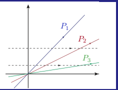

Parton Distributions
Matrix element
Pseudo-PDF
Ioffe-time
Collinear PDF
TMD
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Relations
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Small P
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Factorizable pPDF
Evolution
Nonfactorizable case

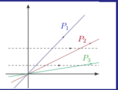
Summary

Quasi-PDFs and Pseudo-PDFs

A.V. Radyushkin

Physics Department, Old Dominion University
&
Theory Center, Jefferson Lab

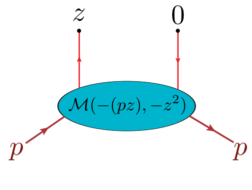
QCD Evolution 2017
May 23, 2017



Parton Densities and Matrix element

- Experimentally, one works with hadrons
- Theoretically, we work with quarks

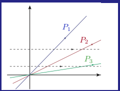
Hadron to Parton Transition



- Can be described in momentum or coordinate space
- Concept of PDFs does not rely on spin complications

$$\langle p|\phi(0)\phi(z)|p\rangle = \mathcal{M}(-(pz), -z^2)$$

- Lorentz: $\langle p|\phi(0)\phi(z)|p\rangle$ depends on z through (pz) and z^2



Pseudo-PDF

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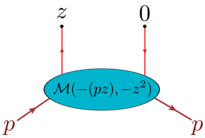
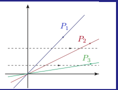


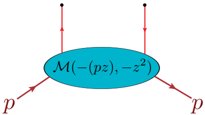
- Pseudo-PDF $\mathcal{P}(x, -z^2)$: Fourier transform with respect to (pz)

$$\mathcal{M}(-(pz), -z^2) = \int_{-1}^{1} dx\, e^{-ix(pz)}\, \mathcal{P}(x, -z^2)$$

- Should be valid in general for a very wide class of functions
- Non-trivial: limits of integration over x
- Support region is dictated by properties of Feynman diagrams
- Is determined by denominators of propagators
- Not affected by numerators present in non-scalar theories



Ioffe-time distribution



$$\mathcal{M}(-(pz), -z^2) = \int_{-1}^{1} dx\, e^{-ix(pz)}\, \mathcal{P}(x, -z^2)$$

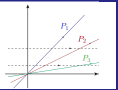
- $(pz) \equiv -\nu$ is Ioffe time [$(pz) = Mz^0$ in rest frame $p = \{M, 0, 0, 0\}$]
- $\mathcal{M}(\nu, -z^2)$ is Ioffe-time distribution

$$\mathcal{M}(\nu, -z^2) = \int_{-1}^{1} dx\, e^{ix\nu}\, \mathcal{P}(x, -z^2)$$

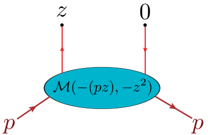
- Inverse transformation

$$\mathcal{P}(x, -z^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu\, e^{-ix\nu}\, \mathcal{M}(\nu, -z^2)$$

- We do not need here $z^2 = 0$ or $p^2 = 0$



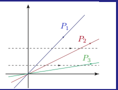
Collinear Parton Distribution



- Take light-like $z = z_-$: collinear parton distribution $f(x) = \mathcal{P}(x, 0)$

$$\mathcal{M}(-p_+z_-, 0) = \int_{-1}^{1} dx\, f(x)\, e^{-ixp_+z_-}$$

- Usual interpretation: parton carries fraction x of hadron p_+
- $z^2 \to 0$ nontrivial in QCD, since $\mathcal{M}(\nu, z^2)$ has $\sim \ln z^2$ singularities
- Reflect perturbative evolution of parton densities
- Within OPE, $\ln z^2$ singularities are subtracted, e.g., by dimensional renormalization $\ln(1/z^2) \to \ln \mu^2$
- Resulting PDFs depend on renormalization scale μ, $f(x) \to f(x, \mu^2)$
- In $\mathcal{P}(x, -z^2)$ pseudo-PDFs, $1/z^2$ serves as cut-off scale



TMDs and quasi-PDFs

- Treat target momentum p as longitudinal $p = (E, \mathbf{0}_\perp, P)$
- Take z with z_- and $z_\perp = \{z_1, z_2\}$ components ($z_+ = 0$), then $(pz) = p_+ z_- \equiv -\nu$; define TMD

$$\mathcal{M}(\nu, z_\perp^2) = \int_{-1}^{1} dx\, e^{ix\nu} \int_{-\infty}^{\infty} d^2k_\perp\, e^{-i(k_\perp z_\perp)} \mathcal{F}(x, k_\perp^2)$$

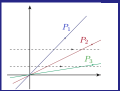
- Parton carries xp_+ and has transverse momentum $k_\perp$
- Rotational invariance in $z_\perp$ plane: this TMD depends on $k_\perp^2$ only
- Take $z = \{0, 0, 0, z_3\}$, define Quasi-PDF

$$\langle p|\phi(0)\phi(z_3)|p\rangle \equiv \mathcal{M}(\underbrace{Pz_3}_{\nu}, \underbrace{z_3^2}_{\nu^2/P^2}) = \int_{-\infty}^{\infty} dy\, e^{iyPz_3}\, Q(y, P)$$

- Inverse Fourier transformation

$$Q(y, P) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu\, e^{-iy\nu}\, \mathcal{M}(\nu, \nu^2/P^2)$$

- $Q(y, P)$ tends to $f(y)$ in $P \to \infty$ limit, as far as $\mathcal{M}(\nu, \nu^2/P^2) \to \mathcal{M}(\nu, 0)$.



Quasi-PDFs vs Pseudo-PDFs

Quasi & Pseudo

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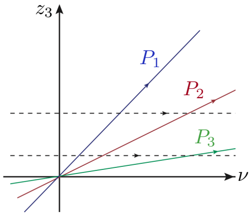
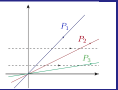
Summary



- Quasi-PDFs $Q(y,P)$: integration of $\mathcal{M}(\nu, z_3^2)$ over $z_3 = \nu/P$ lines
- Becomes horizontal $z_3 = 0$ line in $P \to \infty$ limit $\to$ PDF
- $Q(y,P)$ has perturbative evolution wrt P for large P
- Support region $-\infty < y < \infty$
- Psedo-PDFs: integration of $\mathcal{M}(\nu, z_3^2)$ over z_3 =const lines
- Always has $-1 \le x \le 1$ support
- $\mathcal{P}(x, z_3^2)$ has perturbative evolution wrt $1/z_3$ for small z_3
  $\sim$ PDF $f(x, C^2/z_3^2)$ for small z_3
  C = matching coefficient, $C_{\overline{MS}} = 2e^{-\gamma_E} \approx 1.12$



Relations between quasi-PDFs and TMDs

- Write definition of quasi-PDF

$$Q(y,P) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \; e^{-iy\nu} \, \mathcal{M}(\nu, \nu^2/P^2)$$

- Write definition of TMDs

$$\mathcal{M}(\nu, z_1^2 + z_2^2) = \int_{-1}^{1} dx \; e^{ix\nu} \int_{-\infty}^{\infty} dk_1 dk_2 \, e^{-ik_1 z_1 - ik_2 z_2} \mathcal{F}(x, k_1^2 + k_2^2)$$

- Take $z_1 = 0$, $z_2 = \nu/P$ and combine expressions to get

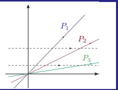
$$Q(y,P)/P = \int_{-1}^{1} dx \int_{-\infty}^{\infty} dk_1 \mathcal{F}(x, k_1^2 + (y-x)^2 P^2)$$

- Introduce momentum distributions in $k_3 \equiv yP$

$$R(k_3, P) = Q(k_3/P, P)/P = \int_{-1}^{1} dx \, \mathcal{R}(x, k_3 - xP)$$

- $\mathcal{R}(x, k_3)$ is TMD $\mathcal{F}(x, \kappa^2)$ integrated over k_1

$$\mathcal{R}(x, k_3) \equiv \int_{-\infty}^{\infty} dk_1 \mathcal{F}(x, k_1^2 + k_3^2)$$



Momentum distributions

- Convolution nature of quasi-PDFs

$$R(k_3, P) = Q(k_3/P, P)/P = \int_{-1}^{1} dx\, \mathcal{R}(x, k_3 - xP)$$

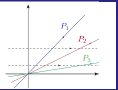
- Take hadron at rest, $p = \{M, 0, 0, 0\}$

$$R(k_3, P = 0) \equiv r(k_3) = \int_{-1}^{1} dx\, \mathcal{R}(x, k_3)$$

- 1D distribution obtained from density
$\mathcal{M}(0, z_3^2) = \langle p|\phi(0)\phi(z_3)|p\rangle|_{\mathbf{p}=\mathbf{0}}$

$$\mathcal{M}(0, z_3^2) = \int_{-\infty}^{\infty} dk_3\, r(k_3)\, e^{ik_3 z_3}$$

- $r(k_3)$ = primordial distribution of k_3 in rest frame
- In moving hadron, parton momentum $k_3 = xP + (k_3 - xP)$ comes
a) from motion of hadron as a whole (part xP)
governed by x-dependence of TMD $\mathcal{F}(x, \kappa^2)$
b) remaining part $k_3 - xP$ governed by κ^2 dependence of TMD
reflecting primordial rest-frame distribution



Factorized distributions

- Two sources of k_3 "look" independent
- Try factorized model

$$\mathcal{R}^{\text{fact}}(x, k_3 - xP) = f(x)r(k_3 - xP)$$

(x integral of $f(x)$ is normalized to 1)

- For original $\mathcal{M}(\nu, -z^2)$ function, this Ansatz corresponds to

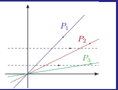
$$\mathcal{M}^{\text{fact}}(\nu, -z^2) = \mathcal{M}(\nu, 0)\mathcal{M}(0, -z^2)$$

- Popular idea: Gaussian dependence of TMD on $k_\perp$. Gives

$$r_G(k_3) = \frac{1}{\sqrt{\pi}\Lambda} e^{-k_3^2/\Lambda^2} \quad , \quad \text{density} : \quad r_G(z_3^3) = e^{-z_3^2\Lambda^2/4}$$

- Factorized Gaussian model for momentum distribution

$$R_G^{\text{fact}}(k_3, P) = \frac{1}{\Lambda\sqrt{\pi}} \int_{-1}^{1} dx\, f(x)\, e^{-(k_3 - xP)^2/\Lambda^2}$$



Numerical results for Gaussian model

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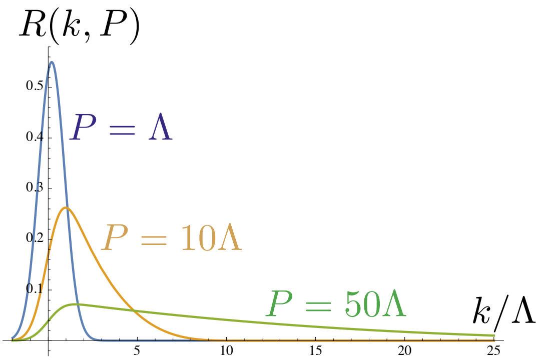
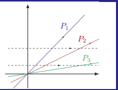
Summary

- Take simple PDF $f(x) = 4(1-x)^3\theta(0 \le x \le 1)$ resembling valence quark distributions



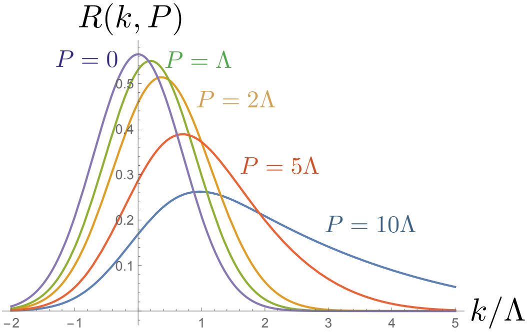
- Changes from a Gaussian shape (for small P) to a shape resembling stretched PDF (for large P)

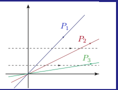


Small Momenta P

- Small-P approximation ($\tilde{x}$ = average x, in our model $\tilde{x} = 0.2$)

$$R(k_3, P) = \int_{-1}^{1} dx\, f(x)\, r(k_3 - xP) \approx r(k_3 - \tilde{x}P)$$



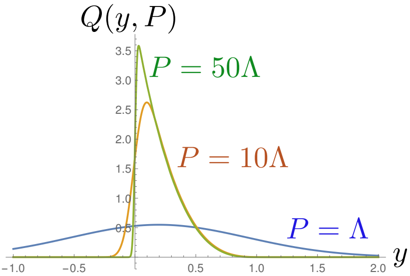


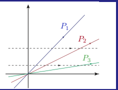
Large Momenta P

- For large P

$$r_G(k_3 - xP) = \frac{1}{\sqrt{\pi}\Lambda} e^{-(k_3 - xP)^2/\Lambda^2} \to \frac{1}{P}\,\delta(x - k_3/P)$$

- Combination $P\,R(k_3, P)$ in large P limit converts into $f(k_3/P) = f(y)$
- For finite P, we have $P\,R(k_3, P) = Q(y = k_3/P, P)$





QCD case

- Matrix element in QCD

$$\mathcal{M}^{\alpha}(z,p) \equiv \langle p|\bar{\psi}(0)\,\gamma^{\alpha}\,\hat{E}(0,z;A)\psi(z)|p\rangle$$

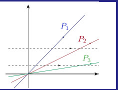
- with standard $0 \to z$ straight-line gauge link $\hat{E}(0,z;A)$
- Decompose into p^{α} and z^{α} parts

$$\mathcal{M}^{\alpha}(z,p) = 2p^{\alpha}\mathcal{M}_p(-(zp),-z^2) + z^{\alpha}\mathcal{M}_z(-(zp),-z^2)$$

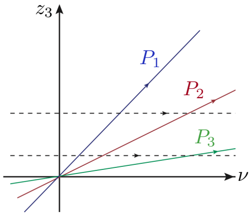
- For TMD: take $z=(z_-,z_\perp)$ and $\alpha=+ \Rightarrow z^{\alpha}$-part drops out
- TMD $\mathcal{F}(x,k_\perp^2)$ is related to $\mathcal{M}_p(\nu,z_\perp^2)$ by scalar formula
- For quasi- PDF: take time component of $\mathcal{M}^{\alpha}(z=z_3,p)$ and define

$$\mathcal{M}^{0}(z_3,p) = 2p^{0}\int_{-1}^{1} dy\, Q(y,P)\; e^{iyPz_3}$$

- $\Rightarrow$ Quasi-PDF $Q(y,P)$ is related to TMD $\mathcal{F}(x,k_\perp^2)$ by scalar formula



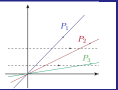
Pseudo-PDFs and Ioffe-time distributions



- Quasi-PDFs $Q(y, P)$: integration of $\mathcal{M}(\nu, z_3^2)$ over $z_3 = \nu/P$ lines
- Have x-convolution structure even if $\mathcal{M}(\nu, z_3^2)$ factorizes, i.e., $\mathcal{M}(\nu, z_3^2) = \mathcal{M}(\nu, 0)\mathcal{M}(0, z_3^2)$
- Fit $\mathcal{M}(Pz_3, z_3^2)$ by $\mathcal{M}(\nu, z_3^2)$ (K. Orginos) and take reduced function

$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)}$$

- In factorized case, gives $\mathcal{M}(\nu, 0)$
  $\Rightarrow$ take its Fourier transform to get PDF $f(x)$
- Bonus: z_3^2-dependence due to self-energy of gauge link cancels in ratio (K. Orginos)



Evolution of Ioffe-time distributions

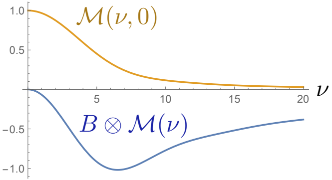
- LO Evolution equation (Braun et al. 1994) for Ioffe-time distribution

$$\frac{d}{d\ln z_3^2}\mathcal{M}(\nu, z_3^2) = \frac{\alpha_s}{2\pi} C_F \int_0^1 du\, B(u)\mathcal{M}(u\nu, z_3^2)$$

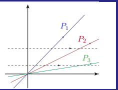
- Nonsinglet evolution kernel

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

- Valence $f(x) = 4(1-x)^3$ corresponds to $\mathcal{M}(\nu, 0) = 12\left[\nu^2 - 4\sin^2(\nu/2)\right]/\nu^4$

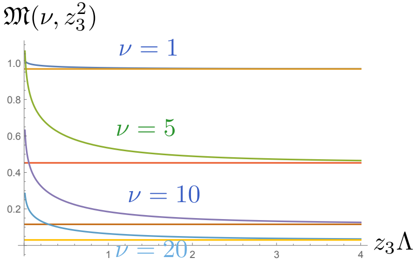


- No perturbative evolution for $\mathcal{M}(0, z_3^2)$ [vector current is conserved]

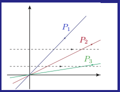


Nonfactorizable case, evolution

- In reality: $\mathfrak{M}(\nu, z_3^2)$ will have residual z_3^2-dependence from a) perturbative evolution visible as $\ln(1/z_3^2\Lambda^2)$ spike for small z_3^2
- Take for illustration $\mathcal{P}^{\text{soft}}(x, z_3^2) = f(x)e^{-z_3^2\Lambda^2/4}$ (corresponding to TMD $\mathcal{F}^{\text{soft}}(x, k_\perp^2) = f(x)e^{-k_\perp^2/\Lambda^2}/\pi\Lambda^2$) and $\alpha_s/\pi = 0.1$ for hard part $\sim \Gamma[0, z_3^2\Lambda^2/4]$

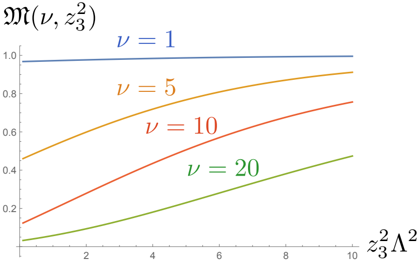


- If $\Lambda = 300\text{MeV}$ we have $z_3\Lambda$ = 1.5 for $z_3 = 1$ fm

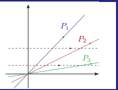


Nonfactorizable case, soft

- $\mathfrak{M}(\nu, z_3^2)$ may also have residual z_3^2-dependence from b) violation of factorization for soft part
- Take for illustration $\mathcal{P}^{\text{soft}}(x, z_3^2) = f(x) e^{-x(1-x) z_3^2 \tilde{\Lambda}^2/4}$
  $\mathcal{F}^{\text{soft}}(x, k_\perp^2) = f(x) e^{-k_\perp^2/x(1-x)\tilde{\Lambda}^2} / [\pi x(1-x)\tilde{\Lambda}^2])$
- To have the same $\langle k_\perp^2 \rangle$ we need $\tilde{\Lambda}^2 = \frac{15}{2}\Lambda^2$

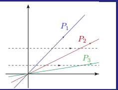


- $z_3^2\Lambda^2$ = 2.25 for $z_3 = 1$ fm



Summary

- Quasi-PDFs are hybrids of PDFs and primordial rest-frame momentum distributions
- Complicated convolution nature of quasi-PDFs necessitates $p_3 \gtrsim$ 3 GeV to wipe out primordial effects
- Alternative approach is to use pseudo-PDFs $\mathcal{P}(x, z_3^2)$ related by Fourier transform to Ioffe-time distributions $\mathcal{M}(\nu, z_3^2)$
- Pseudo-PDFs have same $-1 \leq x \leq 1$ support as PDFs
- Their z_3^2-dependence for small z_3^2 is governed by a usual evolution equation
- Using ratio $\mathcal{M}(\nu, z_3^2)/\mathcal{M}(0, z_3^2)$ of Ioffe-time distributions one divides out z_3^2-dependence of primordial rest-frame distribution
- Ratio excludes z_3^2-dependence coming from gauge link self-energy corrections



Primordial TMDs

- QCD operator $\mathcal{O}^{\alpha}(0, z; A)$ involves straight-line link
- Our TMD differs from stapled-link TMDs used in Drell-Yan and SIDIS processes
- Stapled links reflect initial or final state interactions inherent in these processes
- The "straight-link" TMDs describe structure of a hadron in non-disturbed or "primordial" state
- Unlikely that such a TMD can be measured in a scattering experiment
- Still, it is a well-defined quantum field theory object
- Hopefully can be measured on the lattice through its connection to pseudo-PDFs