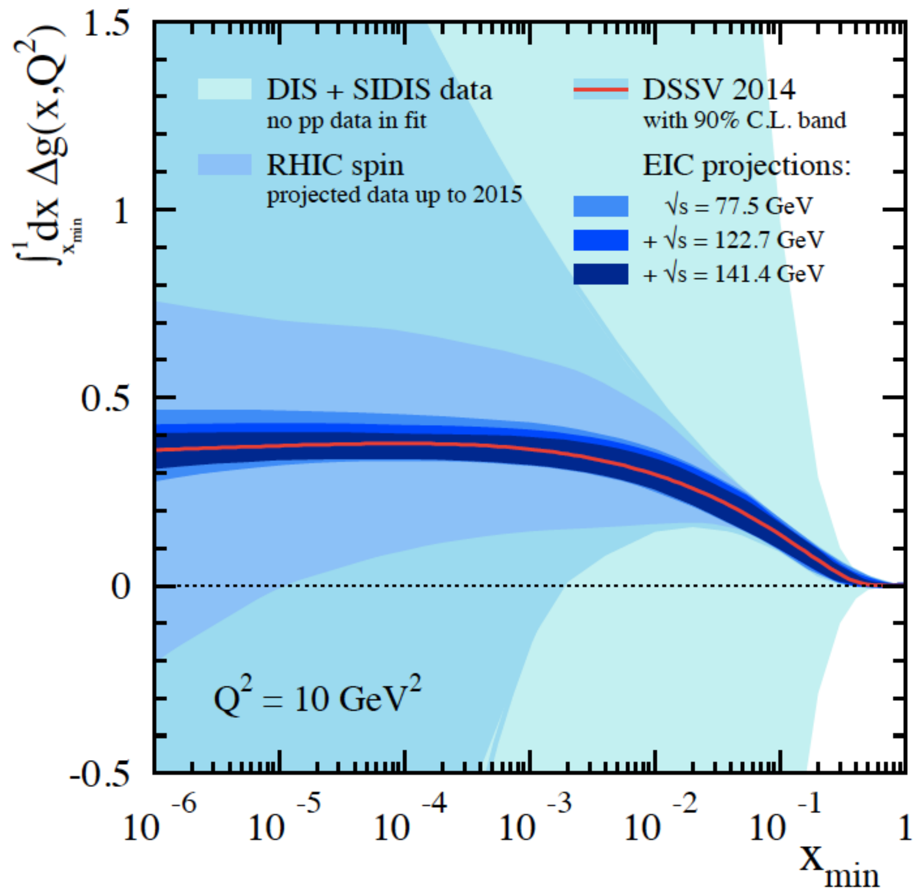
$$\Delta G(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \tfrac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

- For comparison, quark helicity PDF asymptotics is (see Matt Sievert's talk)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

How much spin is at small x?



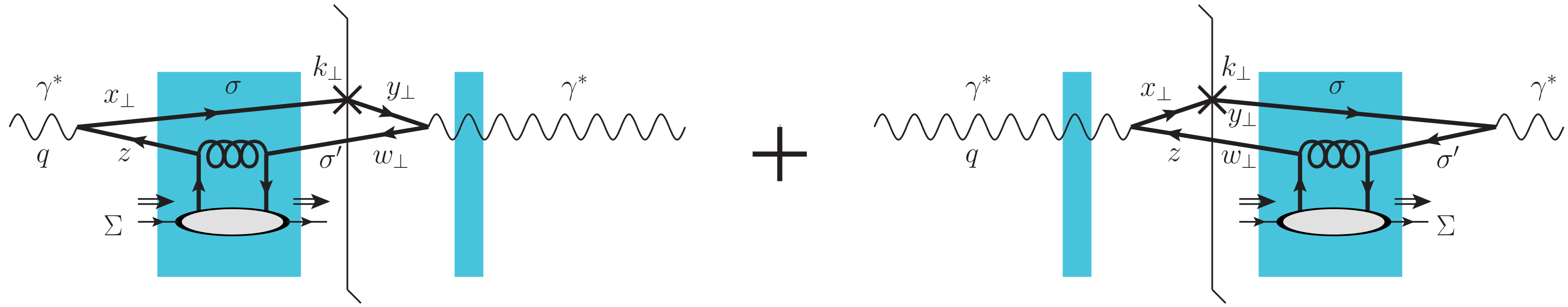


- E. Aschenaur et al, arXiv:1509.06489 **[hep-ph]**
- Uncertainties are very large at small x!

Quark Helicity Evolution at Small x
flavor-singlet case

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph],
arXiv:1610.06197 [hep-ph], arXiv:1610.06188 [hep-ph],
arXiv:1703.05809 [hep-ph]

Quark Helicity Observables at Small x



- One can show that the g_1 structure function and quark helicity PDF (Δq) and TMD at small-x can be expressed in terms of the polarized dipole amplitude (flavor singlet case):

$$g_1^S(x, Q^2) = \frac{N_c\, N_f}{2\,\pi^2 \alpha_{EM}} \int\limits_{z_i}^{1} \frac{dz}{z^2(1-z)} \int dx_{01}^2 \left[\frac{1}{2} \sum_{\lambda\sigma\sigma'} |\psi^T_{\lambda\sigma\sigma'}|^2_{(x_{01}^2, z)} + \sum_{\sigma\sigma'} |\psi^L_{\sigma\sigma'}|^2_{(x_{01}^2, z)} \right] G(x_{01}^2, z),$$

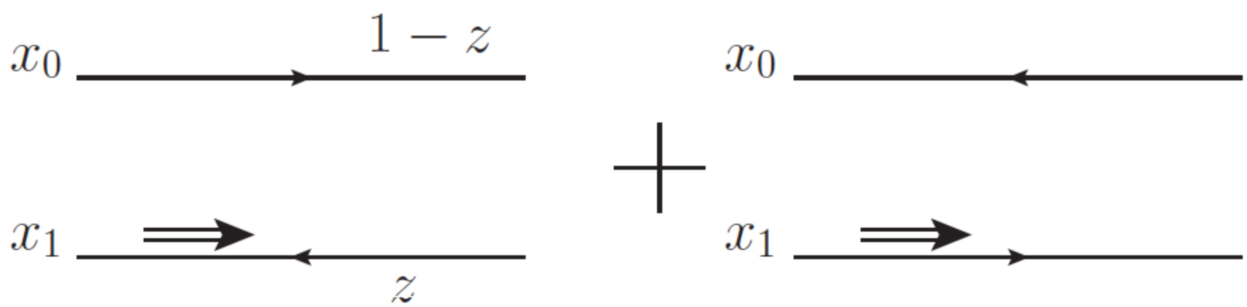
$$\Delta q^S(x, Q^2) = \frac{N_c\, N_f}{2\pi^3} \int\limits_{z_i}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\frac{1}{zQ^2}} \frac{dx_{01}^2}{x_{01}^2}\, G(x_{01}^2, z),$$

$$g_{1L}^S(x, k_T^2) = \frac{8\, N_c\, N_f}{(2\pi)^6} \int\limits_{z_i}^{1} \frac{dz}{z} \int d^2 x_{01}\, d^2 x_{0'1}\, e^{-i \underline{k} \cdot (\underline{x}_{01} - \underline{x}_{0'1})}\, \frac{\underline{x}_{01} \cdot \underline{x}_{0'1}}{x_{01}^2 x_{0'1}^2}\, G(x_{01}^2, z)$$

- Here s is cms energy squared, $z_i = \Lambda^2/s$, $\quad G(x_{01}^2, z) \equiv \int d^2 b\, G_{10}(z)$

Polarized Dipole

- All flavor singlet small-x helicity observables depend on one object, “polarized dipole amplitude”:



$$G_{10}(z) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger} \right] + \mathrm{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle (z)$$

unpolarized quark

polarized quark (“polarized Wilson line”): eikonal propagation, non-eikonal spin-dependent interaction

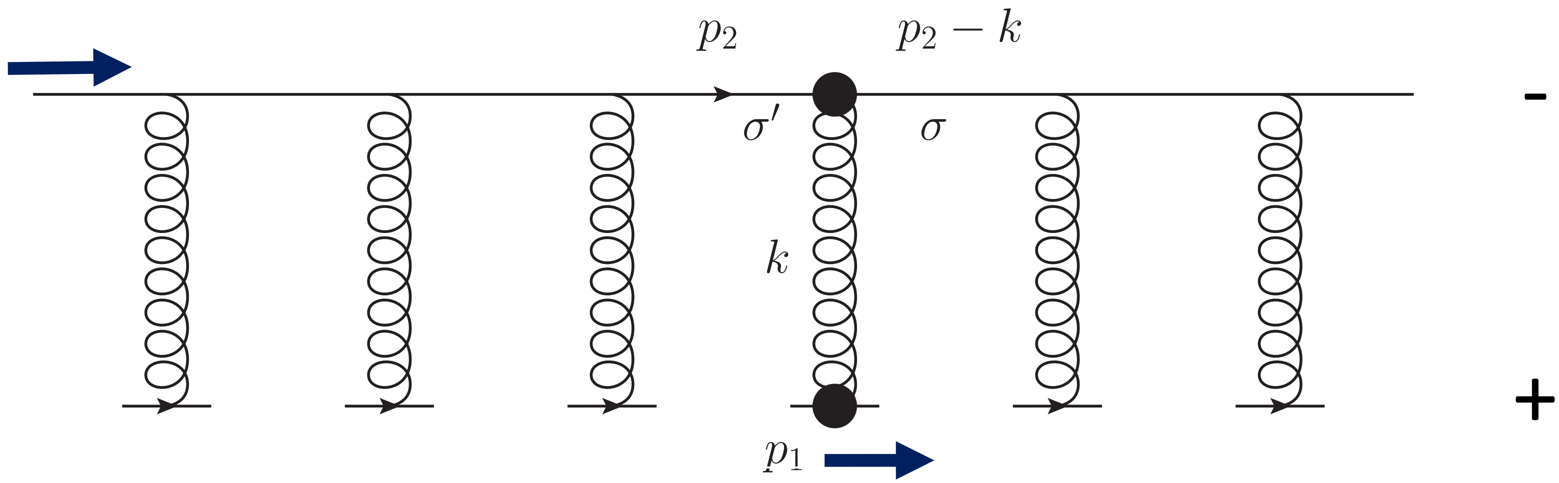
$$V_{\underline{x}} \equiv \mathcal{P} \exp \left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle (z) \equiv zs \left\langle \mathcal{O} \right\rangle (z)$$

"Polarized Wilson line"

To obtain an explicit expression for the "polarized Wilson line" operator due to a sub-eikonal gluon exchange (as opposed to the sub-eikonal quarks exchange), consider multiple Coulomb gluon exchanges with the target:



Most gluon exchanges are eikonal spin-independent interactions, while one of them is a spin-dependent sub-eikonal exchange. (cf. Mueller '90, McLerran, Venugopalan '93)

“Polarized Wilson line”

- A simple calculation in $A^- = 0$ gauge yields the “polarized Wilson line”:

$$V_{\underline{x}}^{pol} = \frac{1}{2\,s} \int\limits_{-\infty}^{\infty} dx^- \, \mathrm{P}\exp\left\{ ig \int\limits_{x^-}^{\infty} dx'^- \, A^+(x'^-, \underline{x}) \right\} ig\, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \; \mathrm{P}\exp\left\{ ig \int\limits_{-\infty}^{x^-} dx'^- \, A^+(x'^-, \underline{x}) \right\}$$

where $\underline{A}_\Sigma(x^-, \underline{x}) = \frac{\Sigma}{2p_1^+} \, \underline{\tilde{A}}(x^-, \underline{x})$

is the spin-dependent gluon field of the plus-direction moving target with helicity Σ.

(A^+ is the unpolarized eikonal field.)

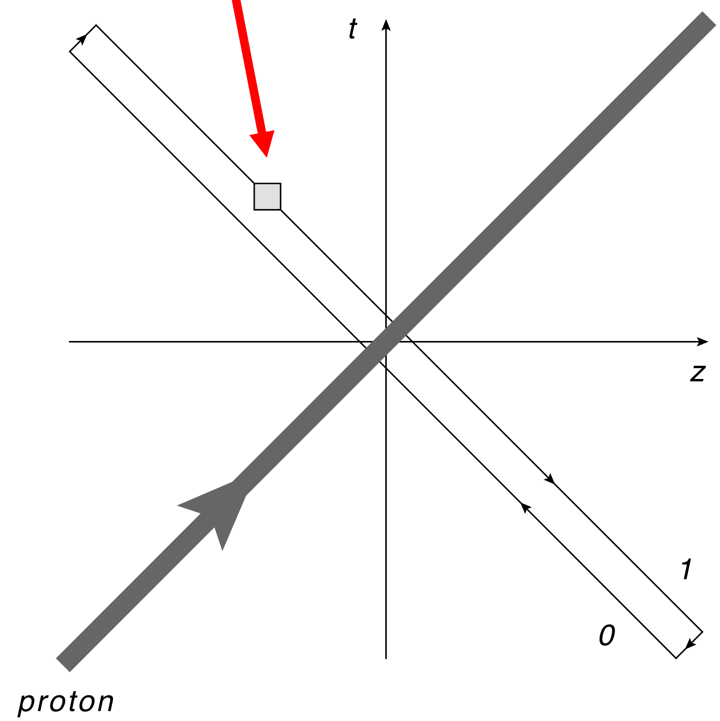
Polarized Dipole Amplitude

- The polarized dipole amplitude is then defined by

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr}\left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty]\right] + \mathrm{c.c.}\right\rangle (z)$$

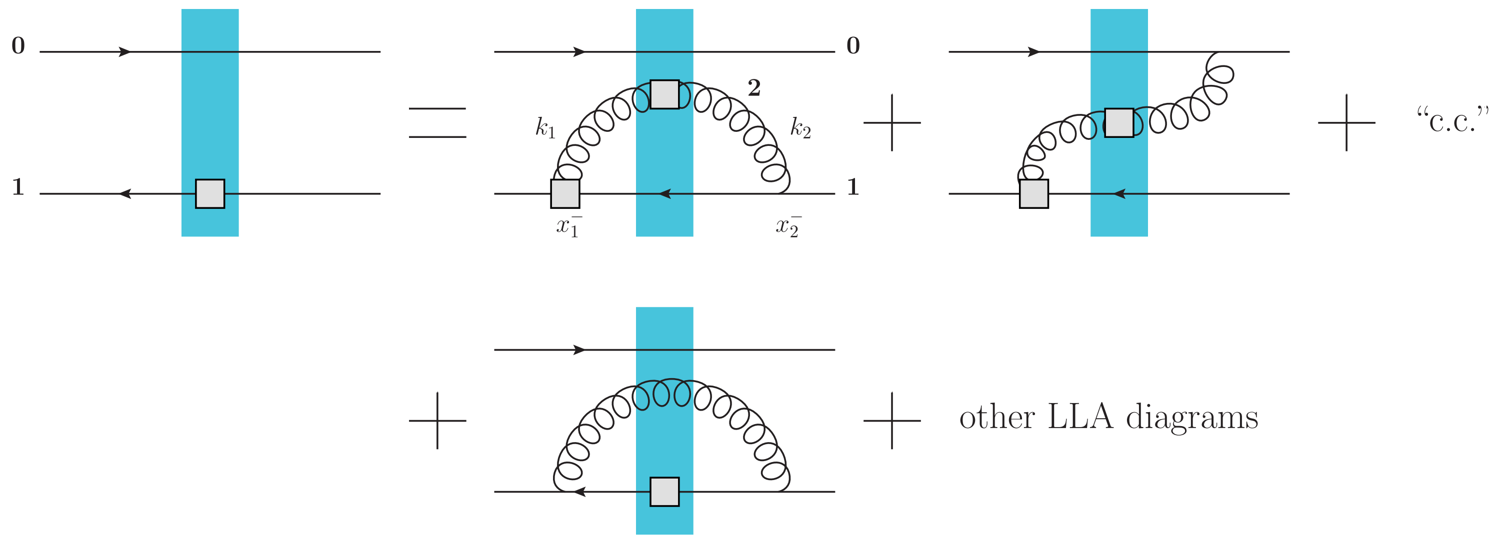
with the standard light-cone Wilson line

$$V_{\underline{x}}[b^-, a^-] = \mathrm{P}\exp\left\{ ig \int\limits_{a^-}^{b^-} dx^- \, A^+(x^-, \underline{x}) \right\}$$



Cross-check: evolution

- One can cross-check the operator and/or the evolution equation we derived for it (see Matt's talk):



- From the first two graphs on the right we get

$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s}{\pi^2} \int^{z} \frac{dz'}{z'} \int \frac{d^2 x_2}{x_{21}^2} \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr} \left[t^b V_0 t^a V_1^\dagger \right] U_2^{pol\, ba} \right\rangle\!\!\right\rangle + \ldots$$

in agreement with our earlier work on quark helicity evolution ('15-'16).

Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ \ln(1/x)$$

- Helicity evolution resummation parameter is double-logarithmic (DLA):

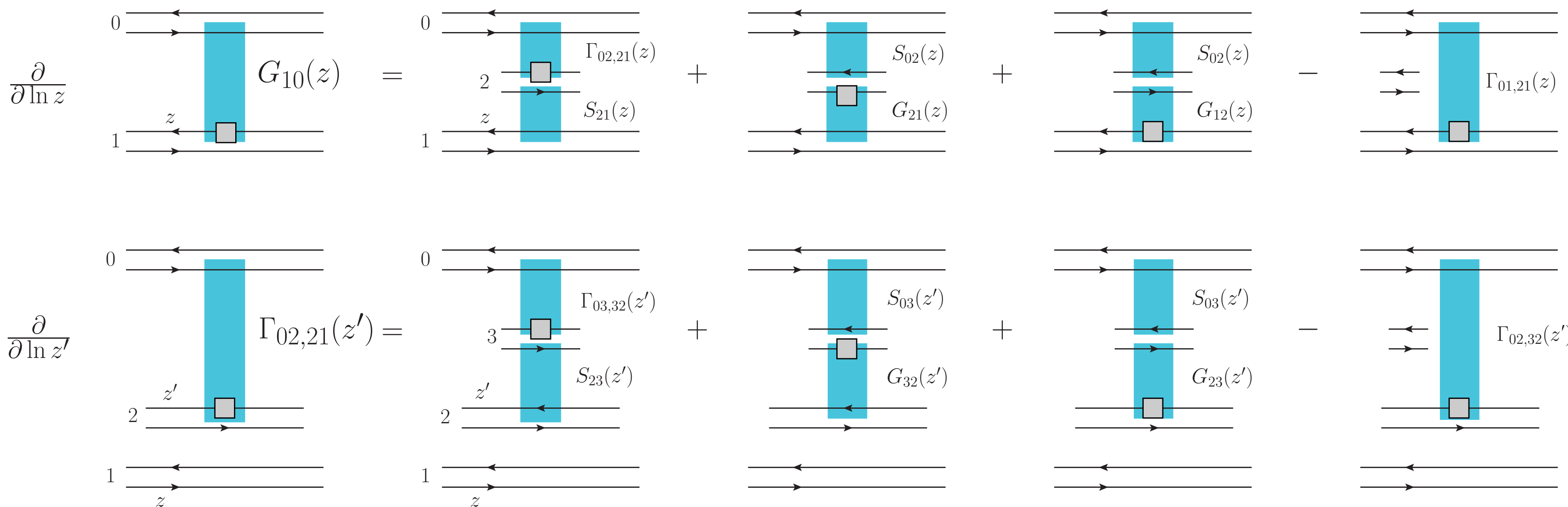
$$\alpha_s \ \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in UV and IR.

- This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.

Polarized Dipole Evolution in the Large-N_c Limit

In the large-N_c limit the equations close, leading to a system of 2 equations:



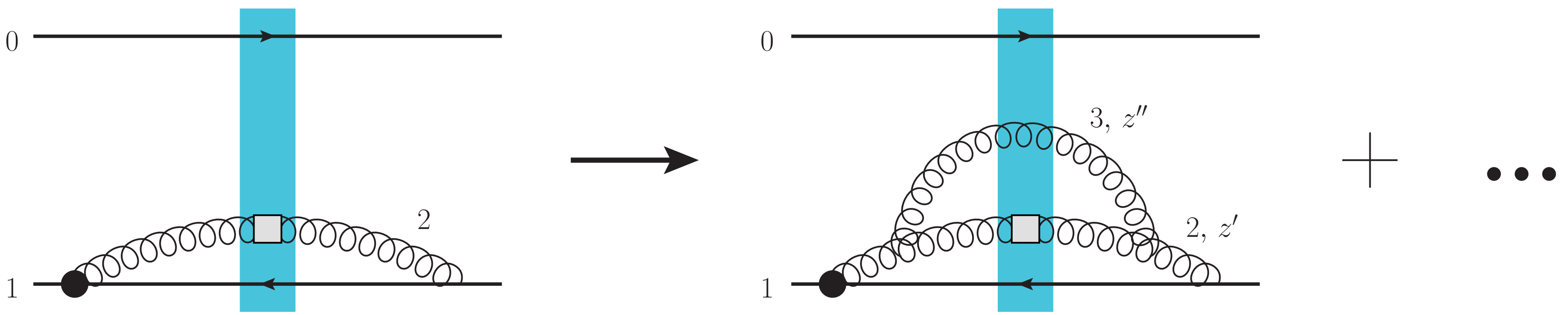
$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\,\Gamma_{02,\,21}(z')\,S_{21}(z') + 2\,G_{21}(z')\,S_{02}(z') + G_{12}(z')\,S_{02}(z') - \Gamma_{01,\,21}(z')\right]$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[2\,\Gamma_{03,\,32}(z'')\,S_{23}(z'') + 2\,G_{32}(z'')\,S_{03}(z'') + G_{23}(z'')\,S_{03}(z'') - \Gamma_{02,\,32}(z'')\right]$$

S = found from BK/JIMWLK, it is LLA

Your friendly “neighborhood” dipole

- There is a new object in the evolution equation – **the neighbor dipole**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may ’know’ about another dipole:



$$x_{21}^2\, z' \gg x_{32}^2\, z''$$

- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02,\,21}(z')$

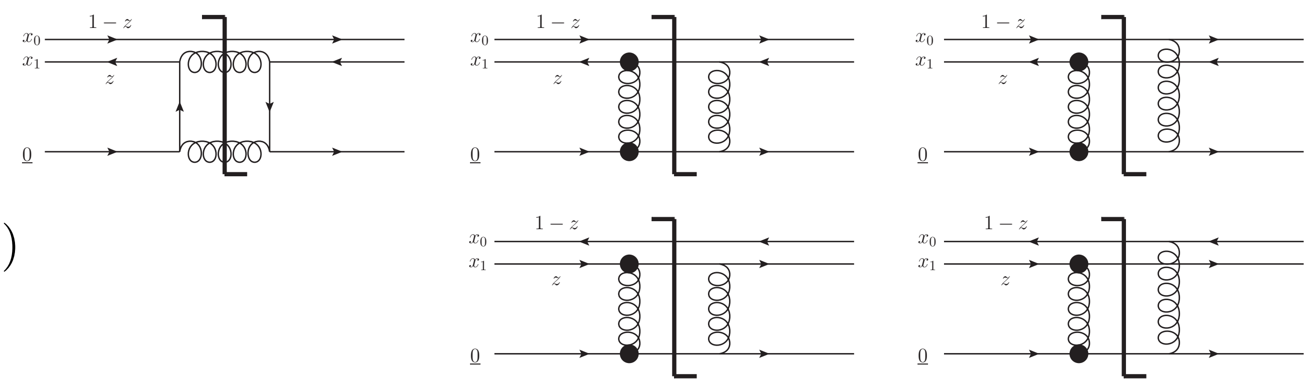
Large-N$_c$ Evolution

- In the strict DLA limit (S=1) and at large N$_c$ we get (here Γ is an auxiliary function we call the 'neighbour dipole amplitude')

$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z') + 3\, G(x_{21}^2, z')\right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\right\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z'') + 3\, G(x_{32}^2, z'')\right]$$

- The initial conditions are given by Born-level graphs



$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2\, \ln(zs\, x_{10}^2)\right]$$

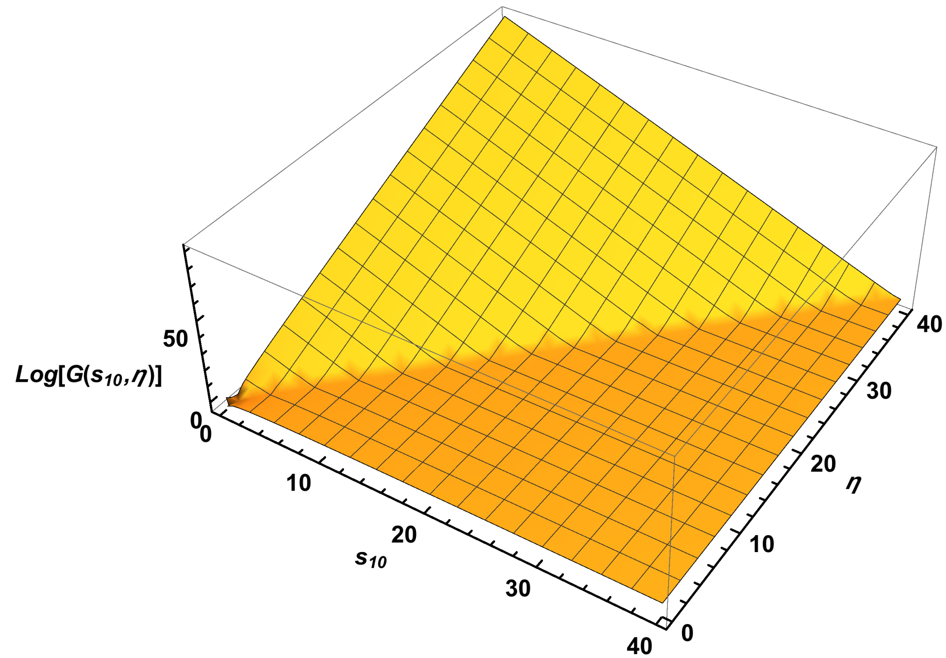
Scaling

- One can solve the helicity evolution equations numerically:

$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2} \qquad s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$

- The solution is well approximated by

$$G(s_{10}, \eta) \propto e^{2.31\,(\eta - s_{10})}$$



- This motivated us to look for the solution in the following scaling form:

$$G(s_{10}, \eta) = G(\eta - s_{10})$$
$$\Gamma(s_{10}, s_{21}, \eta') = \Gamma(\eta' - s_{10}, \eta' - s_{21})$$

Analytic Solution and Intercept

- The (dominant part of the) scaling solution is

$$G(\zeta) \approx \frac{1}{3} e^{\frac{4}{\sqrt{3}}\zeta}$$

$$\Gamma(\zeta, \zeta') \approx \frac{1}{3} e^{\frac{4}{\sqrt{3}}\zeta'} \left(4 e^{\frac{\zeta - \zeta'}{\sqrt{3}}} - 3 \right)$$

$$= G(\zeta') \left(4 e^{\frac{\zeta - \zeta'}{\sqrt{3}}} - 3 \right)$$

- The corresponding helicity intercept is

$$\alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.3094 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- This is in complete agreement with the numerical solution!

$$\alpha_h^q \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

see Matt's talk

Gluon Helicity TMDs

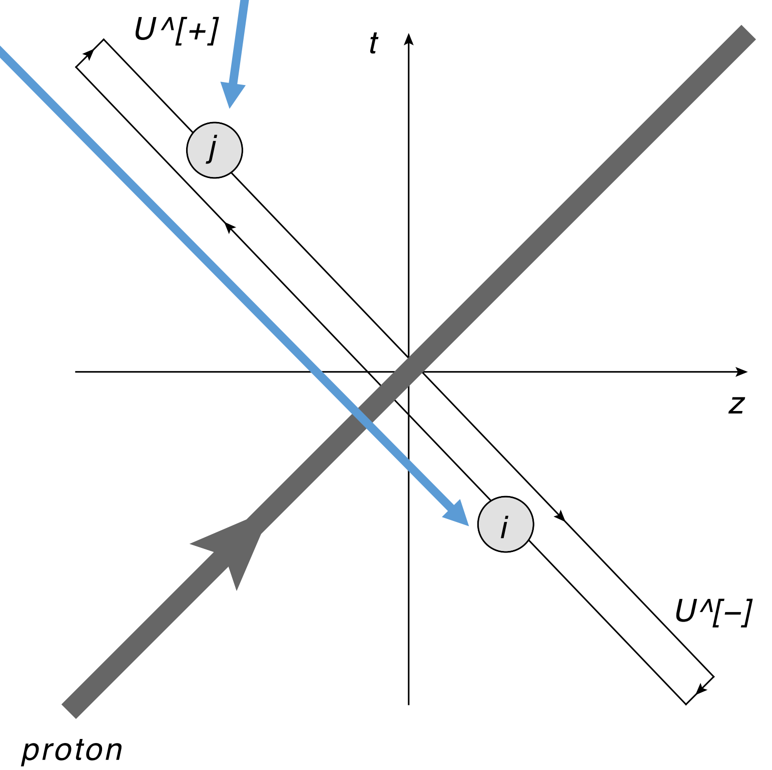
Dipole Gluon Helicity TMD

- Now let us repeat the calculation for gluon helicity TMDs.

- We start with the definition of the gluon dipole helicity TMD:

$$g_1^G(x,k_T^2) = \frac{-2i\,S_L}{x\,P^+} \int \frac{d\xi^-\,d^2\xi}{(2\pi)^3}\, e^{ixP^+\,\xi^- - i\underline{k}\cdot\underline{\xi}} \left\langle P,S_L|\epsilon_T^{ij}\,\mathrm{tr}\left[F^{+i}(0)\,\mathcal{U}^{[+]\dagger}[0,\xi]\,F^{+j}(\xi)\,\mathcal{U}^{[-]}[\xi,0]\right]|P,S_L\right\rangle_{\xi^+=0}$$

- Here $U^{[+]}$ and $U^{[-]}$ are future and past Wilson line staples (hence the name 'dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



Dipole Gluon Helicity TMD

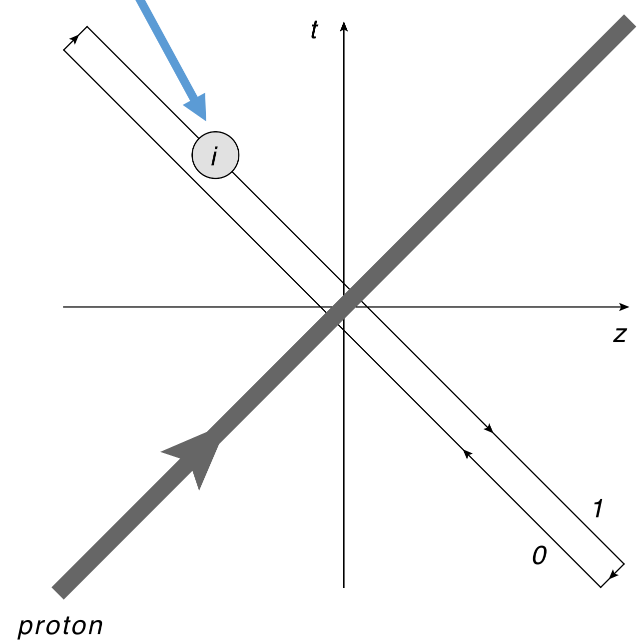
- At small x, the definition of dipole gluon helicity TMD can be massaged into

$$g_1^{G\,dip}(x, k_T^2) = \frac{8i\,N_c\,S_L}{g^2 (2\pi)^3} \int d^2 x_{10}\, e^{i \underline{k} \cdot \underline{x}_{10}}\; k_\perp^i \epsilon_T^{ij} \left[\int d^2 b_{10}\, G_{10}^j (zs = \frac{Q^2}{x}) \right]$$

- Here we obtain a new operator, which is a transverse vector ($A^-=0$ gauge):

$$G_{10}^i (z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \tilde{A}^i (x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

- Note that $k_\perp^i \, \epsilon_T^{ij}$ can be thought of as a transverse curl acting on $G_{10}^i (z)$ and not just on $\tilde{A}^i (x^-, \underline{x})$ -- different from the polarized dipole amplitude!



Dipole TMD vs dipole amplitude

- Note that the operator for the dipole gluon helicity TMD

$$G^i_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \tilde{A}^i(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

is different from the polarized dipole amplitude

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

- We conclude that the dipole gluon helicity TMD does not depend on the polarized dipole amplitude! (Hence the ‘dipole’ name may not even be valid for such TMD.)
- This is different from the unpolarized gluon TMD case.

Dictionary

- We seem to have two operators:

- Quark helicity

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

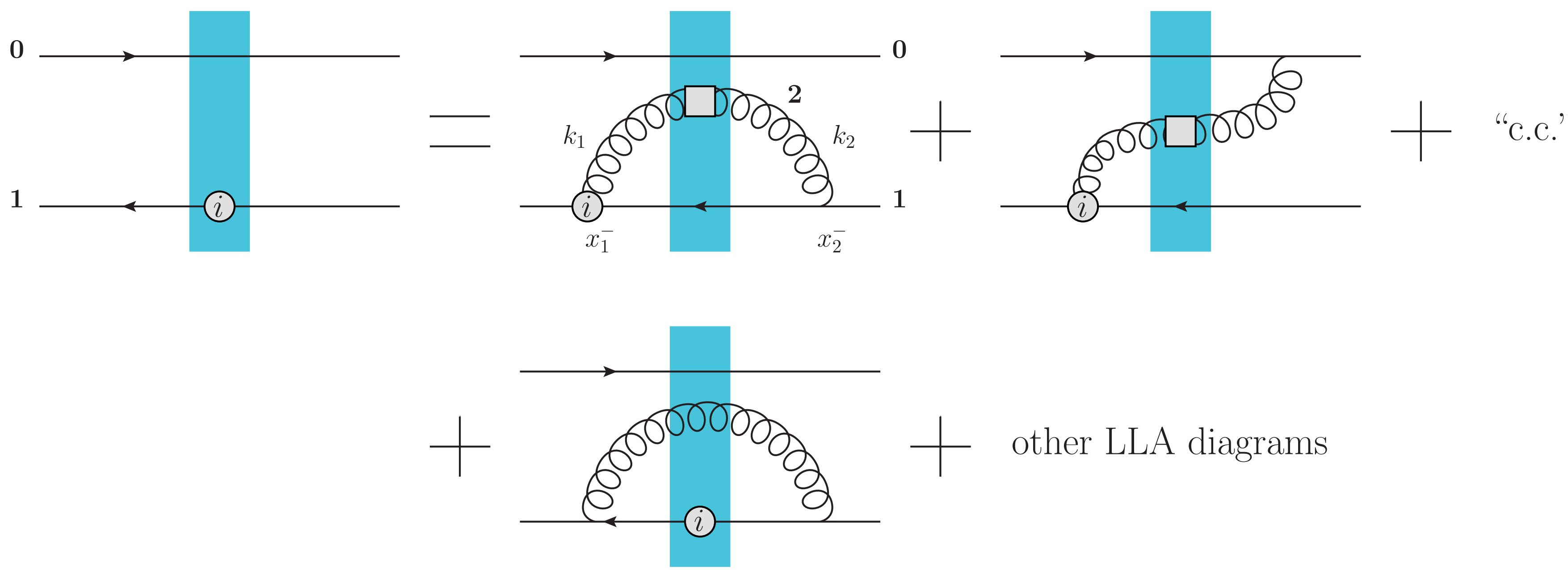
- Gluon helicity

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \tilde{A}^i(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

Gluon Helicity TMDs: Small-x Evolution

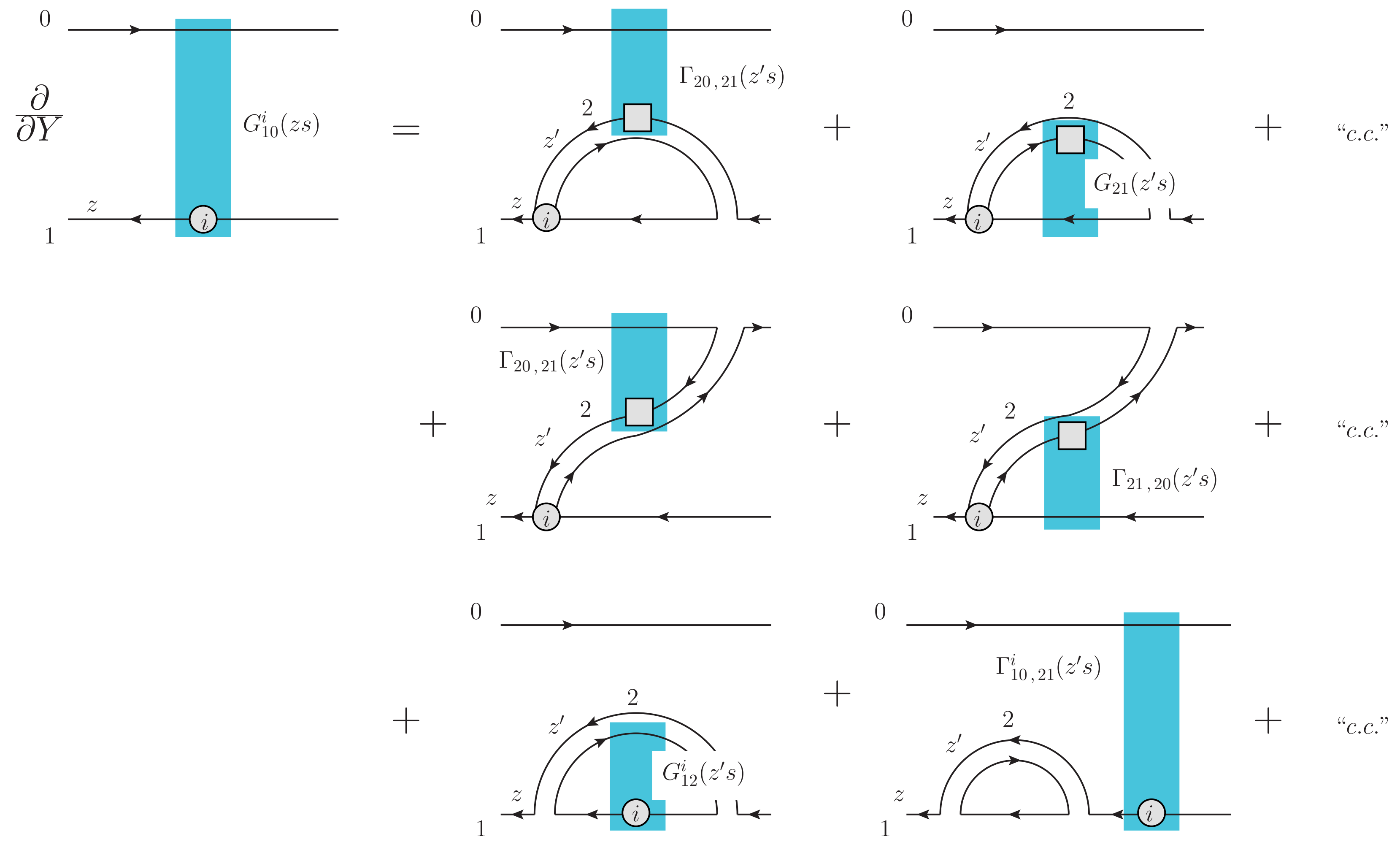
Evolution Equation

- To construct evolution equation for the operator G^i governing the gluon helicity TMD we resum similar (to the quark case) diagrams:



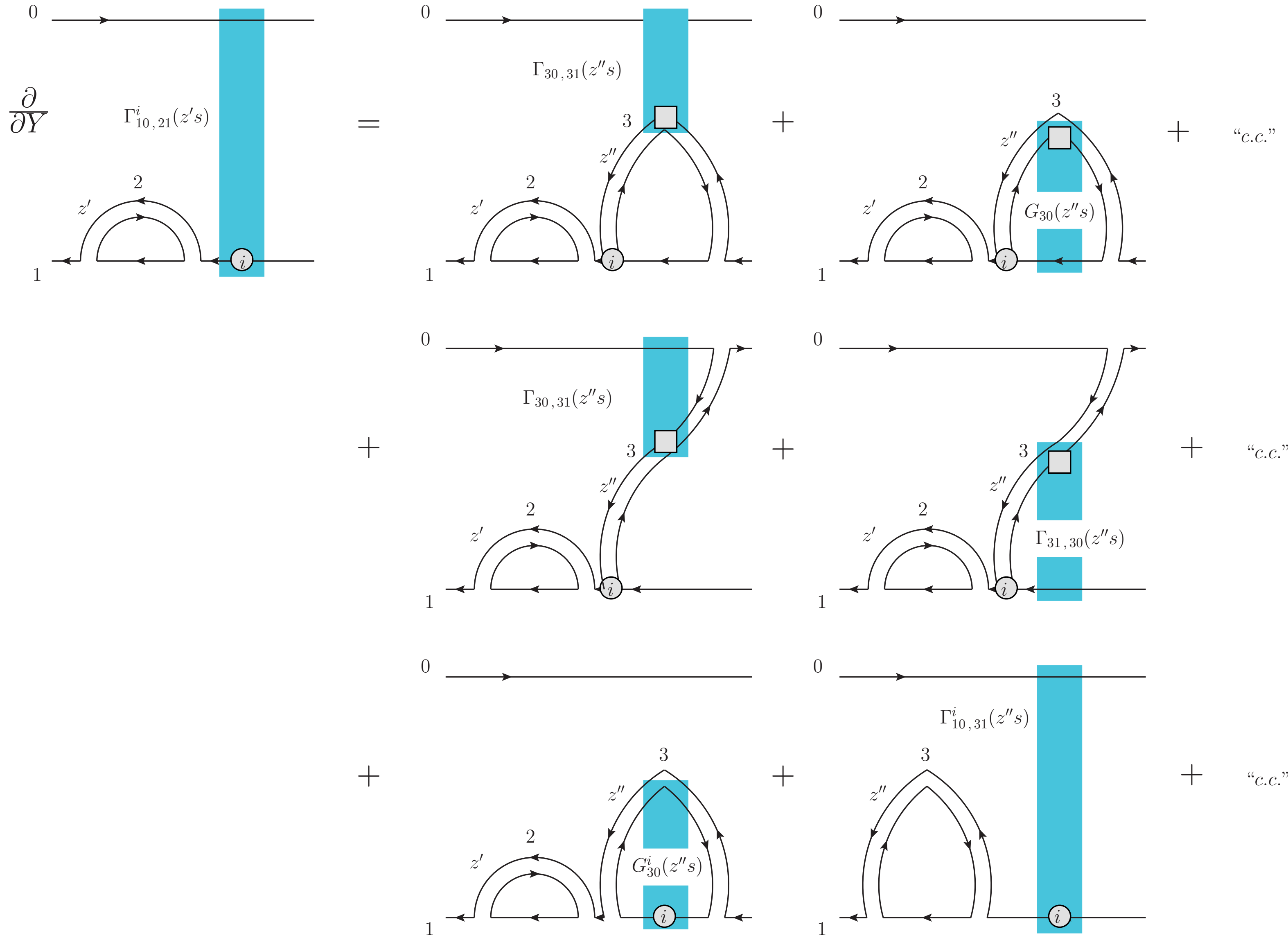
Large-N_c Evolution: Diagrams

- At large-N_c the equations are



Large-N_c Evolution: Diagrams

- and



Large-N_c Evolution: Equations

- This results in the following evolution equations:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{21})^j_\perp}{x_{21}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{20})^j_\perp}{x_{20}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's) \right]$$

$$\Gamma^i_{10\,21}(z's) = G^{i\,(0)}_{10}(z's) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{31})^j_\perp}{x_{31}^2} \left[\Gamma^{gen}_{30\,,\,31}(z''s) + G_{31}(z''s) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{30})^j_\perp}{x_{30}^2} \left[\Gamma^{gen}_{30\,,\,31}(z''s) + \Gamma^{gen}_{31\,,\,30}(z''s) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2\,,\,x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{31}^2}{x_{31}^2} \left[G^i_{13}(z''s) - \Gamma^i_{10\,,\,31}(z''s) \right].$$

Large-N_c Evolution: Equations

- Here

$$\Gamma_{gen}(x_{20}, x_{21}, z's) = \theta(x_{20} - x_{21})\,\Gamma(x_{20}, x_{21}, z's) + \theta(x_{21} - x_{20})\,G(x_{20}, z's)$$

is an object which we know from the quark helicity evolution, as the latter gives us G and Γ.

- Note that our evolution equations mix the gluon (G^i) and quark (G) small-x helicity evolution operators:

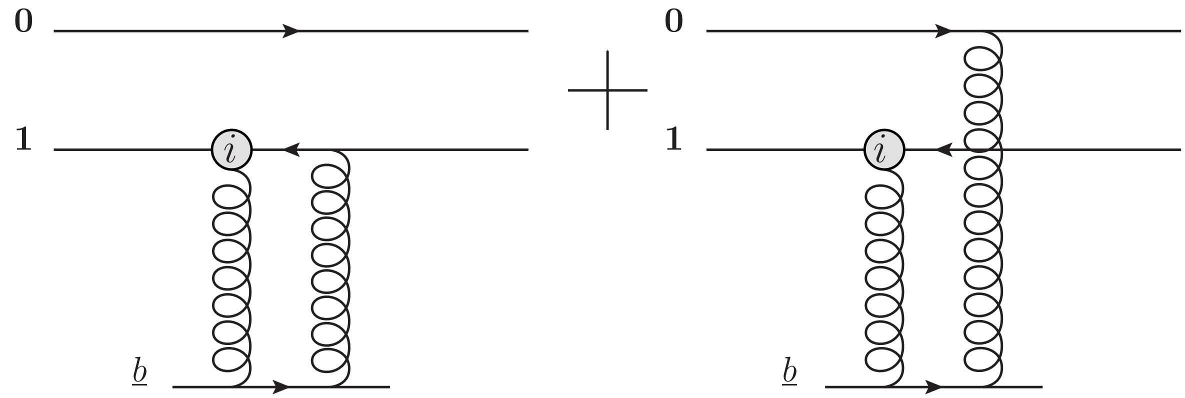
$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij}\,(x_{21})_\perp^j}{x_{21}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's)\right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij}\,(x_{20})_\perp^j}{x_{20}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's)\right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's)\right]$$

Initial Conditions

- Initial conditions for this evolution are given by the lowest order t-channel gluon exchanges:



$$\int d^2b\, G_{10}^{i\,(0)}(z) = \int d^2b\, \Gamma_{10,21}^{i\,(0)}(z) = -2\,\frac{\alpha_s^2 C_F}{N_c}\,\pi\,\epsilon^{ij}\,x_{10}^j\,\ln\frac{1}{x_{10}\,\Lambda}$$

- Note that these initial conditions have no ln s, unlike the initial conditions for the quark evolution:

$$G^{(0)}(x_{10}^2, z) = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = -2\,\frac{\alpha_s^2 C_F}{N_c}\pi\,\ln(zs\,x_{10}^2)$$

Large-N_c Evolution: Power Counting

- The kernel mixing G^i or Γ^i with G and Γ is LLA:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{21})^j_\perp}{x^2_{21}} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{20})^j_\perp}{x^2_{20}} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x^2_{10} s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x^2_{10}} \frac{dx^2_{21}}{x^2_{21}} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's) \right]$$

LLA

DLA

- But, the initial conditions for G and Γ have an extra ln s, making the two terms comparable (order-α_s^2 in $\alpha_s \, ln^2 s \sim 1$ DLA power counting).

Gluon Helicity TMDs: Small-x Asymptotics

Large-N_c Evolution Equations: Solution

- To solve the equations, first decompose the relevant object as follows:

$$\int d^2b\, G^i_{10}(z) = x^i_{10}\, G_1(x^2_{10}, z) + \epsilon^{ij}\, x^j_{10}\, G_2(x^2_{10}, z)$$

$$\int d^2b\, \Gamma^i_{10}(z) = x^i_{10}\, \Gamma_1(x^2_{10}, z) + \epsilon^{ij}\, x^j_{10}\, \Gamma_2(x^2_{10}, z)$$

- It turns out that only G_2 and Γ_2 contribute to evolution and to the gluon helicity TMD.

Large-N_c Evolution Equations: Solution

- Plugging in the analytic solution for the quark helicity operators (Matt's talk), we get

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) - \frac{\alpha_s N_c}{3\pi} \frac{1}{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}} \left(zs x_{10}^2\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}} \ln\frac{1}{x_{10}\Lambda}$$

$$- \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2}\, \Gamma_2(x_{10}^2, x_{21}^2, z's),$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) - \frac{\alpha_s N_c}{3\pi} \frac{1}{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}} \left(z' s x_{10}^2\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}} \ln\frac{1}{x_{10}\Lambda}$$

$$- \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left[x_{10}^2\,,\, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{31}^2}{x_{31}^2}\, \Gamma_2(x_{10}^2, x_{31}^2, z'' s)$$

Large-N_c Evolution Equations: Scaling

- Just like in the quark helicity evolution case, the equations simplify once we recognize the following scaling property:

$$G_2(x_{10}^2, zs) = G_2\left(\sqrt{\frac{\alpha_s\, N_c}{2\pi}}\, \ln(zs x_{10}^2)\right)$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = \Gamma_2\left(\sqrt{\frac{\alpha_s\, N_c}{2\pi}}\, \ln(z's x_{10}^2), \sqrt{\frac{\alpha_s\, N_c}{2\pi}}\, \ln(z's x_{21}^2)\right)$$

- The equations become

$$G_2(\zeta) = -\frac{1}{2}\sqrt{\frac{\alpha_s\, N_c}{6\pi}} e^{\frac{4}{\sqrt{3}}\zeta} - \int\limits_0^\zeta d\xi \int\limits_0^\xi d\xi'\, \Gamma_2(\xi, \xi'),$$

$$\Gamma_2(\zeta, \zeta') = -\frac{1}{2}\sqrt{\frac{\alpha_s\, N_c}{6\pi}} e^{\frac{4}{\sqrt{3}}\zeta} - \int\limits_0^{\zeta'} d\xi \int\limits_0^\xi d\xi'\, \Gamma_2(\xi, \xi') - \int\limits_{\zeta'}^{\zeta} d\xi \int\limits_0^{\zeta'} d\xi'\, \Gamma_2(\xi, \xi')$$

Large-N_c Evolution Equations: Solution

- These equations can be solved in the asymptotic high-energy region using a combination of ODE solving and Laplace transform, yielding

$$G_2(\zeta \gg 1) = -\frac{1}{3}\sqrt{\frac{2\,\alpha_s\,N_c}{\pi}}\,\frac{19\sqrt{3}}{64}\,e^{\frac{13}{4\sqrt{3}}\,\zeta},$$

$$\Gamma_2(\zeta \gg 1, \zeta' \gg 1) = -\frac{1}{3}\sqrt{\frac{2\,\alpha_s\,N_c}{\pi}}\left[\frac{\sqrt{3}}{4}\,e^{\frac{4}{\sqrt{3}}\zeta - \frac{\sqrt{3}}{4}\,\zeta'} + \frac{3\sqrt{3}}{64}\,e^{\frac{4}{\sqrt{3}}\zeta' - \frac{\sqrt{3}}{4}\,\zeta}\right]$$

- The small-x gluon helicity intercept is

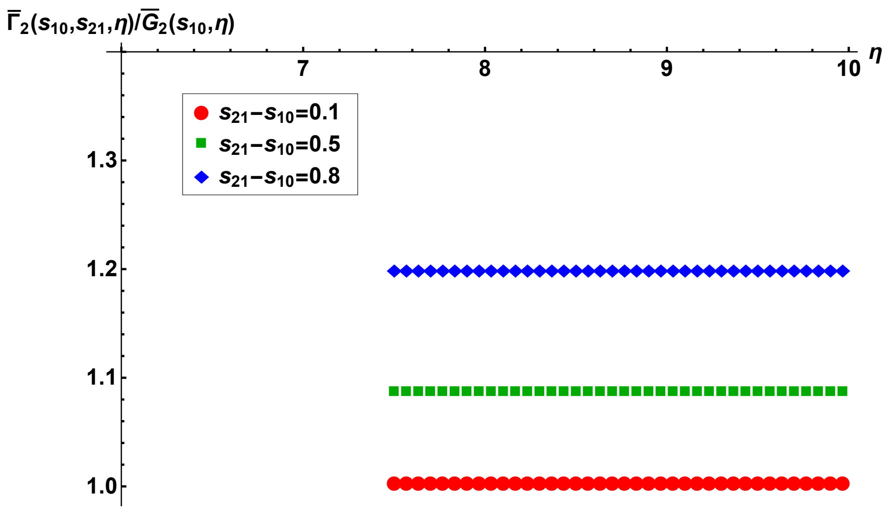
$$\alpha_h^G = \frac{13}{4\sqrt{3}}\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}} \approx 1.88\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}}$$

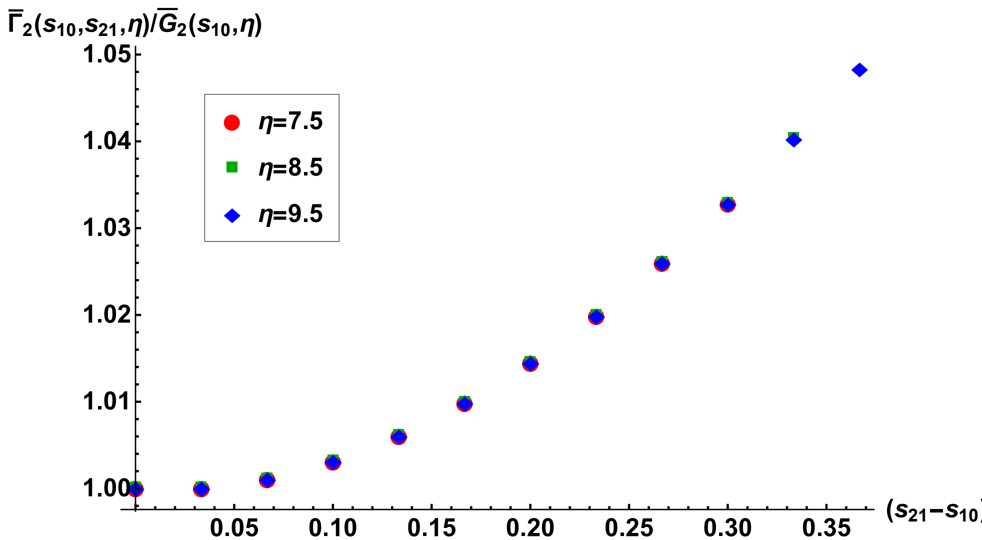
- We obtain the small-x asymptotics of the gluon helicity distributions:

$$\Delta G(x, Q^2) \sim g_{1L}^{G\,dip}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}}}$$

Scaling Solution Cross-Check

- One can check the scaling property $\frac{\Gamma_2}{G_2} = f(s_{21} - s_{10})$ of our analytic solution in the numerical solution of our equations:





$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$$

$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$

$$G_2(s_{10}, \eta) = G_2(\eta - s_{10})$$

$$\Gamma_2(s_{10}, s_{21}, \eta') = \Gamma_2(\eta' - s_{10}, \eta' - s_{21})$$

Conclusions

- We thus conclude that the small-x asymptotics of gluon helicity (at large N_c) is

$$\Delta G(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \tfrac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

while the quark helicity asymptotics is (Matt's talk)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

- Swell…
- May use our approach to constrain the quark and gluon spin (and OAM) at small x (in progress, a long-term goal).

Backup Slides

Large-N_c Evolution: Equations

- This results in the following evolution equations:

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\underline{x}_{10} \cdot \underline{x}_{21}}{x_{10}^2 \, x_{21}^2} \left[\Gamma_{gen}(x_{20}^2, x_{21}^2, z's) + G(x_{21}^2, z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\underline{x}_{10} \cdot \underline{x}_{20}}{x_{10}^2 \, x_{20}^2} \left[\Gamma_{gen}(x_{20}^2, x_{21}^2, z's) + \Gamma_{gen}(x_{21}^2, x_{20}^2, z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \, \Gamma_2(x_{10}^2, x_{21}^2, z's)$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\underline{x}_{10} \cdot \underline{x}_{31}}{x_{10}^2 \, x_{31}^2} \left[\Gamma_{gen}(x_{30}^2, x_{31}^2, z''s) + G(x_{31}^2, z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\underline{x}_{10} \cdot \underline{x}_{30}}{x_{10}^2 \, x_{30}^2} \left[\Gamma_{gen}(x_{30}^2, x_{31}^2, z''s) + \Gamma_{gen}(x_{31}^2, x_{30}^2, z''s) \right]$$

$$- \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2 \, , \, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{31}^2}{x_{31}^2} \, \Gamma_2(x_{10}^2, x_{31}^2, z''s)$$

Large-N_c Evolution: Equations

- This results in the following evolution equations:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{21})^j_\perp}{x_{21}^2} \, \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{20})^j_\perp}{x_{20}^2} \, \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 \, x_{20}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's) \right]$$

$$\Gamma^i_{10\,21}(z's) = G^{i\,(0)}_{10}(z's) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{31})^j_\perp}{x_{31}^2} \, \left[\Gamma^{gen}_{30\,,\,31}(z''s) + G_{31}(z''s) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \times \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{30})^j_\perp}{x_{30}^2} \, \left[\Gamma^{gen}_{30\,,\,31}(z''s) + \Gamma^{gen}_{31\,,\,30}(z''s) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \frac{x_{10}^2}{x_{31}^2 \, x_{30}^2} \left[G^i_{13}(z''s) - \Gamma^i_{10\,,\,31}(z''s) \right]$$