High-Precision QCD
with the Lattice

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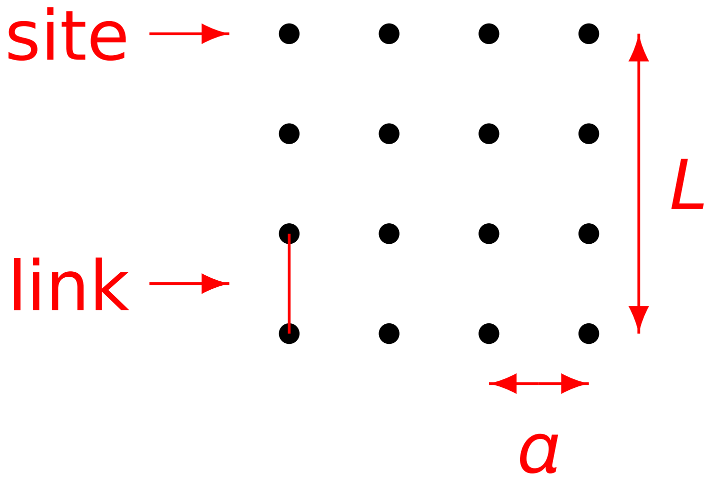
Lattice QCD

Lattice Approximation

Continuous

Space & Time ⟶



⇒ Fields $\psi(x)$, $A_\mu(x)$ specified only at grid sites (or links); interpolate for other points.

⇒ Solving QCD → multidimensional integration (billions of variables ⇒ Monte Carlo):

$$\boxed{\int \mathcal{D}A_\mu \ldots \mathrm{e}^{-\int L dt} \longrightarrow \int \prod_{x_j \in \mathrm{grid}} dA_\mu(x_j) \ldots \mathrm{e}^{-a\sum L_j}}$$

Lattice Simulations

- Parameters: choose bare α_s then tune, for example,

$$\begin{array}{rcl} m_u = m_d & \longleftrightarrow & m_\pi^2 \\ m_s & \longleftrightarrow & 2m_K^2 - m_\pi^2 \\ m_c & \longleftrightarrow & (3m_\psi + m_{\eta_c})/4 \\ m_b & \longleftrightarrow & (3m_\Upsilon + m_{\eta_b})/4 \\ a & \longleftrightarrow & f_\pi \text{ or } m_{\Upsilon'} - m_\Upsilon \ \ldots \end{array}$$

✦ Tunings decouple.

✦ Experimental errors negligible.

✦ Small e/m, isospin errors.

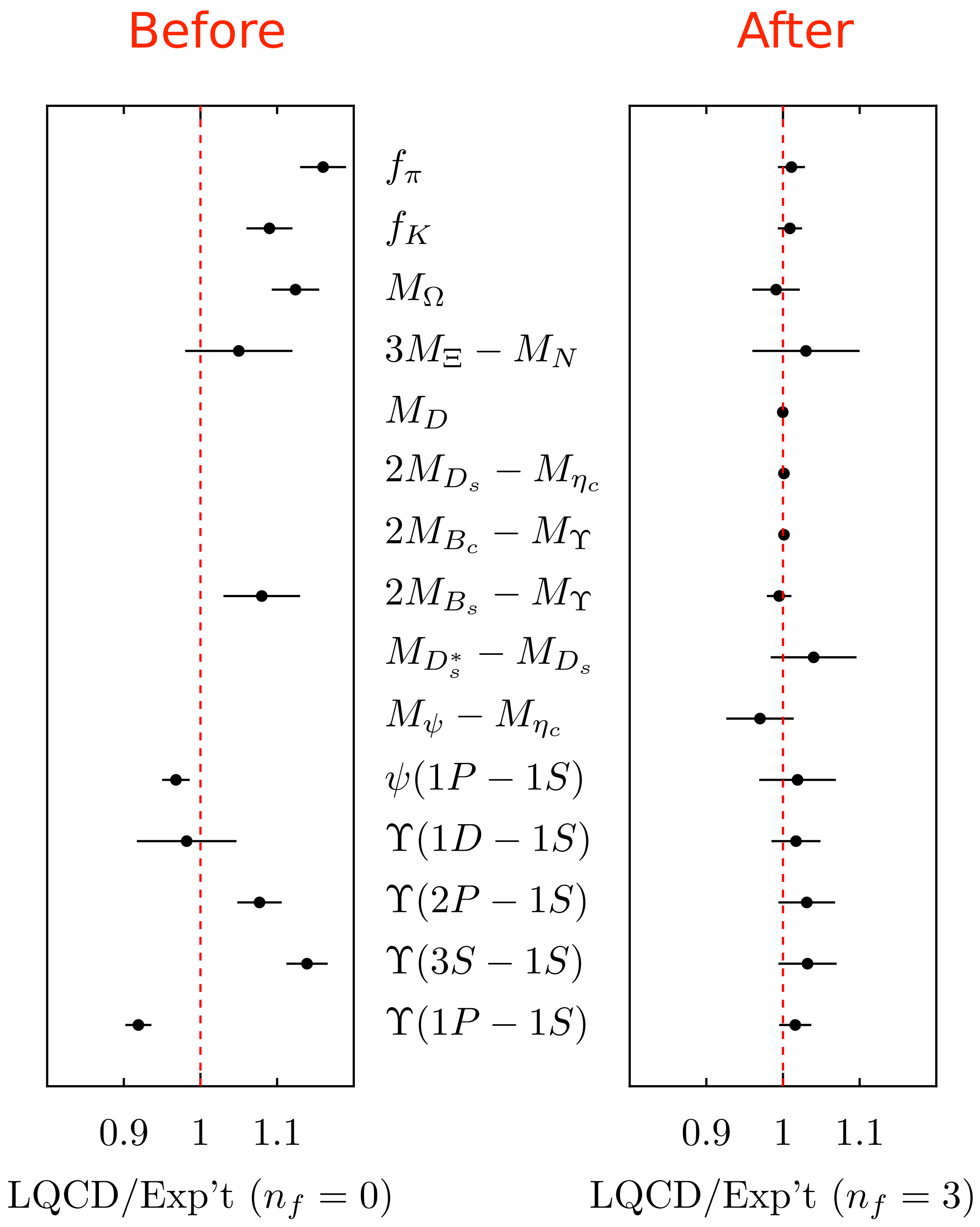
- Generate Monte Carlo results for multiple lattice spacings (masses, volumes ...). Extrapolate to physical values.

- Use vacuum expectation values of numerous operators to extract lots of physics with no free parameters!

Lattice QCD Comes of Age

Before 2004: Unrealistic treatment of light sea quarks ⇒ large uncontrolled systematic errors.

After 2004: New lattice quark actions ⇒ realistic simulations of light sea quarks ⇒ 1% or better errors possible for first time in history.



[Davies et al (2004).]

Lattice QCD = an Effective Field Theory

- Finite lattice spacing ⇒ UV cutoff $\Lambda = \pi/a$.
- Effective Lagrangian:

$$\mathcal{L}^{(a)}_{\mathrm{lat}} \approx \sum_{\mu\nu} \tfrac{1}{2}\mathrm{Tr}F^2_{\mu\nu} + c_2\, a^2\, \mathrm{Tr}F_{\mu\nu}(D^2_\mu + D^2_\nu)F_{\mu\nu} + \cdots$$

$$+ \sum_q \overline{\psi}_q (iD\cdot\gamma - m_q)\psi_q + d_2\, a^2 \sum_\mu \overline{\psi}_q iD^3_\mu \gamma_\mu \psi_q + \cdots$$

- Wrong but suppressed by $(ap)^2$ where p = typical momentum.
- Break Lorentz invariance, *etc.*
- Remove by taking $a \to 0$.
- Remove with correction terms.

Three Examples

Example: QCD Parameters (α_s, $\bar{m}_q$) from *jj* Correlators

Heavy Quark Pseudoscalar Correlator

- Compute $h\bar{h}$ (heavy-quark) correlator:

$\overline{\psi}_h\gamma_5\psi_h$

$$G(t) = a^6 \sum_{\mathbf{x}} (am_{0h})^2 \langle 0 | j_5(\mathbf{x}, t) j_5(0,0) | 0 \rangle$$

Euclidean t

- Mass factors imply UV finite (PCAC because HISQ).
- Implies:

$$\boxed{G_{\mathrm{contin}}(t) = G_{\mathrm{lat}}(t) + \mathcal{O}(a^2) \qquad \text{for all t}}$$

[Follana et al (HPQCD, Karlsruhe) 0805.2999

McNeile et al (HPQCD) 1004.4285

Chakraborty et al (HPQCD) 1408.4169]

α_s and $\overline{m}_q$ from Moments

Low-n moments perturbative ($E_{\text{threshold}} - E \approx m_{\eta h} \gg \Lambda_{\text{QCD}}$):

$$G_n = \sum_t (t/a)^n G(t) \rightarrow \frac{\partial^n}{\partial E^n} \Pi(E=0)$$

Implies ($n \geq 4$):

from continuum pert'n theory

(known to 3rd order for n=4..10)

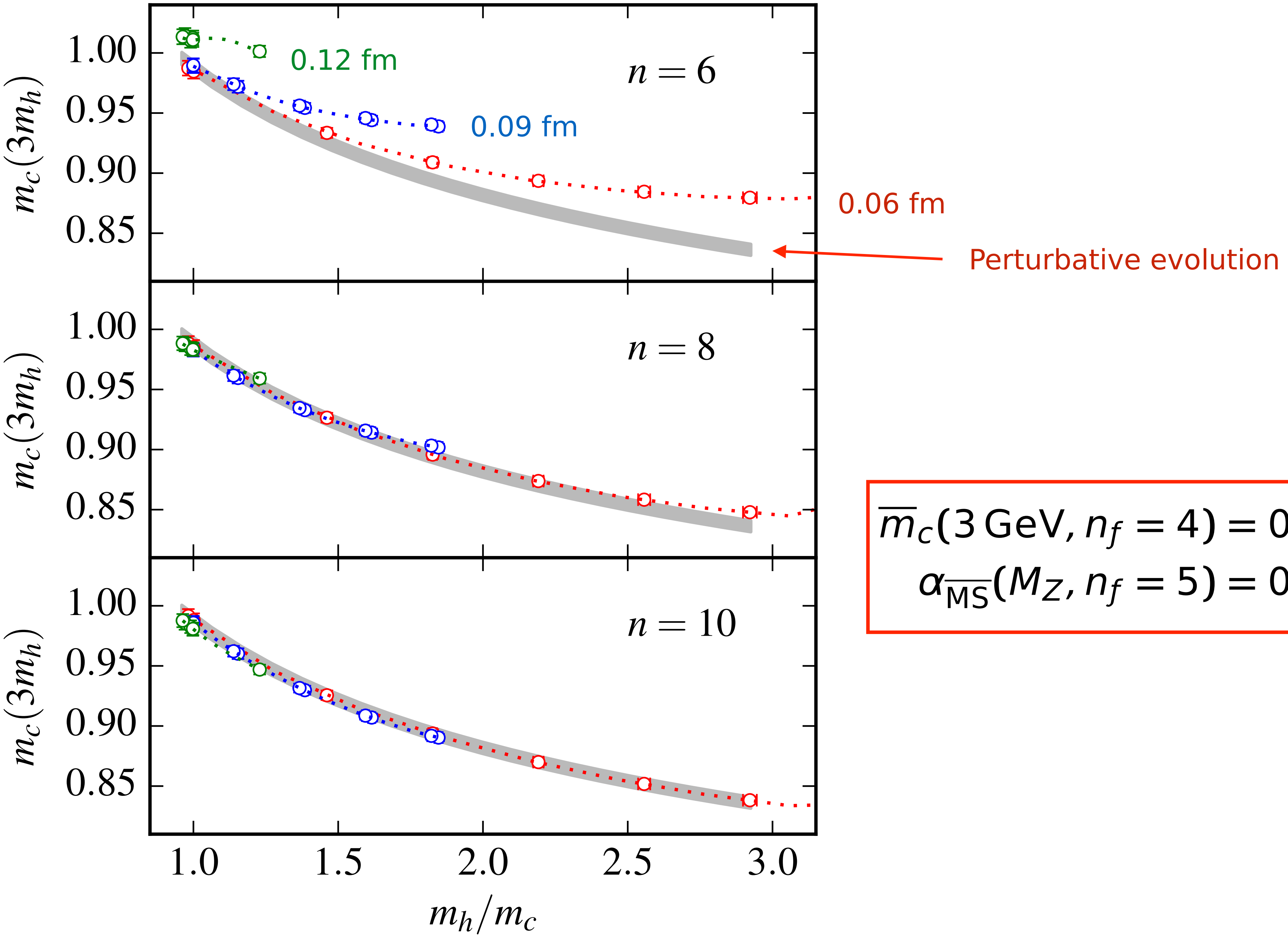
$$G_n = \frac{g_n\left(\alpha_{\overline{\text{MS}}}, \mu/m_h\right)}{(a\overline{m}_h(\mu))^{n-4}}$$

from simulations

solve for coupling

and mass

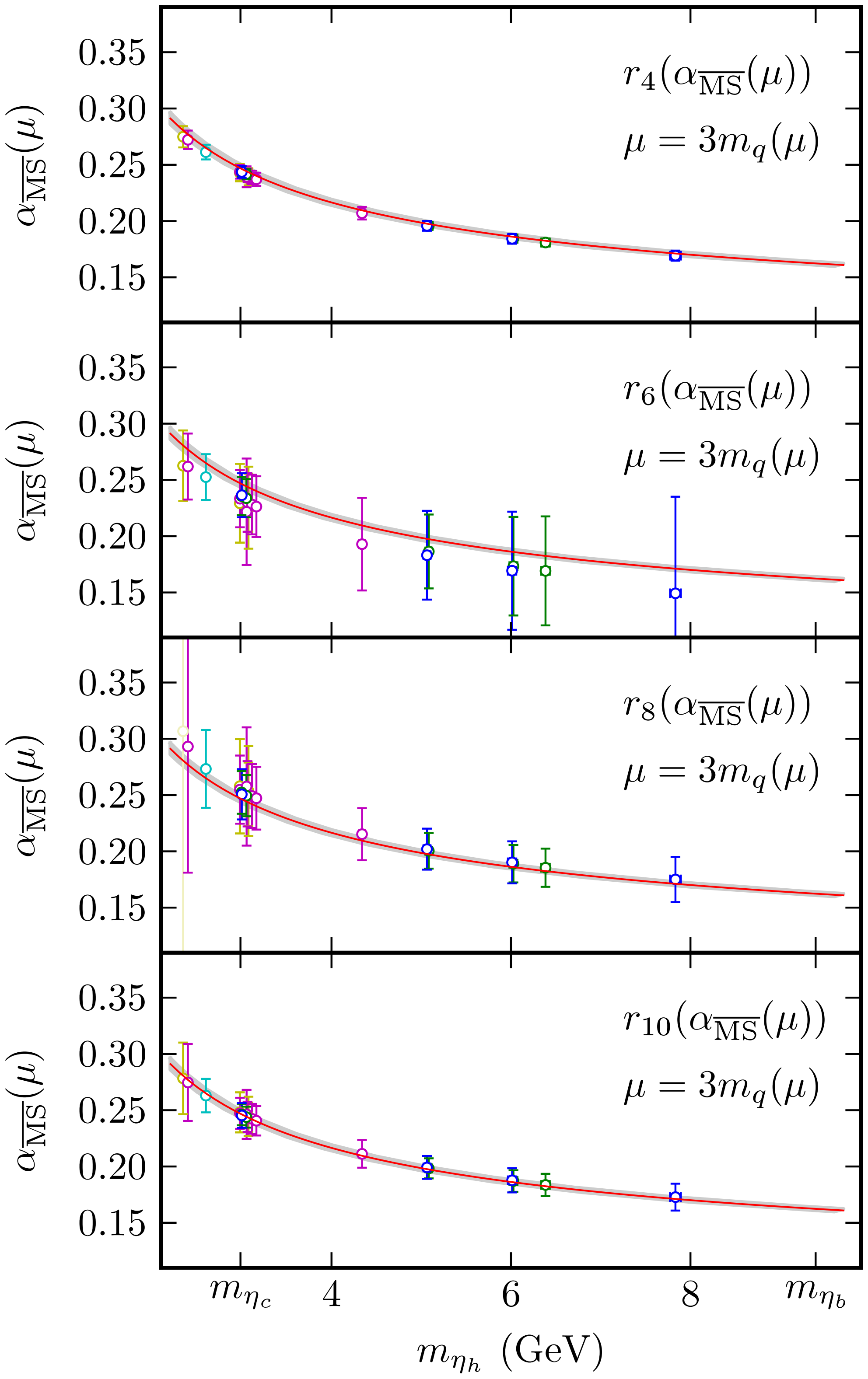
Results (n_f=4 [1408.4169])



$$\overline{m}_c(3\,\text{GeV}, n_f = 4) = 0.9851(63)\,\text{GeV}$$

$$\alpha_{\overline{\text{MS}}}(M_Z, n_f = 5) = 0.11822(74)$$

Results (n_f=3 [1004.4285])



Lattice spacings to 0.045fm

⇒ m_b possible.

$$\overline{m}_c(3\,\text{GeV}, n_f = 4) = 0.986(6)\,\text{GeV}$$

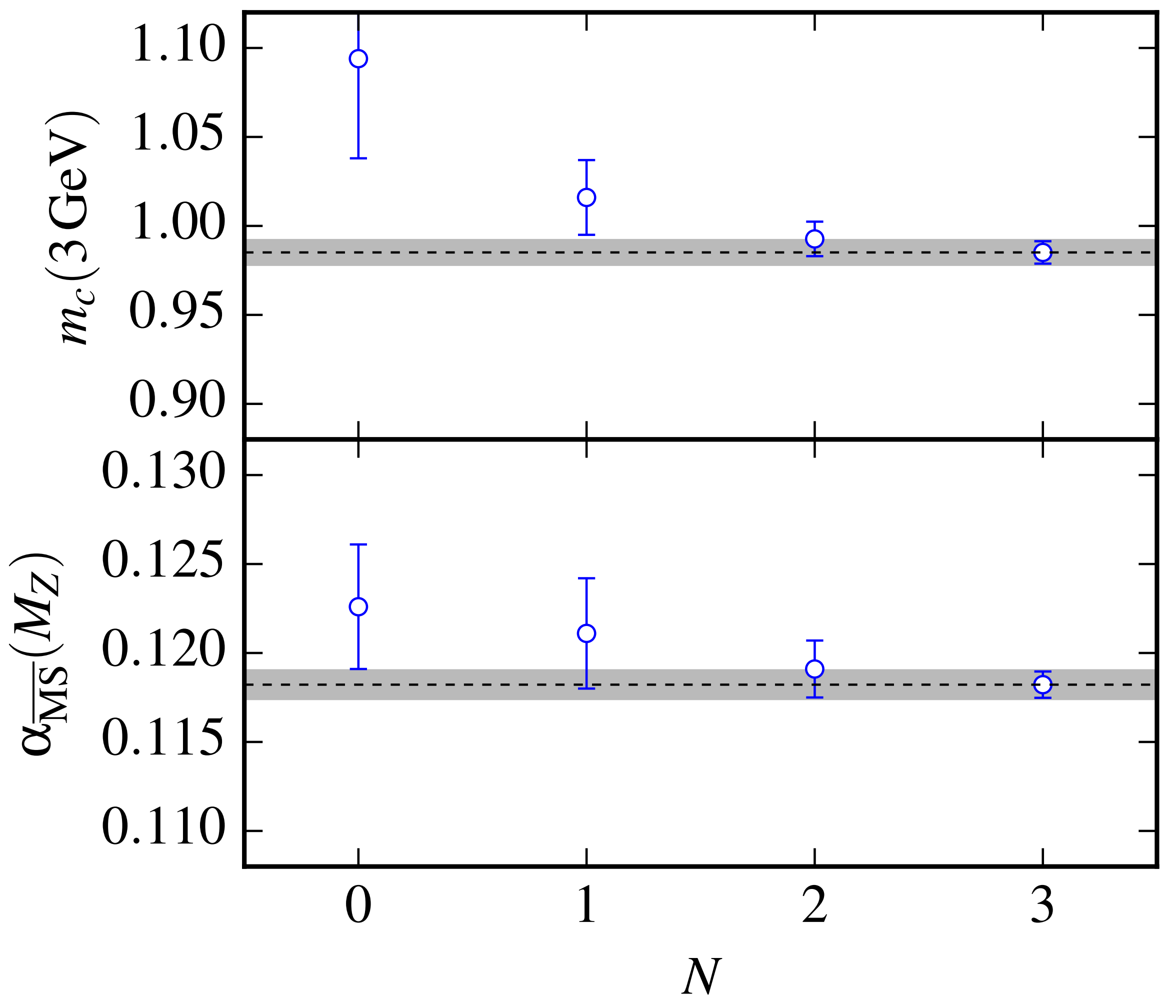
$$\overline{m}_b(\overline{m}_b, n_f = 5) = 4.164(23)\,\text{GeV}$$

$$\overline{m}_b(\mu, n_f)/\overline{m}_c(\mu, n_f) = 4.53(4)$$

$$\alpha_{\overline{\rm MS}}(M_Z, n_f = 5) = 0.1183(7)$$

Sanity Checks

convergence of pert'n theory

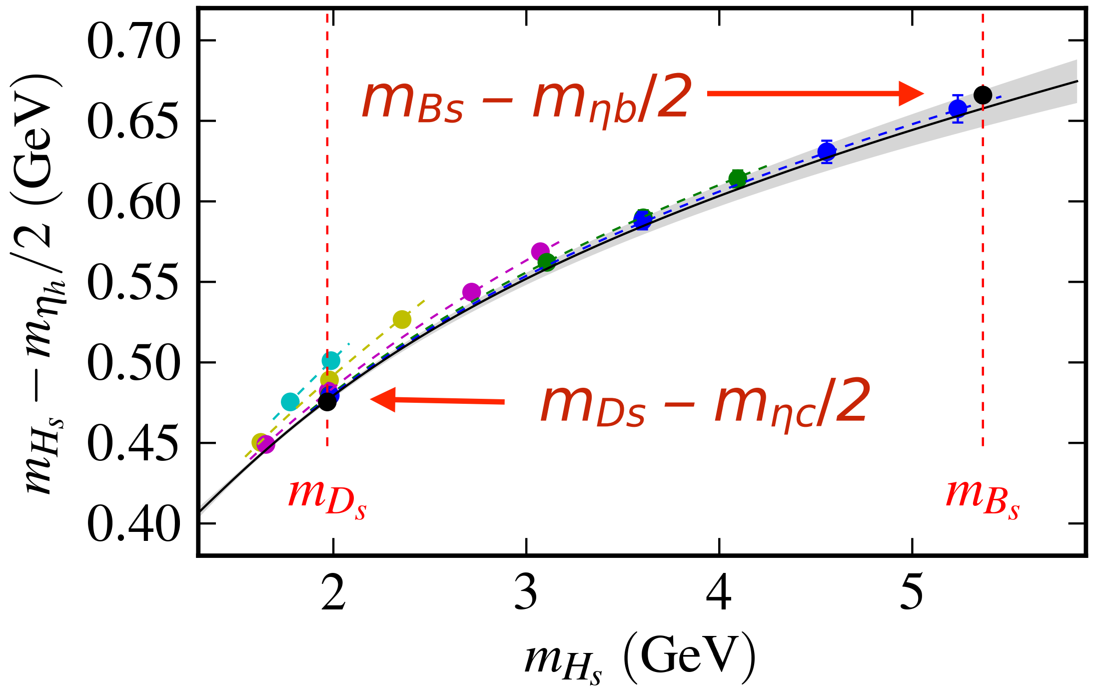


Test evolution by letting β_0 and γ_0 float, as fit parameters ⇒

$$\beta_0 = 0.675(54) \quad \text{(exact 0.663)}$$
$$\gamma_0 = 0.292(19) \quad \text{(exact 0.318)}$$

Nonperturbative determination of $\bar{m}_b/\bar{m}_c$ = 4.49(4) agrees with perturbative value 4.53(4).

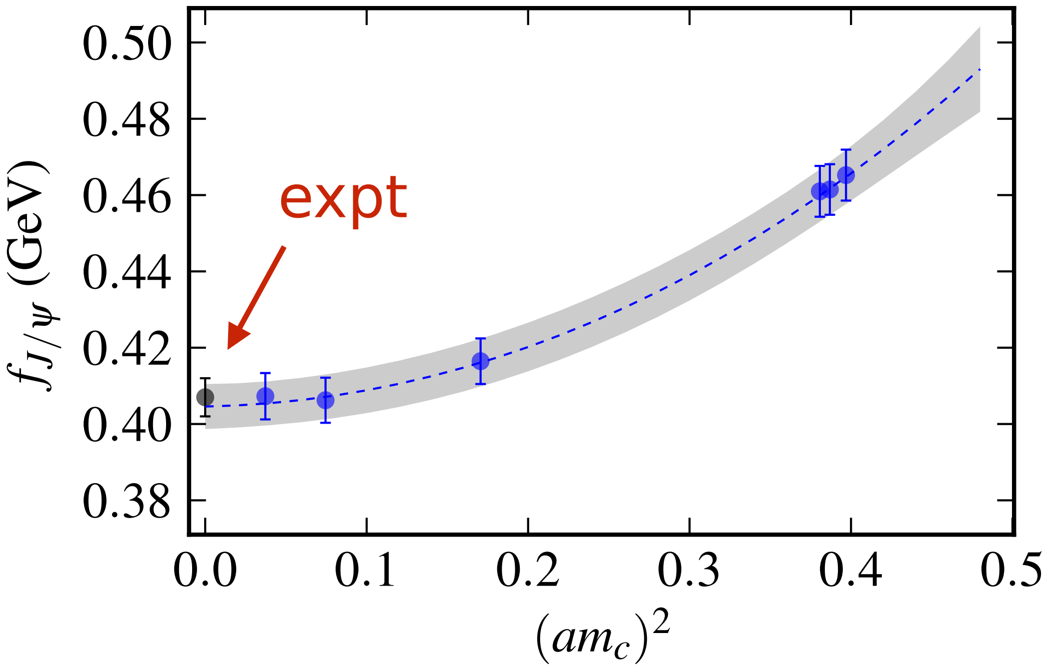
Sanity Checks


$m_{Bs} - m_{\eta b}/2$

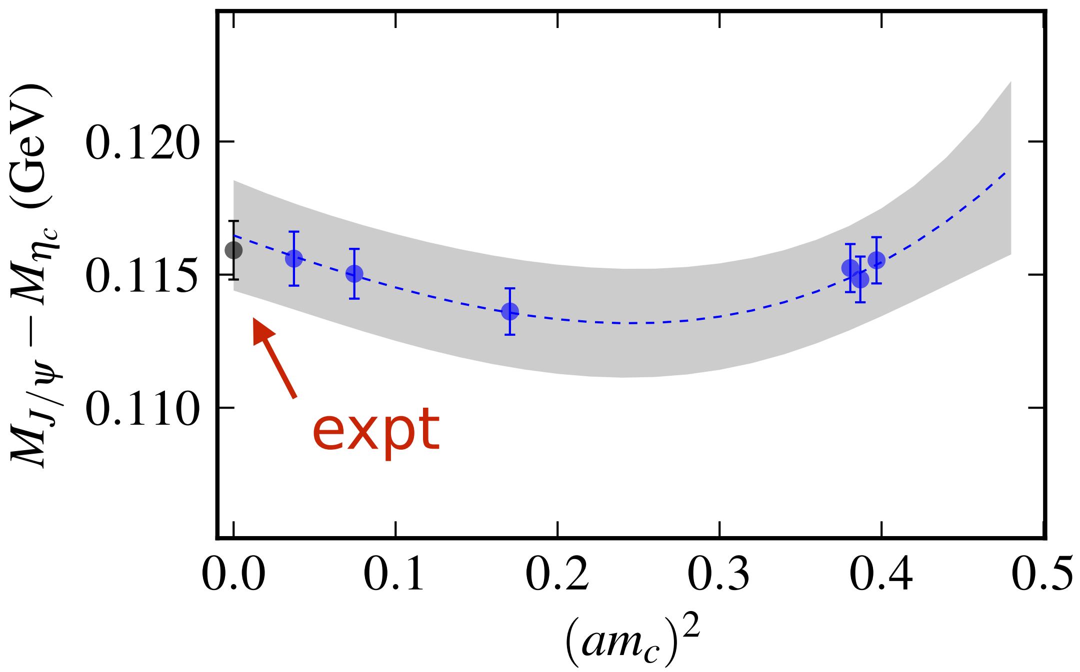
$m_{Ds} - m_{\eta c}/2$

No free parameters!

$\psi \rightarrow e\bar{e}$ decay const


expt

ψ hyperfine splitting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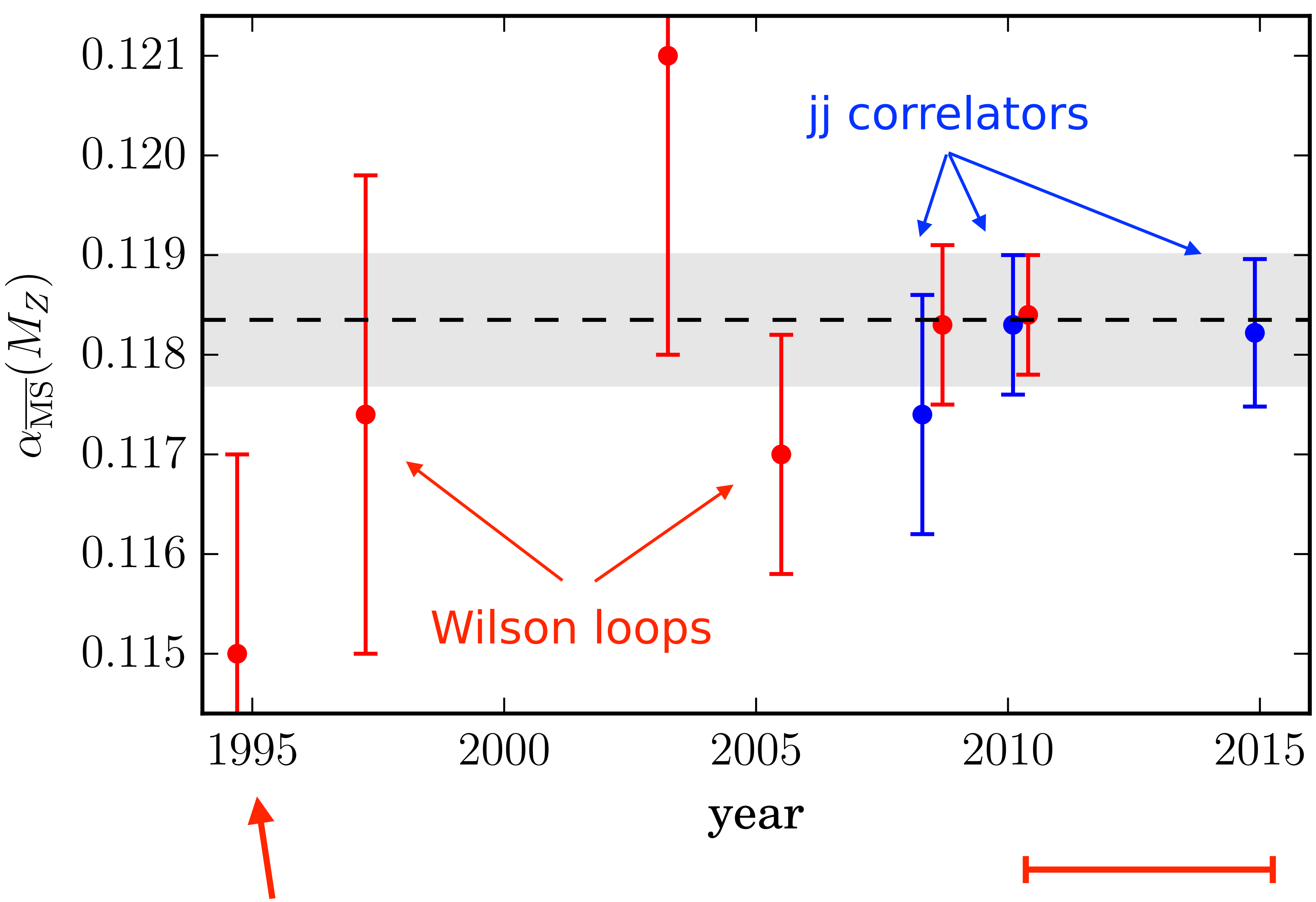
Sanity Check: HPQCD $\alpha_{\overline{\mathrm{MS}}}$ History



π/a = 4.8 GeV n_f = 0, 2 → 3

simple discretization

2nd order in α_s

(4 data points)

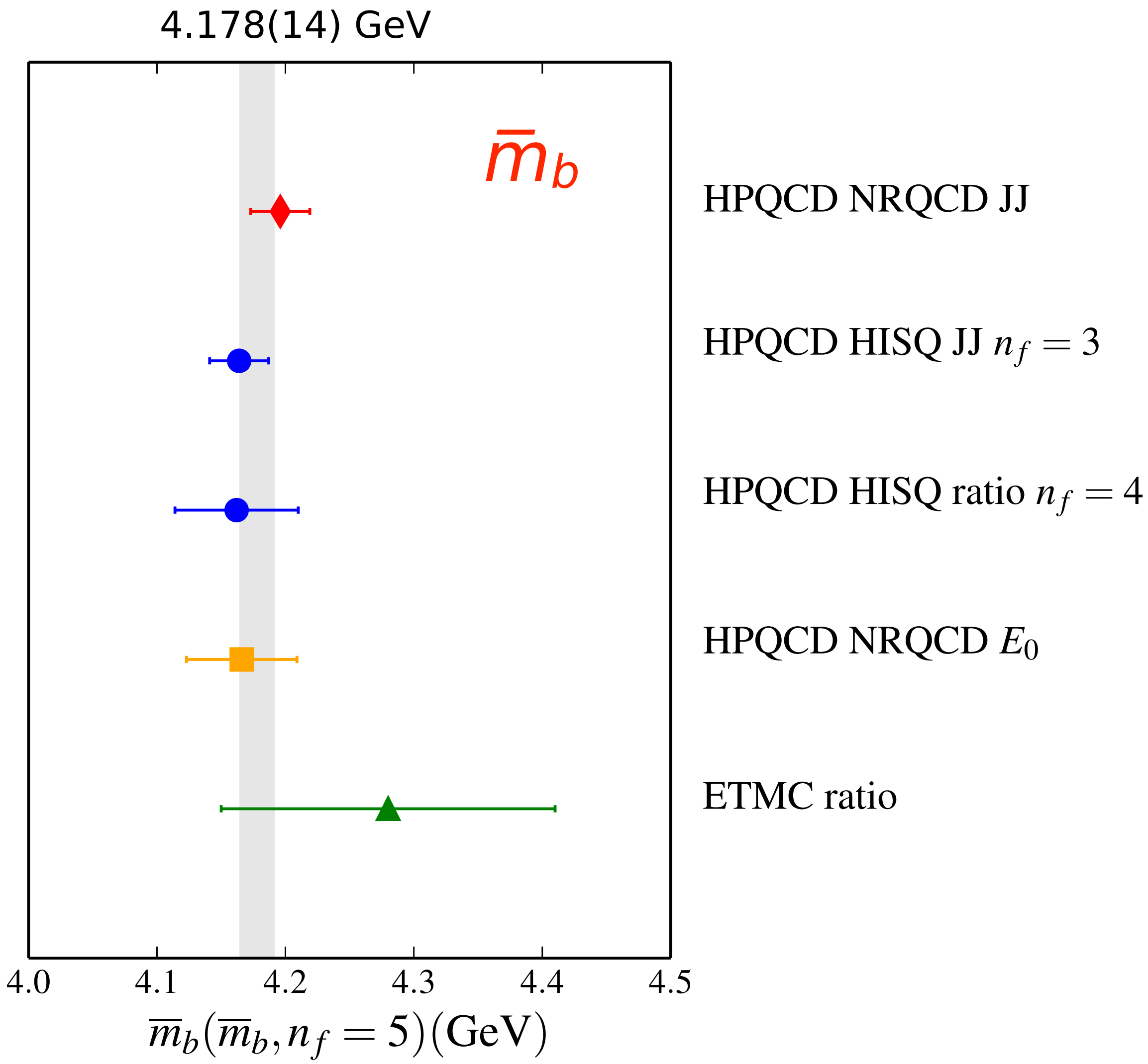
π/a = 4–14 GeV n_f =3,4

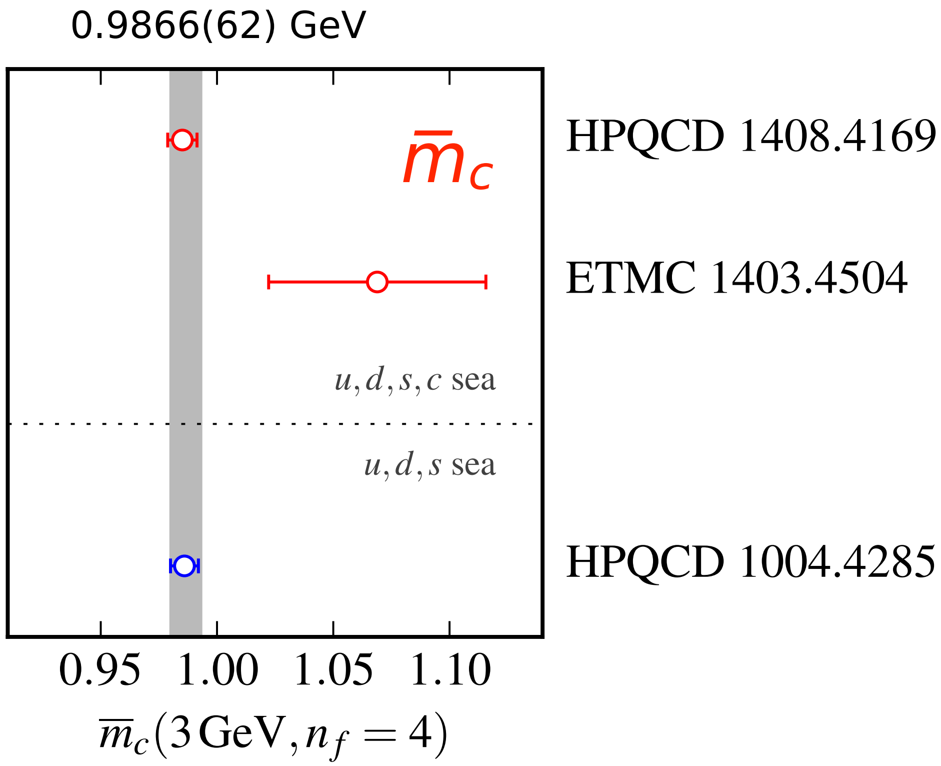
highly improved discretization

3rd + approx. 4th order in α_s

(444 data points)

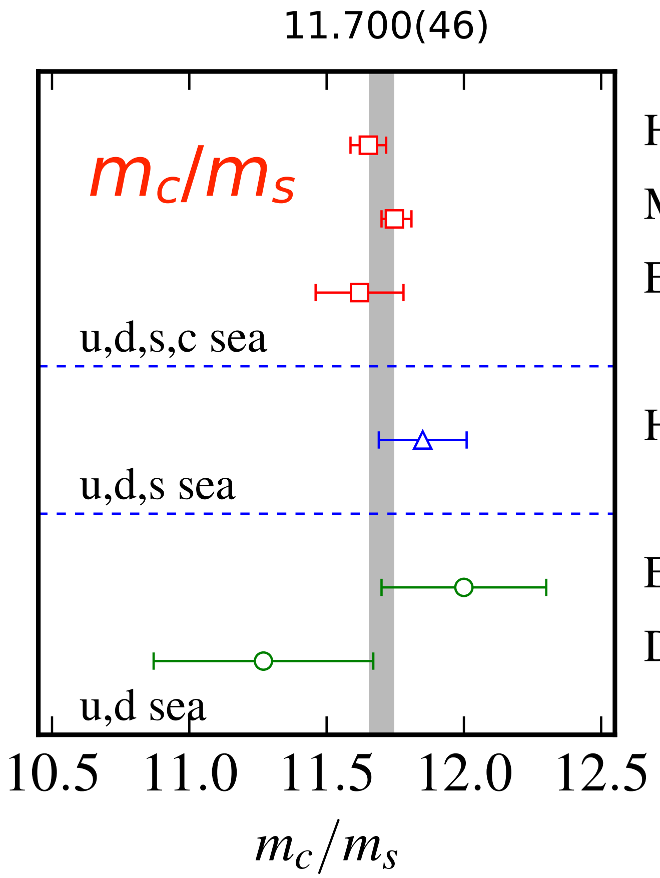
Current LQCD Results ($n_f \geq 3$)

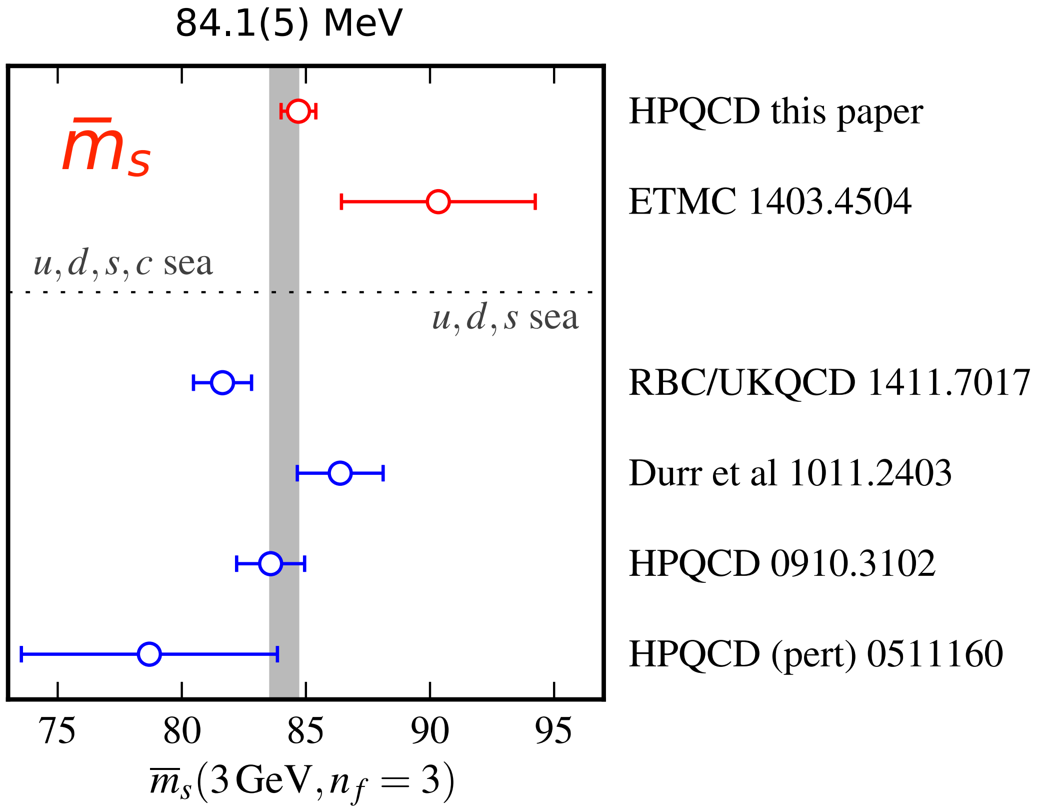


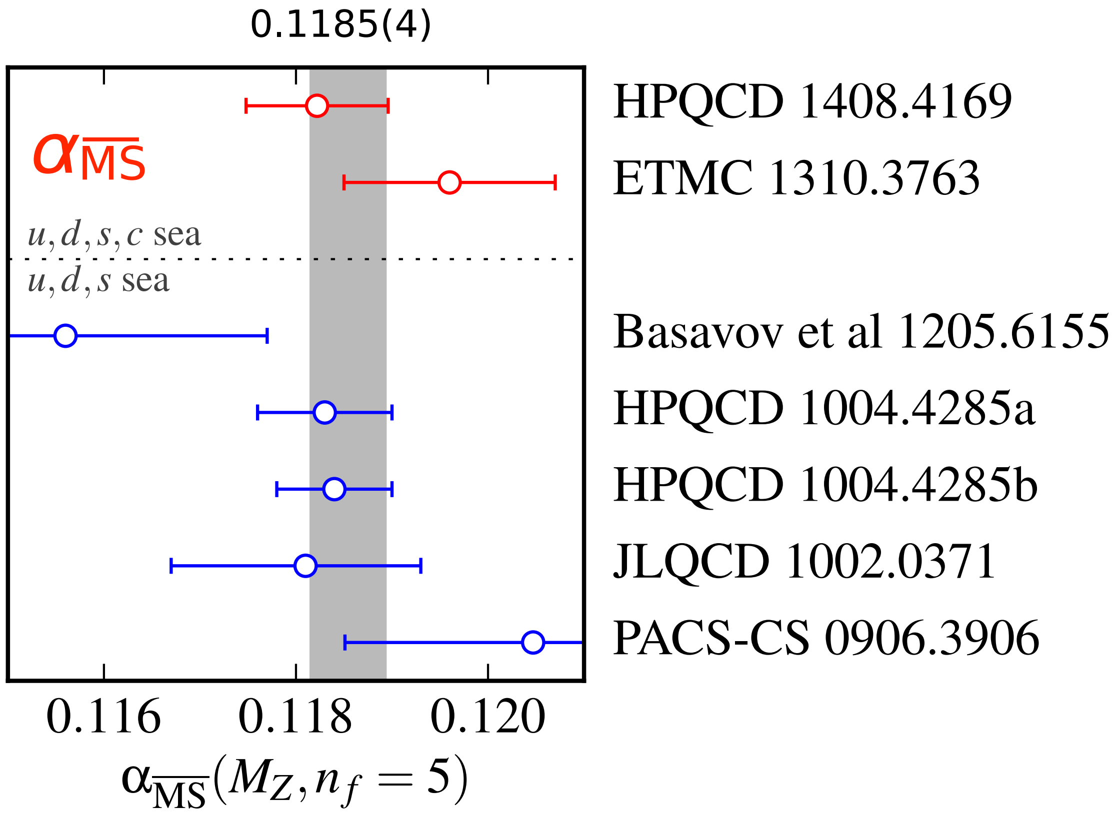


$$\boxed{m_b / m_c = 4.49(4)}$$

Current LQCD Results ($n_f \geq 3$)



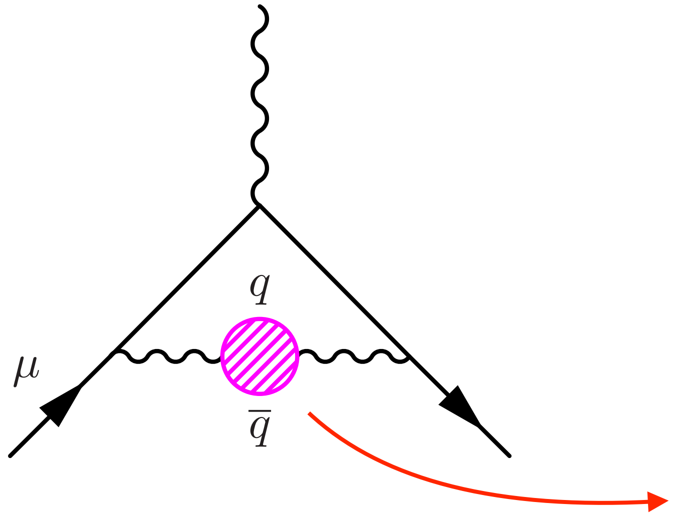




Example: HVP Contribution to Muon's *g*-2

Hadronic Vacuum Polarization in *g*-2

- Dominant QCD correction to μ's g–2:



for $q = u,d$

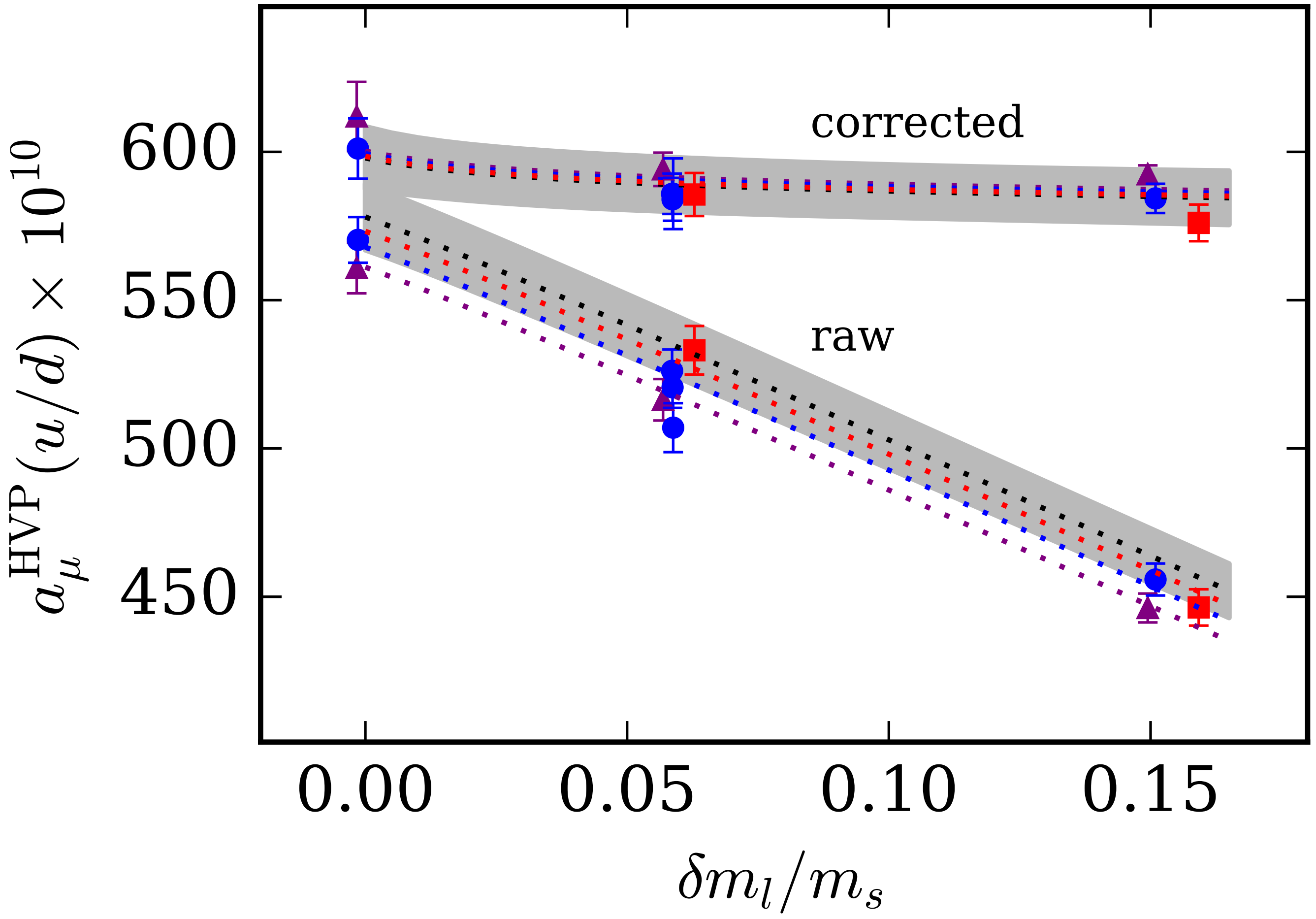
$$\hat{\Pi}(k^2) = \sum_{n=1}^{\infty} \Pi_n \, k^{2n}$$

→ (2, 2) Padé approximant

- Best current theory from e^+e^- data: 0.7% ($\pm 4\mathrm{x}10^{-10}$).
- Need ≤0.25% errors to compete with new experiment.
- New physics?

Results from *u,d*

1601.03071



vary u/d quark mass ⟶

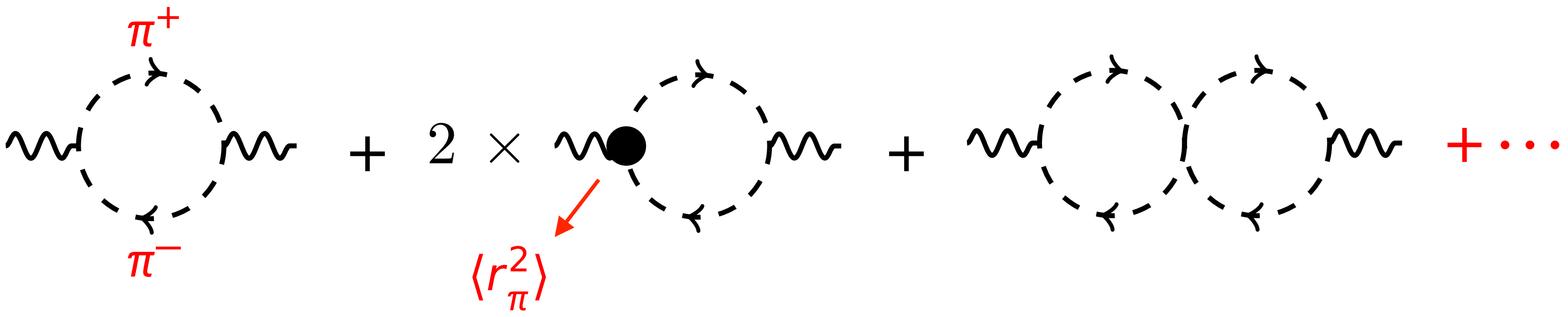
Corrections for:

- finite lattice volume + staggered quarks (chiral pert'n theory);
- δm_ℓ dependence (rescale m_ρ);
- finite lattice spacing, etc.

[Chakraborty et al (HPQCD) 1601.03071 and 1512.03270]

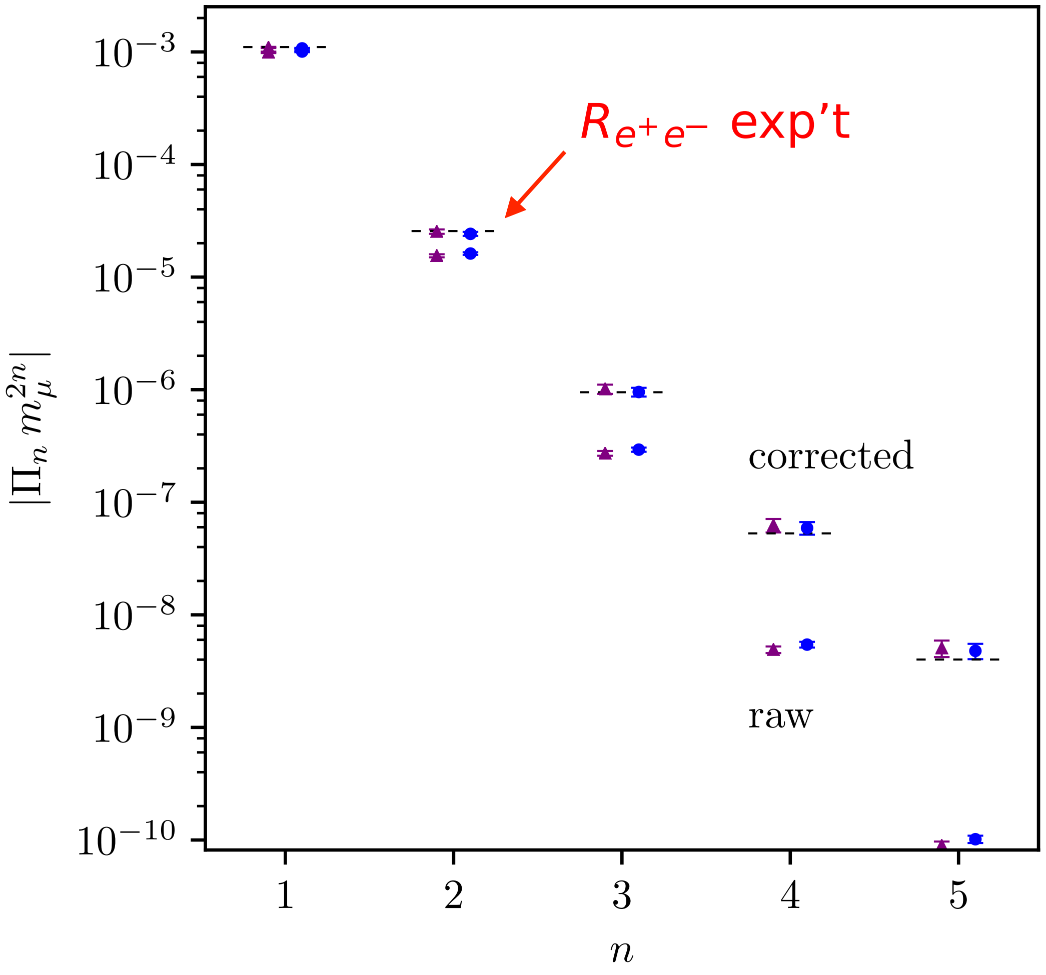
Finite-Volume Corrections

- Use chiral perturbation theory for infrared corrections:



- Test with Π_n:

1601.03071



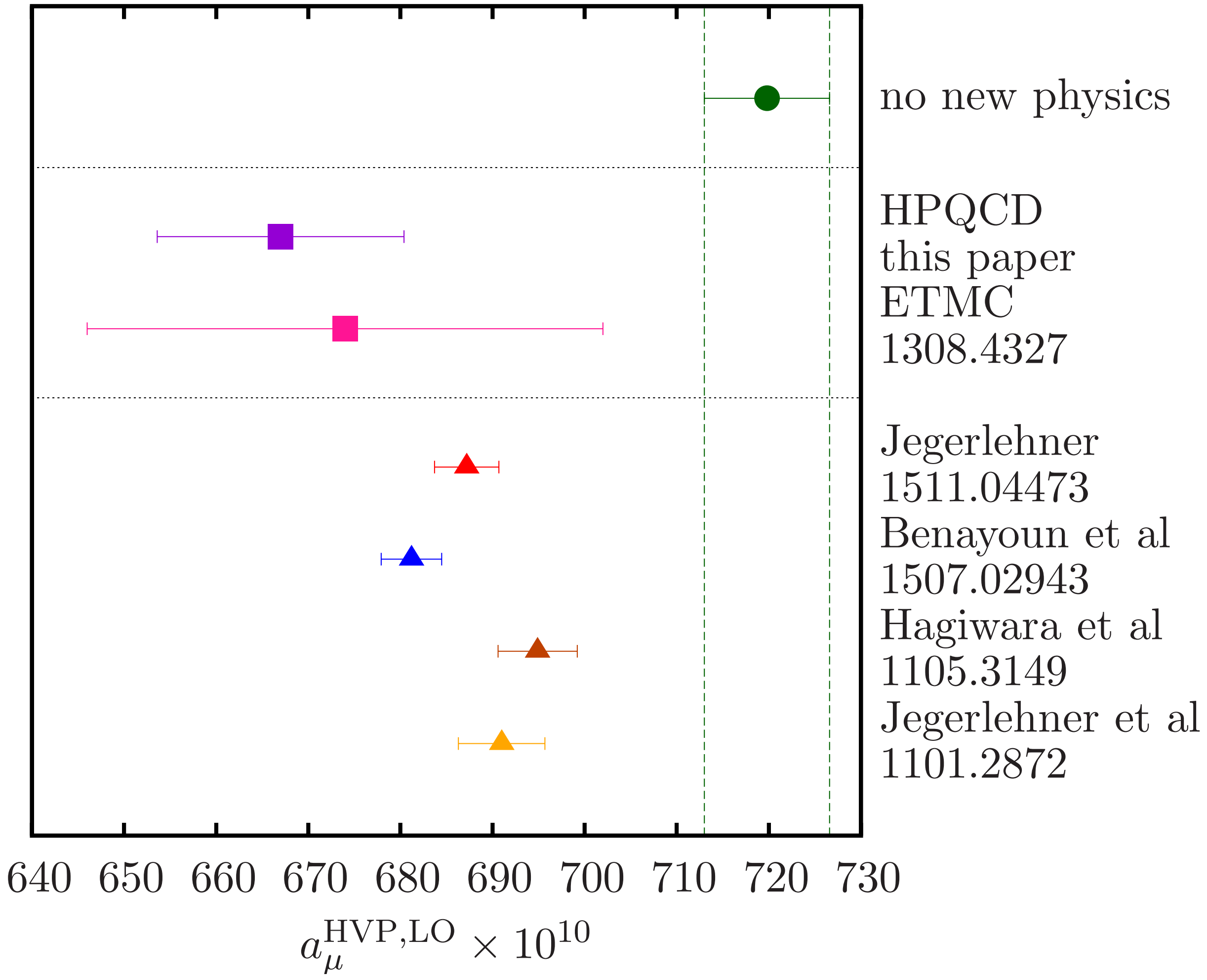
Answer: Total HVP Summary (HPQCD)

$$a_\mu(\mathrm{HVP, LO}) \times 10^{10} = \begin{cases} 599(11) & \text{from } u/d \ \ (1601.03071) \\ 53.4(6) & \text{from } s \ \ (1403.1178) \\ 14.4(4) & \text{from } c \ \ (1403.1778) \\ 0.27(4) & \text{from } b \ \ (1408.5768) \\ 0(9) & \text{from disc.} \ \ (1512.03270) \end{cases}$$

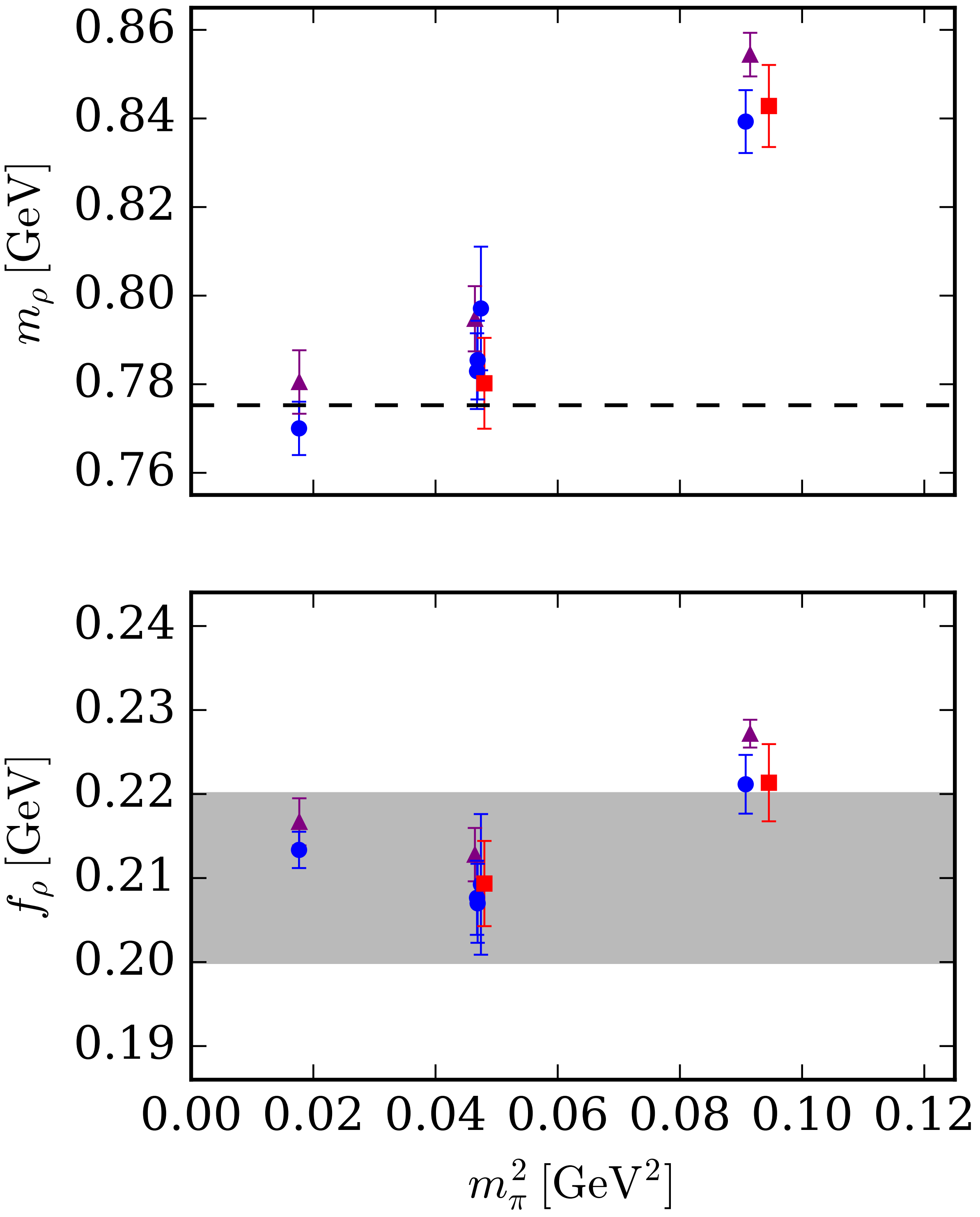
$$= 667(6)(12)$$

Lattice

QED + Isospin

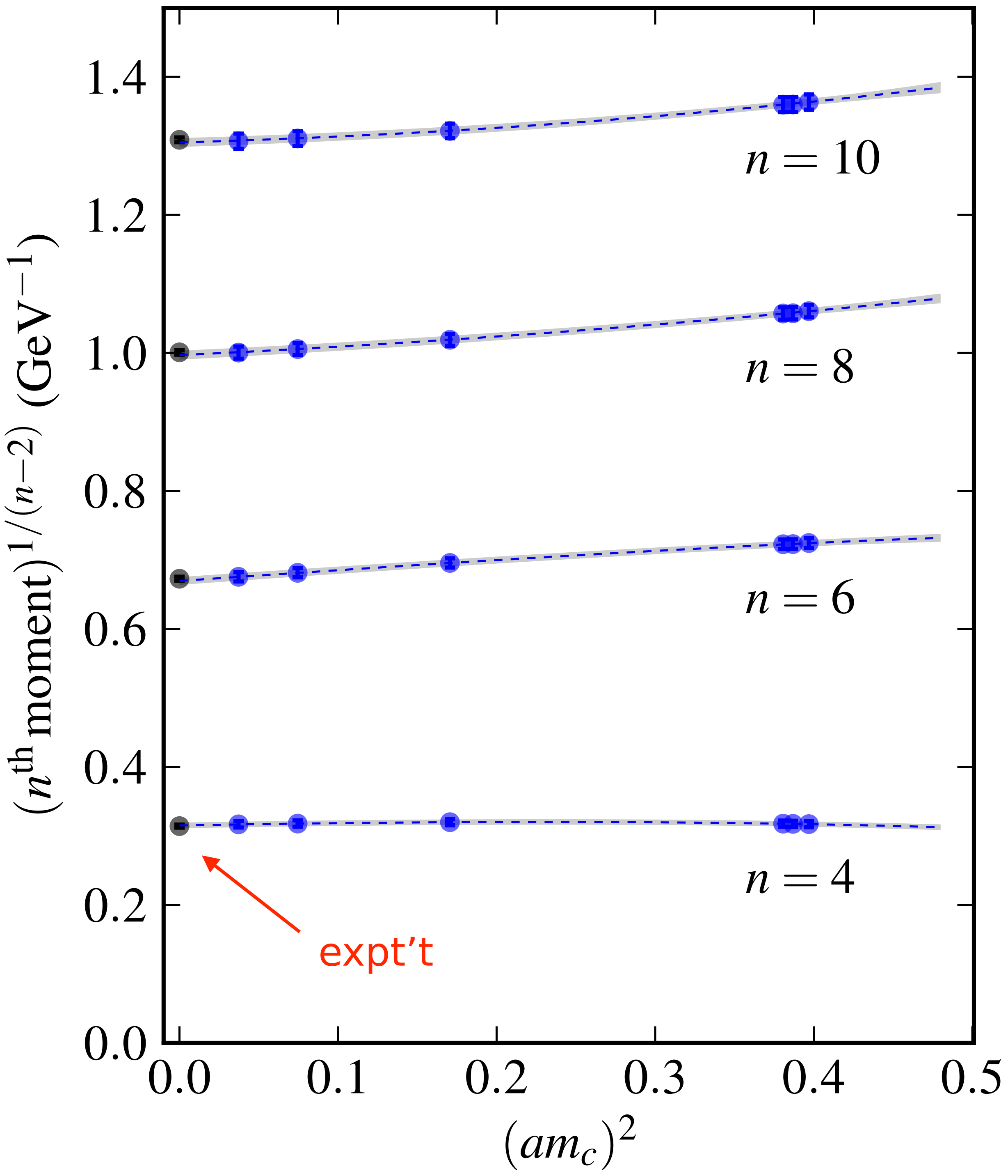


Sanity Checks: ρ Physics



expt't

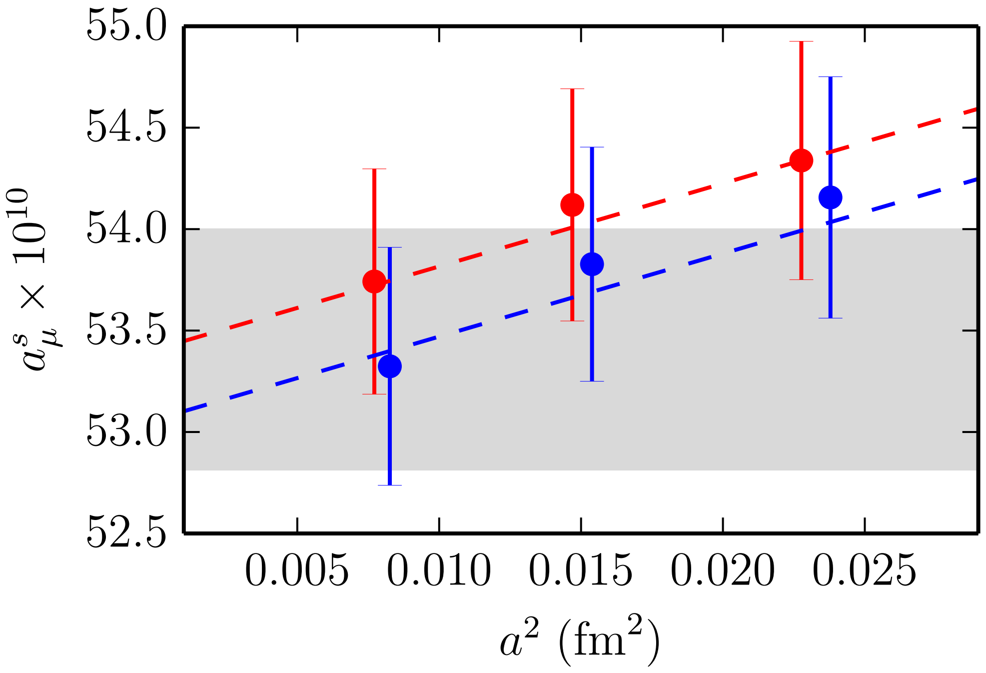
Sanity Checks: *c* Moments



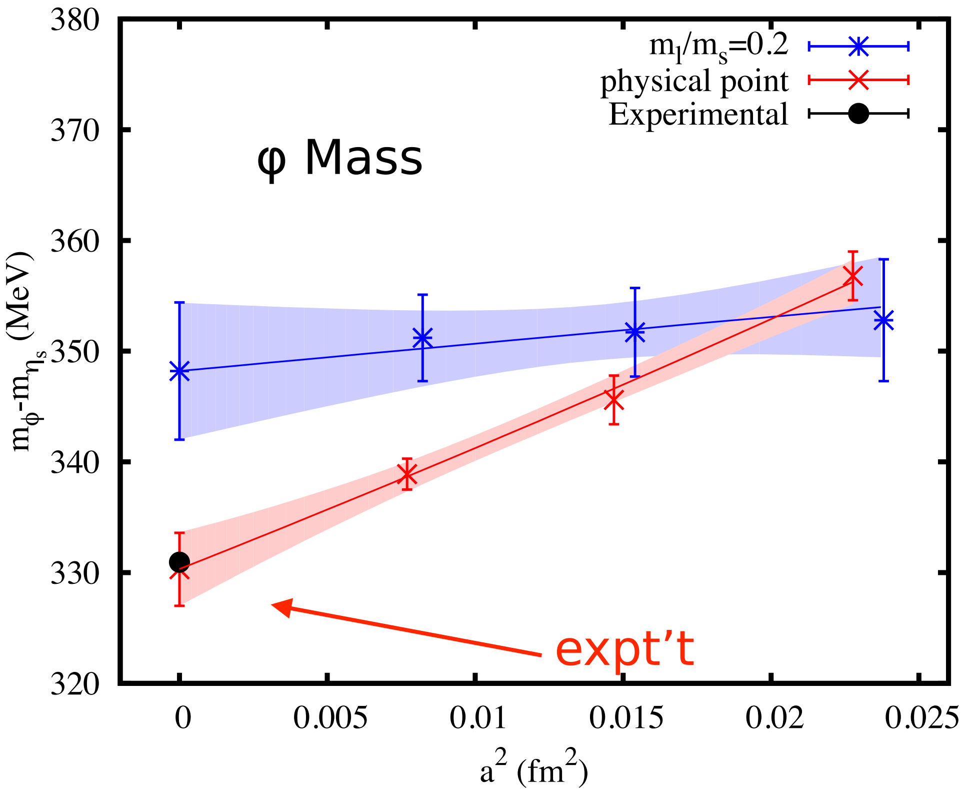
[Donald et al (HPQCD) 1208.2855, 1004..4285]

Sanity Checks: *s* Physics

[Chakraborty et al (HPQCD) 1403.1778]



← m_ℓ 5x too big



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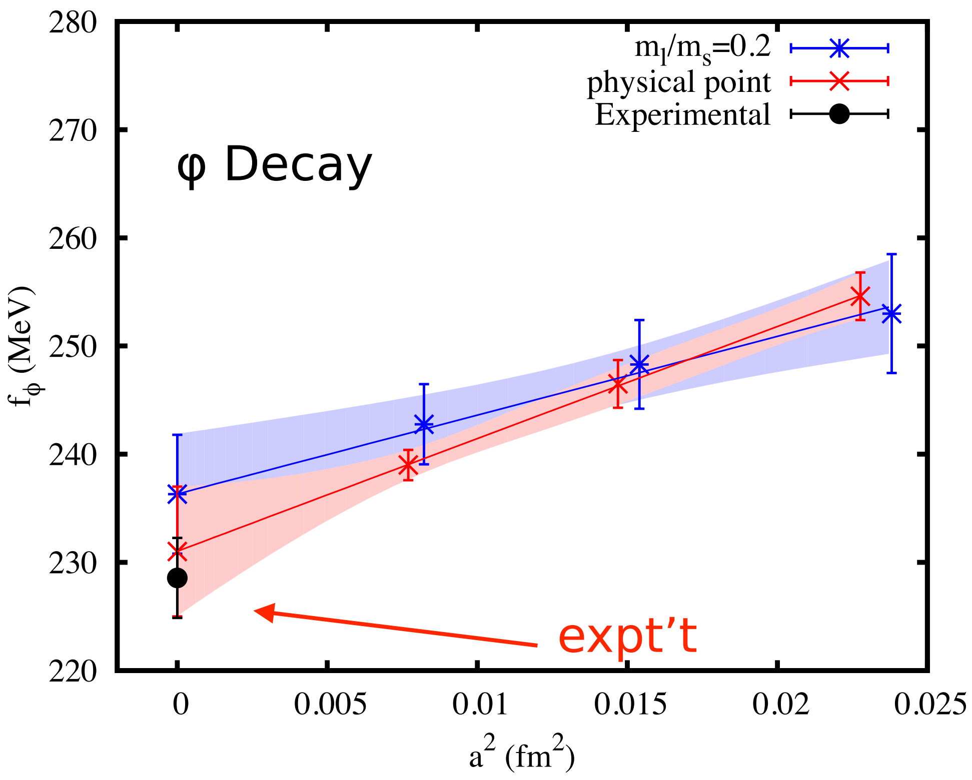
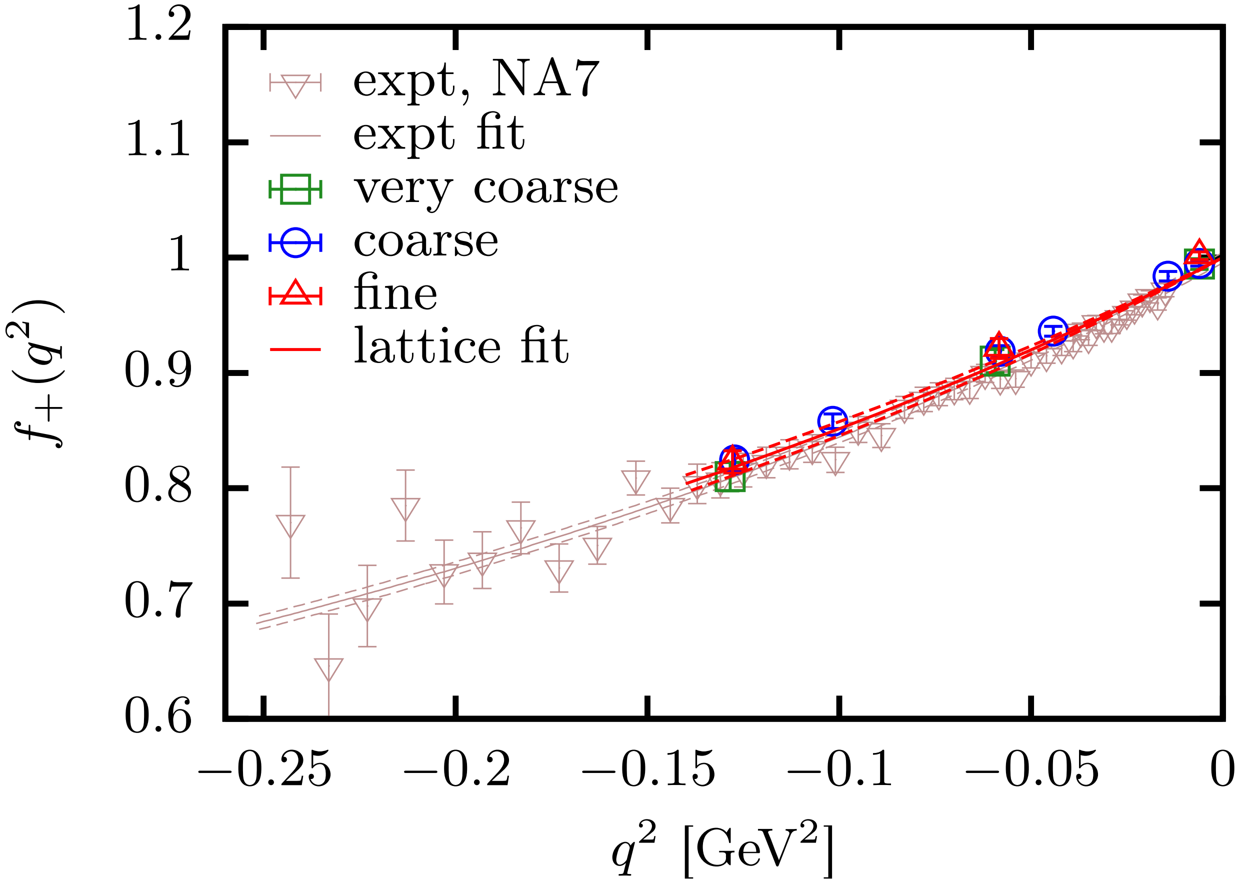


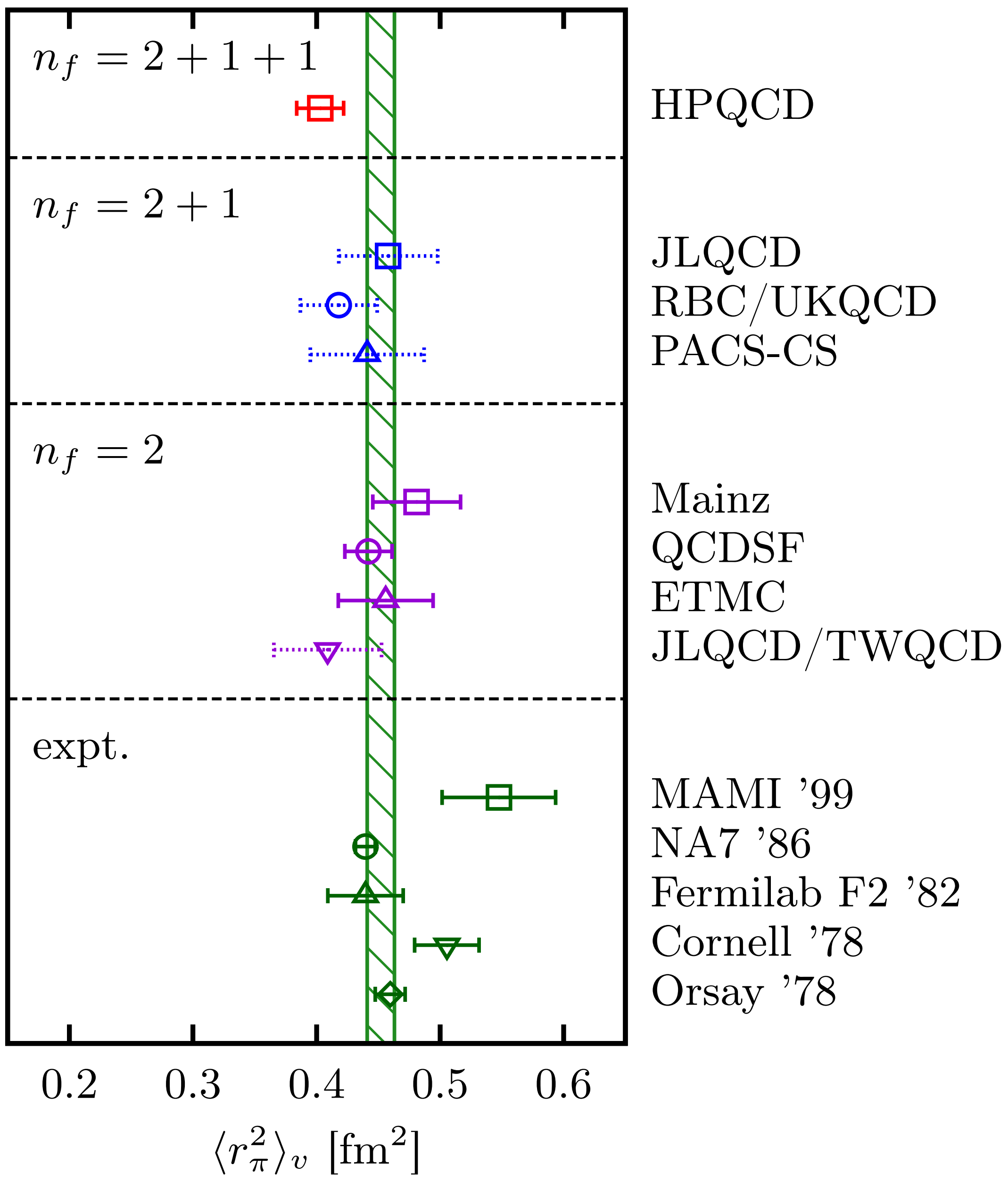
expt't

Example: Meson Form Factor

π Form Factor at Low q^2

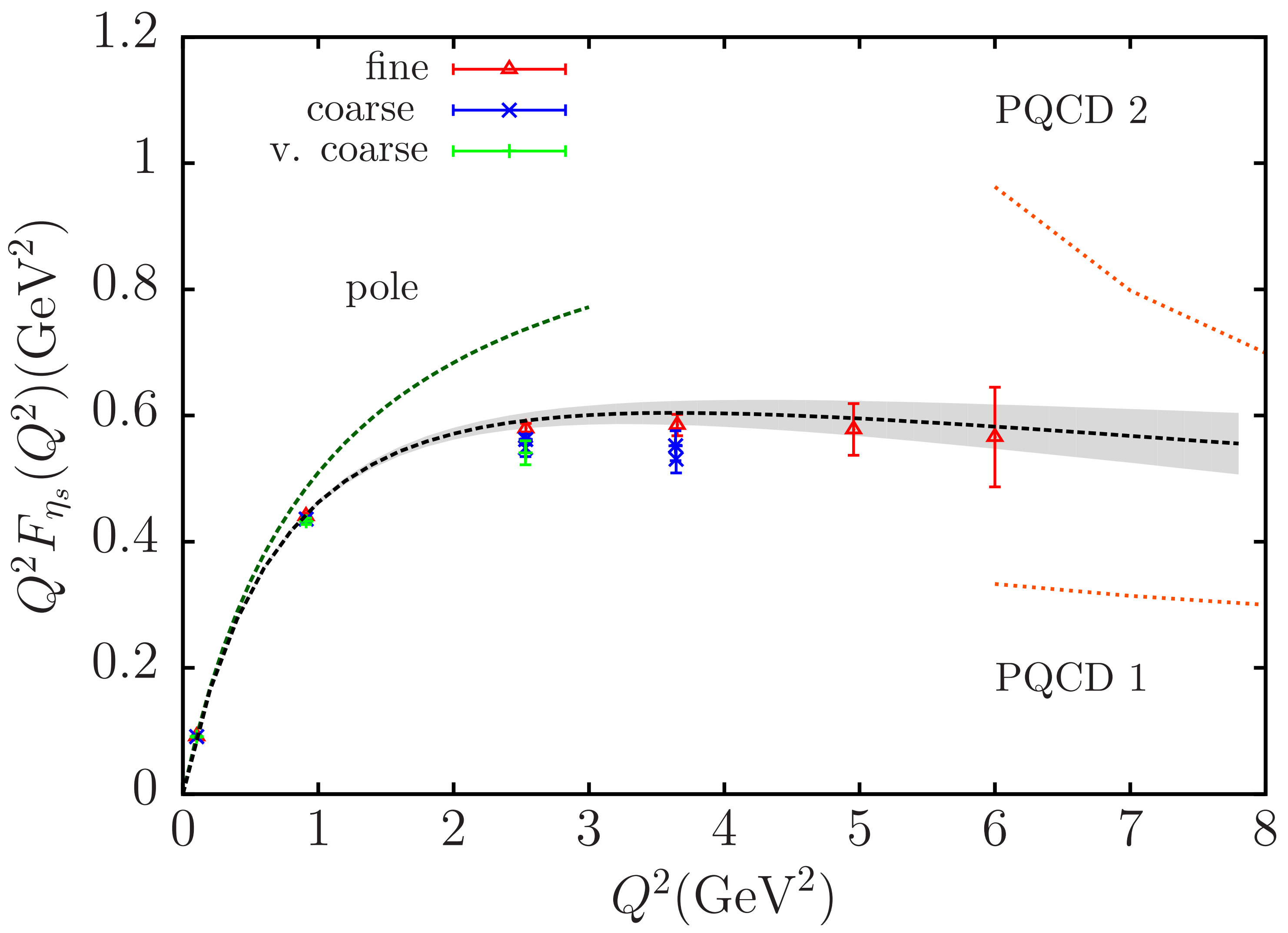
1511.07382





[Koponen et al (HPQCD) 1511.07382]

η_s Form Factor at High q^2

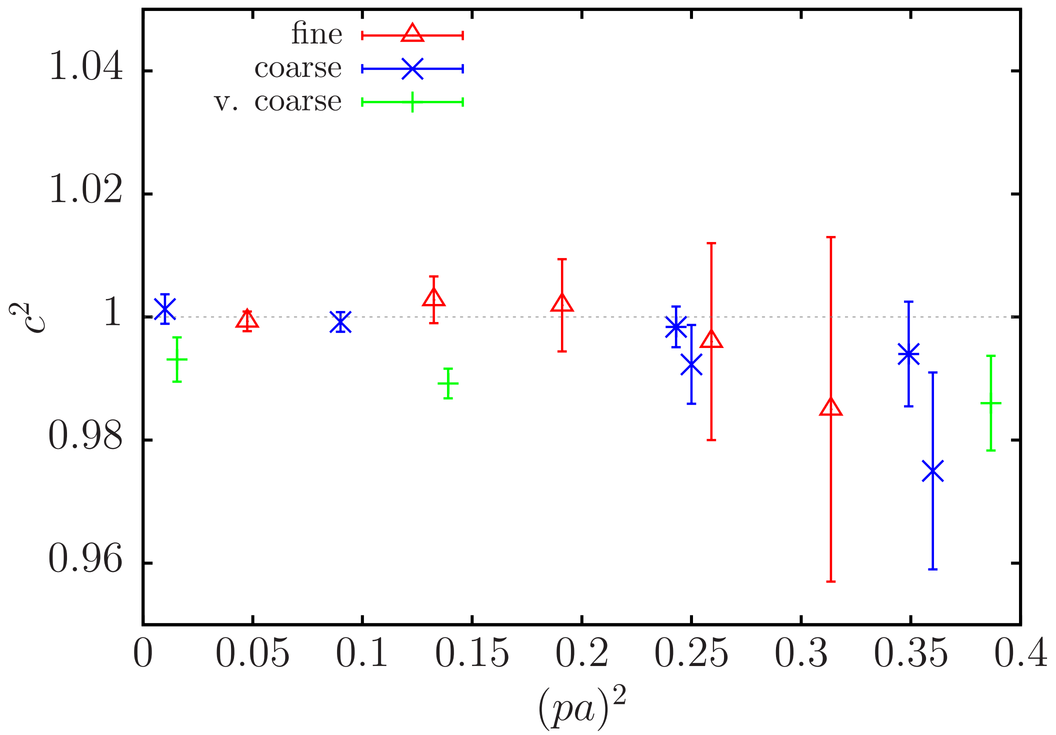


η_s = pion whose valence quarks have mass m_s.

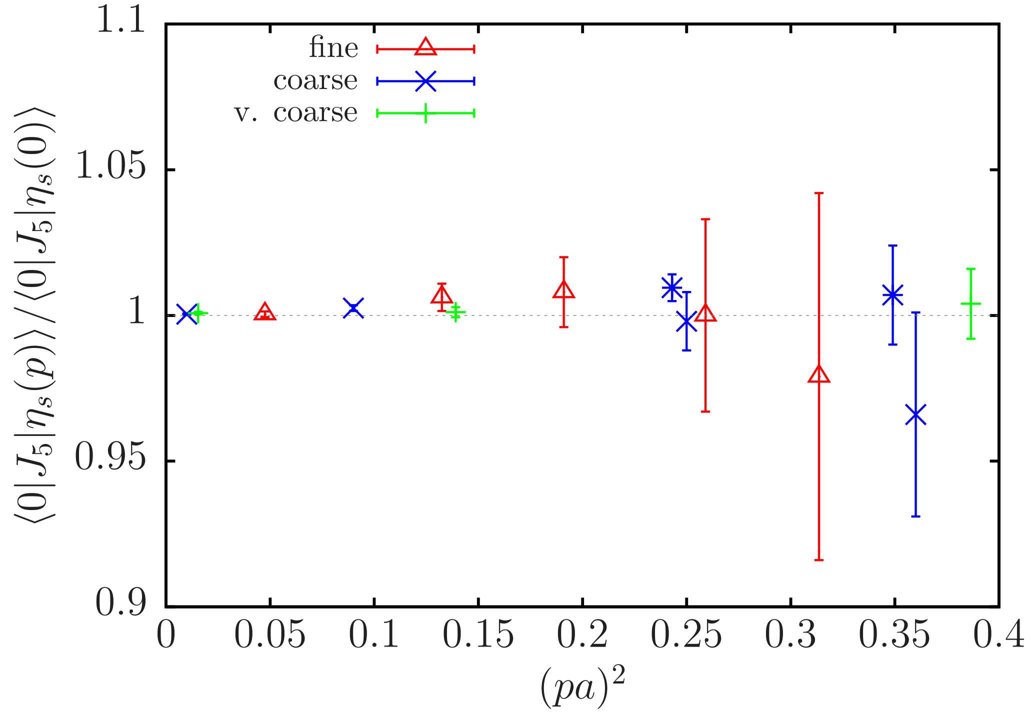
- Rescale by $(f_\pi/f_{\eta s})^2$ to obtain estimate for pion form factor at q^2=6.
- New perturbative QCD tests; important, e.g., for $B \to \pi\ \ell\nu$.
- Demo/prep for (much more costly) analysis of K and π form factors.

[Koponen et al (HPQCD) 1701.04250]

Sanity Checks: $(pa)^2$ Errors (Lor. Inv'ce)



$$c^2(p) \equiv \frac{E(\mathbf{p})^2 - m^2}{\mathbf{p}^2}$$



$$\frac{\langle 0|J_5\,|\eta_s(\mathbf{p})\rangle}{\langle 0|J_5\,|\eta_s(0)\rangle}$$

Conclusions

- Lattice QCD now a standard tool for strong interaction physics, both theoretical and experimental.
  - Most accurate strong-interaction calculations in history.
  - Landmark in history of quantum field theory: high-precision quantitative verification of nonperturbative technology (for a real theory).
  - Essential for weak interaction phenomenology, Beyond the Standard Model physics, … — QCD backgrounds.
  - New source of “data”.

- Problems that remain: hadronization of jets, quark matter, axial gauge theories, SUSY …
  - Need methods that don’t rely upon Monte Carlo integration.