Rapidity evolution of gluon TMD from low to moderate x

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JLAB & ODU

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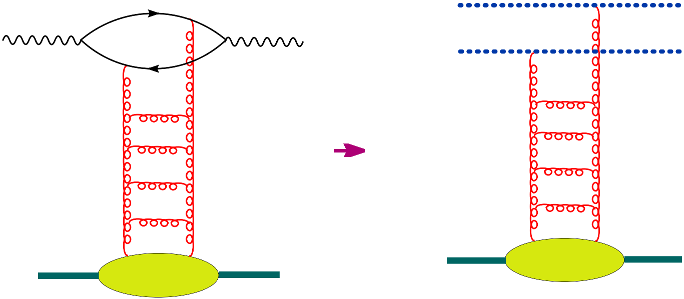
Outline

- Reminder: rapidity factorization and evolution of color dipoles
- Method of calculation: shock-wave approach + light-cone expansion.
- One loop: real corrections and virtual corrections.
- One-loop evolution of gluon TMD
- DGLAP, Sudakov and BK limits of TMD evolution equation
- Gluon TMDs in particle production
- Conclusions and outlook

DIS at high energy: Wilson lines and color dipoles

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un + x_\perp)\Big]$$ Wilson line

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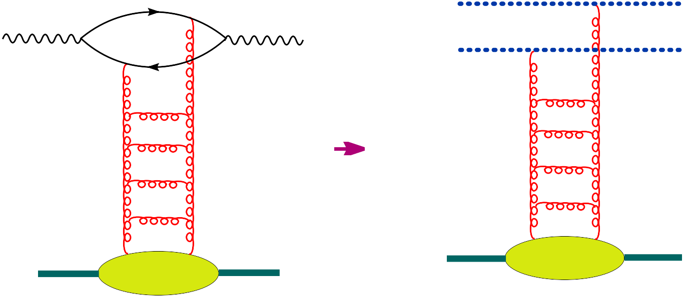
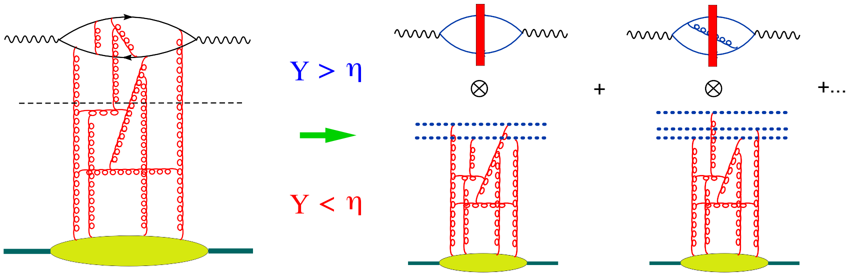


$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig \int_{-\infty}^{\infty} du\, n^\mu A_\mu(un + x_\perp)\Big] \qquad \text{Wilson line}$$

Formally, $\rightarrow$ means the operator expansion in Wilson lines

Rapidity factorization: OPE in Wilson lines



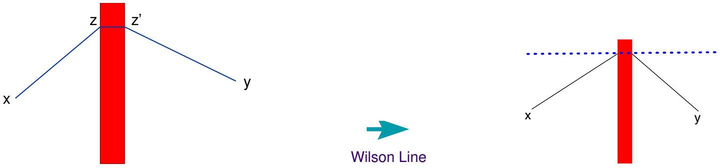
η - rapidity factorization scale

Rapidity Y > η - coefficient function ("impact factor")
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big], \quad A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Spectator frame: propagation in the shock-wave background.

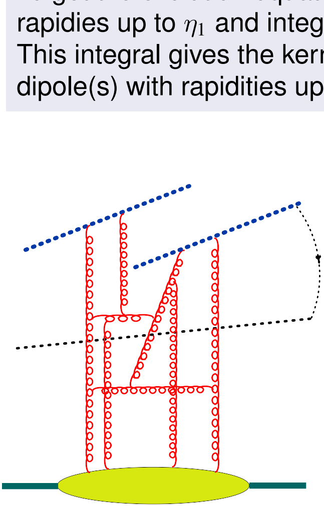
Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.



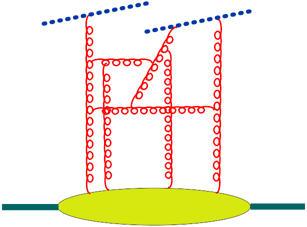
[$x \to z$: free propagation]$\times$
[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave]$\times$
[$z \to y$: free propagation]

Reminder: evolution of color dipoles at small x

To get the evolution equation for color dipoles, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



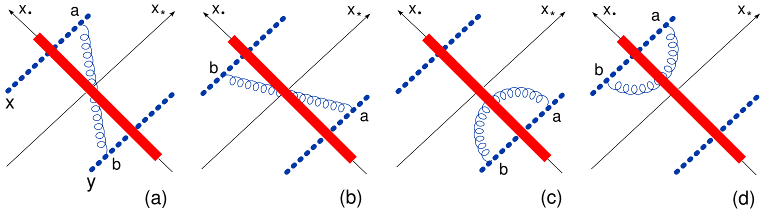
$$\alpha_s(\eta_1 - \eta_2) K_{\mathrm{evol}} \;\otimes$$



Rapidity evolution of color dipoles in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD ⇒ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD ⇒ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

NLO kernels for BK and JIMWLK are now known (even NNLO BK for $\mathcal{N}=4SYM$!)

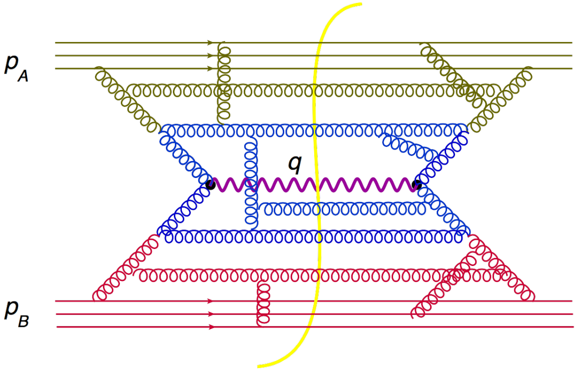
Rapidity factorization for particle production

Sudakov variables:

$$k \;=\; \alpha p_1 + \beta p_2 + k_\perp, \qquad p_1 \simeq p_A,\;\; p_2 \simeq p_B,\;\; p_1^2 = p_2^2 = 0$$

Dimensionless light-cone coordinates

$$x_* \;\equiv\; x_\mu p_2^\mu \;=\; \sqrt{\frac{s}{2}}x_+, \quad x_\bullet \;\equiv\; x_\mu p_1^\mu \;=\; \sqrt{\frac{s}{2}}x_-$$



"Projectile" fields: $|\beta| < \sigma_a$

"Central" fields

"Target" fields: $|\alpha| < \sigma_b$

We integrate over "central" fields in the background of projectile and target

Matrix element between hadron states $\Rightarrow \sum_X = 1$

"Hadronic tensor"

$$W(p_A,p_B,q) \stackrel{\rm def}{=} \sum_X \int d^4x\, e^{-iqx}\langle p_A,p_B|F^2(x)|X\rangle\langle X|F^2(0)|p_A,p_B\rangle$$
$$= \int d^4x\, e^{-iqx}\langle p_A,p_B|F^2(x)F^2(0)|p_A,p_B\rangle$$

Double functional integral for W

$$W(p_A,p_B,q) = \sum_X \int d^4x\, e^{-iqx}\langle p_A,p_B|F^2(x)|X\rangle\langle X|F^2(0)|p_A,p_B\rangle$$
$$= \lim_{\substack{t_f\to\infty\\ t_i\to-\infty}} \int d^4x\, e^{-iqx}\int^{\tilde{A}(t_f)=A(t_f)} D\tilde{A}_\mu DA_\mu \int^{\tilde{\psi}(t_f)=\psi(t_f)} D\tilde{\bar{\psi}}D\tilde{\psi}D\bar{\psi}D\psi\ \Psi^*_{p_A}(\vec{\tilde{A}}(t_i),\tilde{\psi}(t_i))$$
$$\times\ \Psi^*_{p_B}(\vec{\tilde{A}}(t_i),\tilde{\psi}(t_i))e^{-iS_{\rm QCD}(\tilde{A},\tilde{\psi})}e^{iS_{\rm QCD}(A,\psi)}\tilde{F}^2(x)F^2(y)\Psi_{p_A}(\vec{A}(t_i),\psi(t_i))\Psi_{p_B}(\vec{A}(t_i),\psi(t_i))$$

"Left" A,ψ fields correspond to the amplitude $\langle X|F^2(0)|p_A,p_B\rangle$,
"right" fields $\tilde{A},\tilde{\psi}$ correspond to amplitude $\langle p_A,p_B|F^2(x)|X\rangle$
The boundary conditions $\tilde{A}(t_f)=A(t_f)$ and $\tilde{\psi}(t_f)=\psi(t_f)$ reflect the sum over intermediate states X.

TMD factorization with power corrections

In the region $s \gg Q^2 \gg Q_\perp^2$ at the tree level

$$
\begin{aligned}
W(p_A,p_B,q) \;=\; & \frac{64/s^2}{N_c^2-1}\int d^2x_\perp\; e^{i(q,x)_\perp}\,\frac{2}{s}\int dx_\bullet dx_*\; e^{-i\alpha_q x_\bullet - i\beta_q x_*} \\
& \times \Big\{ \langle p_A|\mathcal{G}_*^{mi}(x_\bullet,x_\perp)\mathcal{G}_*^{mj}(0)|p_A\rangle\langle p_B|\mathcal{F}_{\bullet i}^{n}(x_*,x_\perp)\mathcal{F}_{\bullet j}^{n}(0)|p_B\rangle \\
& + \frac{32}{Q^2}\frac{N_c^2\Delta^{ij,kl}}{(N_c^2-4)(N_c^2-1)}\int_{-\infty}^{x_\bullet} d\frac{2}{s}x'_\bullet\; d^{abc}\langle p_A|\mathcal{G}_{*i}^{a}(x_\bullet,x_\perp)\mathcal{G}_{*j}^{b}(x'_\bullet,x_\perp)\mathcal{G}_{*r}^{c}(0)|p_A\rangle \\
& \qquad\qquad \times \int_{-\infty}^{x_\bullet} d\frac{2}{s}x'_\bullet\; d^{mnl}\langle p_B|\mathcal{F}_{\bullet k}^{m}(x_*,x_\perp)\mathcal{F}_{\bullet l}^{n}(x'_*,x_\perp)\mathcal{F}_{\bullet r}^{n}(0)|p_B\rangle \Big\}
\end{aligned}
$$

$$\Delta^{ij,kl} \equiv g^{ij}g^{kl} - g^{ik}g^{jl} - g^{il}g^{jk}$$

$$
\begin{aligned}
\mathcal{G}_{*i}^{b}(z_\bullet,z_\perp) \;&\equiv\; \big([-\infty_\bullet,z_\bullet]_z^{A_*}\big)^{ab} F_{*i}^{b}(z_\bullet,z_\perp), \\
\mathcal{F}_{\bullet i}^{a}(z_*,z_\perp) \;&\equiv\; \big([-\infty_*,z_*]_z^{A_\bullet}\big)^{ab} F_{\bullet i}^{b}(z_*,z_\perp)
\end{aligned}
$$

Rapidity evolution: one loop

We study evolution of $\tilde{\mathcal{F}}_i^{a\eta}(x_\perp, x_B)\mathcal{F}_j^{a\eta}(y_\perp, x_B)$ with respect to rapidity cutoff η

$$\mathcal{F}_i^{a(\eta)}(z_\perp, x_B) \equiv \frac{2}{s}\int dz_*\ e^{ix_Bz_*}[\infty, z_*]_z^{am}F_{\bullet i}^m(z_*, z_\perp)$$
$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

At first we study gluon TMDs with Wilson lines stretching to $+\infty$ (like in SIDIS).
Matrix element of $\tilde{\mathcal{F}}_i^a(k'_\perp, x'_B)\mathcal{F}^{ai}(k_\perp, x_B)$ at one-loop accuracy:
diagrams in the "external field" of gluons with rapidity $< \eta$.

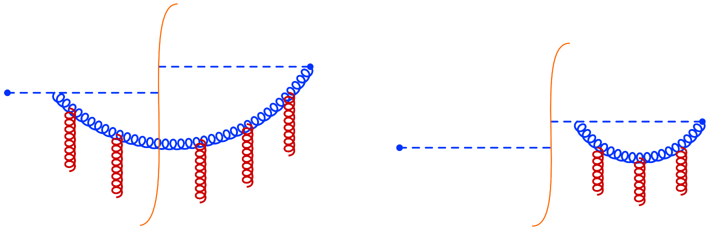
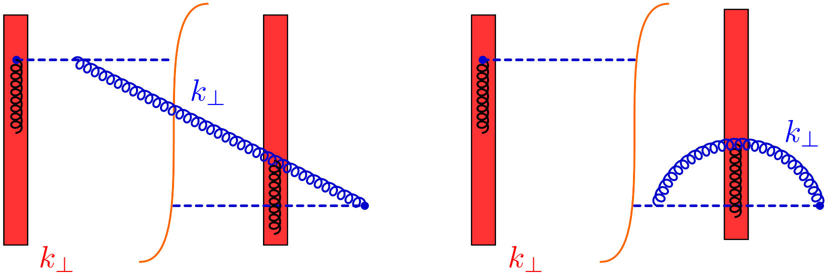


Figure : Typical diagrams for one-loop contributions to the evolution of gluon TMD.
(Fields $\tilde{\mathcal{A}}$ to the left of the cut and $\mathcal{A}$ to the right.)

Shock-wave formalism and transverse momenta

$\alpha \gg \alpha$ and $k_\perp \sim k_\perp \Rightarrow$ shock-wave external field



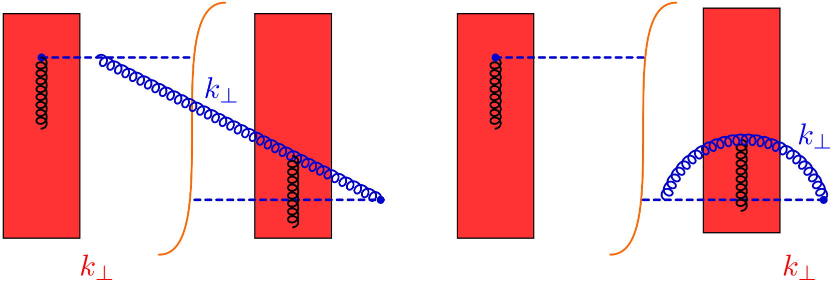
Characteristic longitudinal scale of fast fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s} \Rightarrow\ x_* \sim \frac{\alpha s}{k_\perp^2}$

Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s} \Rightarrow\ x_* \sim \frac{\alpha s}{k_\perp^2}$

If $\alpha \gg \alpha$ and $k_\perp^2 \leq k_\perp^2 \Rightarrow x_* \gg x_*$
$\Rightarrow$ Diagrams in the shock-wave background at $k_\perp \sim k_\perp$

Problem: different transverse momenta

$\alpha \gg \alpha$ and $k_\perp \gg k_\perp$ ⇒ the external field may be wide



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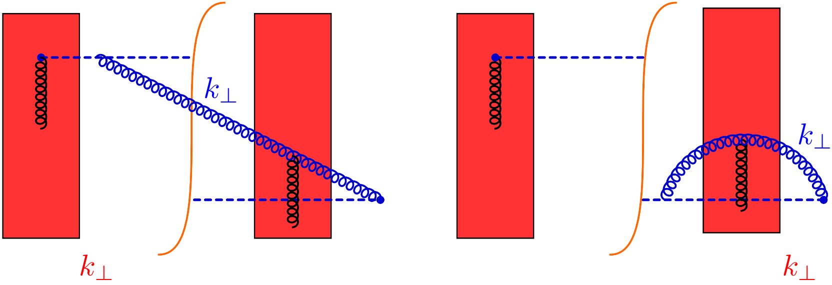
If $\alpha \gg \alpha$ and $k_\perp^2 \gg k_\perp^2$ ⇒ $x_* \sim x_*$ ⇒ shock-wave approximation is invalid

Problem: different transverse momenta

$\alpha \gg \alpha$ and $k_\perp \gg k_\perp$ $\Rightarrow$ the external field may be wide



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Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta s}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$ $\Rightarrow$ $x_* \sim \frac{\alpha s}{k_\perp^2}$

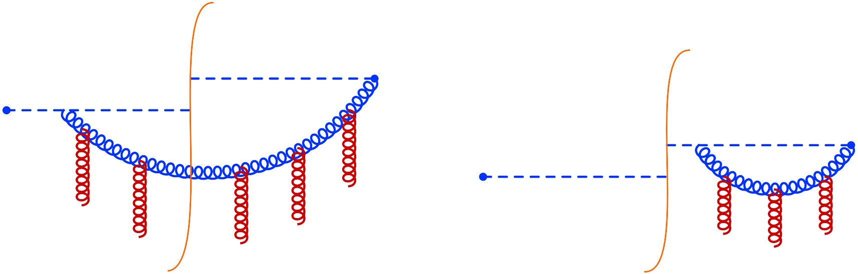
If $\alpha \gg \alpha$ and $k_\perp^2 \gg k_\perp^2$ $\Rightarrow x_* \sim x_* \Rightarrow$ shock-wave approximation is invalid

Fortunately, at $k_\perp^2 \gg k_\perp^2$ we can use another approximation

$\Rightarrow$ Light-cone expansion of propagators at $k_\perp \gg k_\perp$

Method of calculation

We calculate one-loop diagrams in the fast-field background



in following way:

if $k_\perp \sim k_\perp \Rightarrow$ propagators in the shock-wave background

if $k_\perp \gg k_\perp \Rightarrow$ light-cone expansion of propagators

We compute one-loop diagrams in these two cases and write down "interpolating" formulas correct both at $k_\perp \sim k_\perp$ and $k_\perp \gg k_\perp$

Shock-wave calculation

Reminder:

$$\tilde{\mathcal{F}}_i^a(z_\perp, x_B) \equiv \frac{2}{s}\int dz_* \; e^{-ix_B z_*} F^m_{\bullet i}(z_*, z_\perp)[z_*, \infty]_z^{ma}$$

At $x_B \sim 1$ $e^{-ix_B z_*}$ may be important even if shock wave is narrow.
Indeed, $x_* \sim \frac{\alpha s}{k_\perp^2} \ll x_* \sim \frac{\alpha s}{k_\perp^2} \Rightarrow$ shock-wave approximation is OK,
but $x_B \sigma_* \sim x_B \frac{\alpha s}{k_\perp^2} \sim \frac{\alpha s}{k_\perp^2} \geq 1 \Rightarrow$ we need to “look inside” the shock wave.

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Technically, we consider small but finite shock wave:
take the external field with the support in the interval $[-\sigma_*, \sigma_*]$ (where $\sigma_* \sim \frac{\alpha s}{k_\perp^2}$), calculate diagrams with points in and out of the shock wave, and check that the σ_*-dependence cancels in the sum of “in” and “out” contributions.

One-loop corrections in the shock-wave background

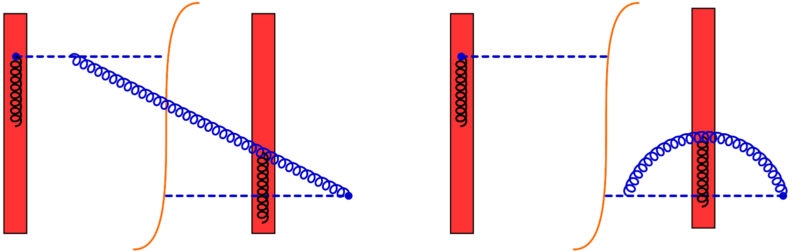

(a) (b)

Figure : Typical diagrams for production (a) and virtual (b) contributions to the evolution kernel.

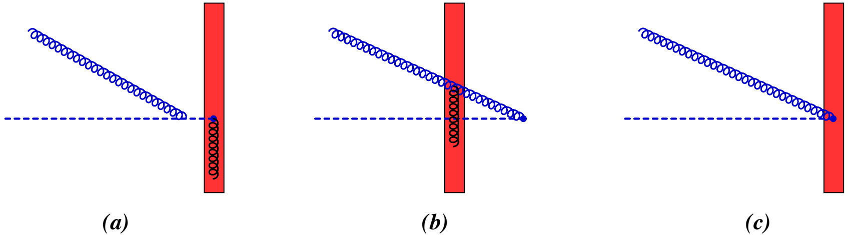


Figure : Lipatov vertex of gluon emission.

Definition

$$L^{ab}_{\mu i}(k, y_\perp, x_B) \;=\; i \lim_{k^2\to 0} k^2 \langle T\{A^a_\mu(k)\mathcal{F}^b_i(y_\perp, x_B)\}\rangle$$

Lipatov vertex in the shock-wave case

Result of calculation (in the background-Feynman gauge)

$$L^{ab}_{\mu i}(k,y_\perp,x_B) \;=\; 2ge^{-i(k,y)_\perp}\Big(\frac{p_{2\mu}}{\alpha s}-\frac{\alpha p_{1\mu}}{k_\perp^2}\Big)[\mathcal{F}_i(x_B,y_\perp)-U_i(y_\perp)]^{ab}$$

$$+\,g(k_\perp|g_{\mu i}\Big(\frac{\alpha x_B s}{\alpha x_B s+p_\perp^2}-U\frac{\alpha x_B s}{\alpha x_B s+p_\perp^2}U^\dagger\Big)+2\alpha p_{1\mu}\Big(\frac{p_i}{\alpha x_B s+p_\perp^2}-U\frac{p_i}{\alpha x_B s+p_\perp^2}U^\dagger\Big)$$

$$+\,\Big[2ix_Bp_{2\mu}\partial_iU-2i\partial^\perp_\mu Up_i+\frac{2p_{2\mu}}{\alpha s}\partial_\perp^2 Up_i\Big]\frac{1}{\alpha x_B s+p_\perp^2}U^\dagger-\frac{2\alpha p_{1\mu}}{p_\perp^2}U_i|y_\perp)^{ab}$$

$U_i\equiv\mathcal{F}_i(0)=i(\partial_iU)U^\dagger$.

Schwinger's notations $(x_\perp|\mathcal{O}(\hat{p}_\perp,\hat{X}_\perp)|y_\perp)\equiv\int d^2p\,\mathcal{O}(p_\perp,x_\perp)e^{-i(p,x-y)_\perp}$

Lipatov vertex in the light-cone case

Result of calculation (in the background-Feynman gauge)

$$L^{ab}_{\mu i}(k,y_\perp,x_B)\rangle ~=~ \frac{2ge^{-i(k,y)_\perp}}{\alpha x_B s+k_\perp^2}\mathcal{F}^{ab}_l(x_B+\frac{k_\perp^2}{\alpha s},y_\perp)$$
$$\times\Big[\frac{\alpha x_B s}{k_\perp^2}\big(\frac{k_\perp^2}{\alpha s}p_{2\mu}-\alpha p_{1\mu}\big)\delta^l_i-\delta^l_\mu k_i+\frac{\alpha x_B s g_{\mu i}k^l}{k_\perp^2+\alpha x_B s}+\frac{2\alpha k_i k^l}{k_\perp^2+\alpha x_B s}p_{1\mu}\Big]$$

NB:

$$k^\mu L^{ab}_{\mu i}(k,y_\perp,x_B) ~=~ 0$$

for both shock-wave and light-cone Lipatov vertices.

It is convenient to write Lipatov vertex in the light-like gauge $p_2^\mu A_\mu=0$ by replacement $\alpha p_1^\mu\rightarrow\alpha p_1^\mu-k^\mu ~=~ -k_\perp^\mu-\frac{k_\perp^2}{\alpha s}$

$$L^{ab}_{\mu i}(k,y_\perp,x_B)^{\rm light-like} ~=~ 2ge^{-i(k,y)_\perp}$$
$$\times\Big[\frac{k^\perp_\mu\delta^l_i}{k_\perp^2}-\frac{\delta^l_\mu k_i+\delta^l_i k^\perp_\mu-g_{\mu i}k^l}{\alpha x_B s+k_\perp^2}-\frac{k_\perp^2 g_{\mu i}k^l+2k^\perp_\mu k_i k^l}{(\alpha x_B s+k_\perp^2)^2}\Big]\mathcal{F}^{ab}_l(x_B+\frac{k_\perp^2}{\alpha s},y_\perp) ~+~ O(p_{2\mu})$$

Lipatov vertex at arbitrary momenta

"Interpolating formula" between the shock-wave and light-cone Lipatov vertices

$$L^{ab}_{\mu i}(k,y_\perp,x_B)^{\mathrm{light-like}}$$
$$= g(k_\perp|\mathcal{F}^j\big(x_B+\frac{k_\perp^2}{\alpha s}\big)\Big\{\frac{\alpha x_B s g_{\mu i}-2k^\perp_\mu k_i}{\alpha x_B s+k_\perp^2}(k_jU+Up_j)\frac{1}{\alpha x_B s+p_\perp^2}U^\dagger$$
$$-2k^\perp_\mu U\frac{g_{ij}}{\alpha x_B s+p_\perp^2}U^\dagger-2g_{\mu j}U\frac{p_i}{\alpha x_B s+p_\perp^2}U^\dagger+\frac{2k^\perp_\mu}{k_\perp^2}g_{ij}\Big\}|y_\perp)^{ab}+O(p_{2\mu})$$

This formula is actually correct (within our accuracy $\alpha_{\mathrm{fast}}\ll\alpha_{\mathrm{slow}}$) in the whole range of x_B and transverse momenta

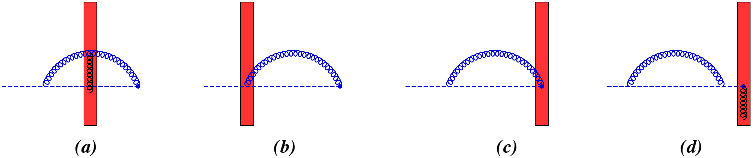


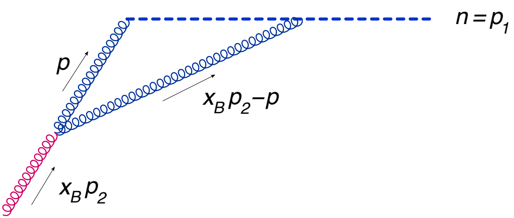
Figure : Virtual gluon corrections.

Result of the calculation (in light-like and background-Feynman gauges)

$$\langle \mathcal{F}_i^n(y_\perp, x_B)\rangle^{\text{Fig. 4}} = -ig^2 f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\bar{}\,\alpha}{\alpha}(y_\perp| - \frac{p^j}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l + U_l)$$
$$\times\,(2\delta_j^k\delta_i^l - g_{ij}g^{kl})U\frac{1}{\alpha x_B s + p_\perp^2}U^\dagger + \mathcal{F}_i(x_B)\frac{\alpha x_B s}{p_\perp^2(\alpha x_B s + p_\perp^2)}|y_\perp)^{kl}$$

NB: with $\alpha < \sigma$ cutoff there is no UV divergence.

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

$$I \;=\; \int \frac{d^4p}{\pi^2 i}\,\frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)}\,\frac{x_Bp_2\cdot n}{(x_Bp_2 - p)^2 - m^2 + i\epsilon}$$

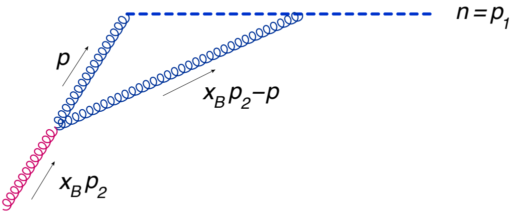
Regularization # 1 (ours): $n = p_1$, $|\alpha| < \sigma$

$$I_1 \;=\; -i\frac{s}{2\pi^2}\int_{-\sigma}^{\sigma} d\alpha \int \frac{d\beta}{\beta - i\epsilon}\int d^2p_\perp \frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon}\,\frac{x_B}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$= \frac{1}{\pi}\int_0^{\sigma} d\alpha \int d^2p_\perp \frac{1}{m^2 + p_\perp^2}\,\frac{1}{\alpha + \frac{m^2 + p_\perp^2}{sx_B}} \;=\; \int_0^{\sigma}\frac{d\alpha}{\pi\alpha}\ln\Big(1 + \frac{\alpha s x_B}{m^2}\Big) \;=\; \frac{1}{2}\ln^2\frac{\sigma s x_B}{m^2} + \frac{\pi^2}{6}$$

Double log of σ, no UV.

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

$$I \;=\; \int \frac{d^4p}{\pi^2 i}\,\frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)}\,\frac{x_Bp_2\cdot n}{(x_Bp_2 - p)^2 - m^2 + i\epsilon}$$

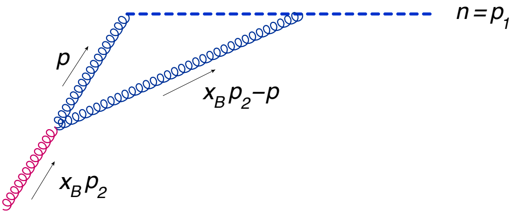
Regularization # 2 (by slope of Wilson line): $n = p_1 + \gamma p_2$, $\gamma \ll 1$

$$I_2 \;=\; -i\frac{s}{2\pi^2}\int d\alpha d\beta \int d^2p_\perp \frac{1}{\beta + \gamma\alpha - i\epsilon}\,\frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon}\,\frac{1}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$\Rightarrow\; I_2 \;\stackrel{(p_2\cdot n)^2 \gg m^2n^2}{\rightarrow}\; \frac{1}{2}\ln^2\frac{x_Bs^2}{m^2n^2} + \frac{\pi^2}{6} \;=\; \frac{1}{2}\ln^2\frac{x_Bs}{m^2\gamma} + \frac{\pi^2}{6}$$

Double log of σ, no UV.

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

$$I = \int \frac{d^4p}{\pi^2 i} \frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)} \frac{x_B p_2 \cdot n}{(x_B p_2 - p)^2 - m^2 + i\epsilon}$$

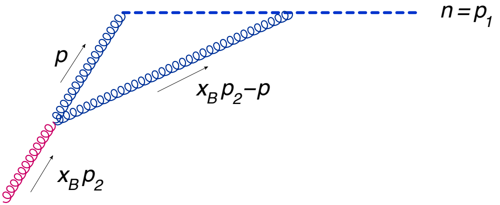
Regularization # 3: $n = p_1$, $\beta > b$

$$I_3 = -i\frac{s}{2\pi^2}\int d\alpha d\beta \int d^2p_\perp \frac{1}{\beta - i\epsilon} \frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon} \frac{x_B}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$\Rightarrow I_3 = \frac{1}{\pi}\int_b^{x_B} \frac{d\beta}{\beta} \int \frac{d^2p_\perp}{m^2 + p_\perp^2} = \ln\frac{x_B}{b} \ln\frac{\mu_{\rm UV}^2}{m^2}$$

UV $\times$ single log of the cutoff

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

$$I = \int \frac{d^4p}{\pi^2 i} \frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)} \frac{x_Bp_2\cdot n}{(x_Bp_2 - p)^2 - m^2 + i\epsilon}$$

Regularization # 1 $\Rightarrow$ Regularization # 3:

$$\text{change of variables} \quad \beta = \frac{x_B(m^2 + p_\perp^2)}{\alpha s x_B + m^2 + p_\perp^2}$$

$$\int_0^\sigma d\alpha \int d^2p_\perp \frac{1}{m^2 + p_\perp^2} \frac{1}{\alpha + \frac{m^2+p_\perp^2}{sx_B}} = \int_b^{x_B} \frac{d\beta}{\beta} \int \frac{d^2p_\perp}{m^2 + p_\perp^2}, \quad b = \frac{x_B(m^2 + p_\perp^2)}{\sigma s x_B + m^2 + p_\perp^2}$$

Evoltuion equation for the gluon TMD operator

A. Tarasov and I.B.

$$\frac{d}{d\ln\sigma}\big(\tilde{\mathcal{F}}^a_i(x_\perp,x_B)\mathcal{F}^a_j(y_\perp,x_B)\big)^{\ln\sigma}$$
$$= -\alpha_s\int đ^2k_\perp\ \mathrm{Tr}\{\tilde{L}_i^{\ \mu}(k,x_\perp,x_B)^{\mathrm{light-like}}L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}}\}$$
$$-\ \alpha_s\mathrm{Tr}\Big\{\tilde{\mathcal{F}}_i(x_\perp,x_B)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U\frac{1}{\sigma x_Bs+p_\perp^2}U^\dagger$$
$$+\ \mathcal{F}_j(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|y_\perp)$$
$$+\ (x_\perp|\tilde{U}\frac{1}{\sigma x_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(x_B)\frac{p^m}{p_\perp^2}$$
$$+\ \tilde{\mathcal{F}}_i(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}\ +\ O(\alpha_s^2)$$

This expression is UV and IR convergent.
It describes the rapidity evolution of gluon TMD operator in for any x_B and transverse momenta!

Evoltuion equation for the gluon TMD

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\big(\tilde{\mathcal{F}}^a_i(x_\perp,x_B)\mathcal{F}^a_j(y_\perp,x_B)\big)^{\ln\sigma}|p\rangle \\
=\ &-\alpha_s\int d\!\!\bar{}^2k_\perp\ \langle p|\mathrm{Tr}\{\tilde{L}_i^{\ \mu}(k,x_\perp,x_B)^{\rm light-like}\theta\big(1-x_B-\frac{k_\perp^2}{\alpha s}\big)L_{\mu j}(k,y_\perp,x_B)^{\rm light-like}\}|p\rangle \\
&-\alpha_s\langle p|\mathrm{Tr}\Big\{\tilde{\mathcal{F}}_i(x_\perp,x_B)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U\frac{1}{\sigma x_Bs+p_\perp^2}U^\dagger \\
&\qquad\qquad +\mathcal{F}_j(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|y_\perp) \\
&+(x_\perp|\tilde{U}\frac{1}{\sigma x_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(x_B)\frac{p^m}{p_\perp^2} \\
&\qquad\qquad +\tilde{\mathcal{F}}_i(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}|p\rangle\ +\ O(\alpha_s^2)
\end{aligned}
$$

The factor $\theta\big(1-x_B-\frac{k_\perp^2}{\alpha s}\big)$ reflects kinematical restriction that the fraction of initial proton's momentum carried by produced gluon should be smaller than $1-x_B$

Light-cone limit

$$\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma} = \frac{\alpha_s}{\pi}N_c\int_{\sigma'}^{\sigma}\frac{d\alpha}{\alpha}\int_0^\infty d\beta\,\Big\{\theta(1-x_B-\beta)$$
$$\times\Big[\frac{1}{\beta}-\frac{2x_B}{(x_B+\beta)^2}+\frac{x_B^2}{(x_B+\beta)^3}-\frac{x_B^3}{(x_B+\beta)^4}\Big]\langle p|\tilde{\mathcal{F}}_i^n(x_B+\beta,x_\perp)$$
$$\times\,\mathcal{F}^{ni}(x_B+\beta,x_\perp)|p\rangle^{\ln\sigma'}-\frac{x_B}{\beta(x_B+\beta)}\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma'}\Big\}$$

In the LLA the cutoff in $\sigma \Leftrightarrow$ cutoff in transverse momenta

$$\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{k_\perp^2<\mu^2} = \frac{\alpha_s}{\pi}N_c\int_0^\infty d\beta\int_{\frac{\mu'^2}{\beta s}}^{\frac{\mu^2}{\beta s}}\frac{d\alpha}{\alpha}\,\{\mathrm{same}\}$$

$\Rightarrow$ DGLAP equation $\Rightarrow$ $(z'\equiv\frac{x_B}{x_B+\beta})$ DGLAP kernel

$$\frac{d}{d\eta}\alpha_s\mathcal{D}(x_B,0_\perp,\eta) = \frac{\alpha_s}{\pi}N_c\int_{x_B}^1\frac{dz'}{z'}\,\Big[\Big(\frac{1}{1-z'}\Big)_+ +\frac{1}{z'}-2+z'(1-z')\Big]\alpha_s\mathcal{D}\Big(\frac{x_B}{z'},0_\perp,\eta\Big)$$

Low-x case: BK evolution of the WW distribution

Low-x regime: $x_B = 0$ + characteristic transverse momenta $p_\perp^2 \sim (x-y)_\perp^{-2} \ll s$
$\Rightarrow$ in the whole range of evolution ($1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$) we have $\frac{p_\perp^2}{\sigma s} \ll 1 \Rightarrow$ the kinematical constraint $\theta\big(1 - \frac{k_\perp^2}{\alpha s}\big)$ can be omitted

$\Rightarrow$ non-linear evolution equation

$$\frac{d}{d\eta}\tilde{U}_i^a(z_1)U_j^a(z_2)$$
$$= -\frac{g^2}{8\pi^3}\mathrm{Tr}\Big\{(-i\partial_i^{z_1} + \tilde{U}_i^{z_1})\Big[\int d^2z_3(\tilde{U}_{z_1}\tilde{U}_{z_3}^\dagger - 1)\frac{z_{12}^2}{z_{13}^2 z_{23}^2}(U_{z_3}U_{z_2}^\dagger - 1)\Big](i\overleftarrow{\partial}_j^{z_2} + U_j^{z_2})\Big\}$$

where $\eta \equiv \ln\sigma$ and $\frac{z_{12}^2}{z_{13}^2 z_{23}^2}$ is the BK kernel

This eqn holds true also at small x_B up to $x_B \sim \frac{(x-y)_\perp^{-2}}{s}$ since in the whole range of evolution $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ one can neglect $\sigma x_B s$ in comparison to $p_\perp^2$ in the denominators $(p_\perp^2 + \sigma x_B s) \Leftrightarrow$ effectively $x_B = 0$.

Sudakov double logs

Sudakov limit: $x_B \equiv x_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \sim$ few GeV.

One can show that the non-linear terms are power suppressed $\Rightarrow$

$$\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle$$
$$= 4\alpha_s N_c\int\frac{d\!\!\!{}^-{}^2p_\perp}{p_\perp^2}\Big[e^{i(p,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_i^a\Big(x_B+\frac{p_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j^a\Big(x_B+\frac{p_\perp^2}{\sigma s},y_\perp\Big)|p\rangle$$
$$-\frac{\sigma x_B s}{\sigma x_B s+p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

Double-log region: $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ and $\sigma x_B s \gg p_\perp^2 \gg (x-y)_\perp^{-2}$

$$\Rightarrow\ \frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma) = -\frac{\alpha_s N_c}{\pi^2}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2p_\perp}{p_\perp^2}\big[1-e^{i(p,z)_\perp}\big]$$

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$$-\;\frac{\sigma x_B s}{\sigma x_B s+p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

Double-log region: $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ and $\sigma x_B s \gg p_\perp^2 \gg (x-y)_\perp^{-2}$

$$\Rightarrow\;\frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma)\;=\;-\frac{\alpha_s N_c}{\pi^2}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2p_\perp}{p_\perp^2}\big[1-e^{i(p,z)_\perp}\big]$$

$\Rightarrow$ Sudakov double logs

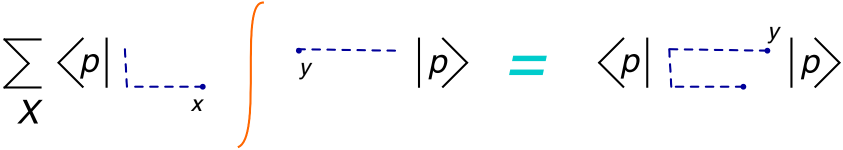
$$\mathcal{D}(x_B,k_\perp,\ln\sigma)\;\sim\;\exp\Big\{-\frac{\alpha_s N_c}{2\pi}\ln^2\frac{\sigma s}{k_\perp^2}\Big\}\mathcal{D}\Big(x_B,k_\perp,\ln\frac{k_\perp^2}{s}\Big)$$

Gluon TMD with gauge links to $-\infty$

$$\alpha_s\mathcal{D}(x_B,z_\perp) \;=\; -\frac{\alpha_s}{2\pi(p\cdot n)x_B}\int du\; e^{-ix_Bu(pn)}\langle p|\tilde{\mathcal{F}}^a_\xi(z_\perp+un)[z_\perp,0]_{-\infty}\mathcal{F}^{a\xi}(0)|p\rangle$$

$$\begin{aligned}\mathcal{F}^a_\xi(z_\perp+un) &\equiv [-\infty n+z_\perp, un+z_\perp]^{am}n^\mu F^m_{\mu\xi}(un+z_\perp)\\ \tilde{\mathcal{F}}^a_\xi(z_\perp+un) &\equiv n^\mu F^m_{\mu\xi}(un+z_\perp)[un+z_\perp,-\infty n+z_\perp]^{ma}\end{aligned}$$

Double functional integral:



One-loop diagrams are the same as before.

Lipatov vertex for gauge links going to $-\infty$

$$L^{ab}_{\mu i}(k, y_\perp, \beta_B)^{\rm light-like}_{-\infty} \;=\; g(k_\perp|U\mathcal{F}^j\big(\beta_B + \frac{k_\perp^2}{\alpha s}\big)$$

$$\Big[\frac{\alpha\beta_B s g_{\mu i} - 2k^\perp_\mu k_i}{\alpha\beta_B s + k_\perp^2}\frac{(p+k)_j}{\alpha\beta_B s + p_\perp^2} - \frac{2k^\perp_\mu g_{ij} + 2g_{\mu j}p_i}{\alpha\beta_B s + p_\perp^2}\Big] + 2U\frac{p^\perp_\mu}{p_\perp^2}\mathcal{F}_i\big(\beta_B + \frac{k_\perp^2}{\alpha s}\big)|y_\perp)^{ab}$$

Lipatov vertex for gauge links going to $-\infty$

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$$\Big[\frac{\alpha\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\alpha\beta_B s+k_\perp^2}\frac{(p+k)_j}{\alpha\beta_B s+p_\perp^2}-\;\frac{2k^\perp_\mu g_{ij}+2g_{\mu j}p_i}{\alpha\beta_B s+p_\perp^2}\Big]+2U\frac{p^\perp_\mu}{p_\perp^2}\mathcal{F}_i\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)|y_\perp)^{ab}$$

Compare to L. vertex for gauge links going to $+\infty$

$$L^{ab}_{\mu i}(k,y_\perp,\beta_B)^{\text{light}-\text{like}}_{+\infty} \;=\; g(k_\perp|\Big\{\frac{\alpha\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\alpha\beta_B s+k_\perp^2}\mathcal{F}^j\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)U\frac{(p+k)_j}{\alpha\beta_B s+p_\perp^2}U^\dagger$$
$$-\,2\mathcal{F}^j\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)U\frac{g_{ij}k^\perp_\mu+g_{\mu j}p_i}{\alpha\beta_B s+p_\perp^2}U^\dagger+\frac{2k^\perp_\mu}{k_\perp^2}\mathcal{F}_i\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)\Big\}|y_\perp)^{ab}$$

Evoltuion equation for the gluon TMD

Replace

$\infty n \to -\infty n$ everywhere

and

$x_B \to -x_B$ in the virtual correction:

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\big(\mathcal{F}_i^a(x_\perp,x_B)\mathcal{F}_j^a(y_\perp,x_B)\big)^{\ln\sigma}|p\rangle \\
=\; & -\alpha_s\int \bar{d}^2k_\perp\;\langle p|\mathrm{Tr}\{L_i^{\ \mu}(k,x_\perp,x_B)^{\mathrm{light-like}}\theta\Big(1-x_B-\frac{k_\perp^2}{\alpha s}\Big)L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}}\}|p\rangle \\
& -\alpha_s\langle p|\mathrm{Tr}\Big\{\mathcal{F}_i(x_\perp,x_B)(y_\perp|U^\dagger\frac{1}{\sigma x_B s-p_\perp^2+i\epsilon}U(2\delta_m^k\delta_j^l-g_{jm}g^{kl})(i\partial_l+U_l)\mathcal{F}_k(x_B)\frac{p^m}{p_\perp^2} \\
&\qquad\qquad +\mathcal{F}_j(x_B)\frac{\sigma x_B s}{p_\perp^2(\sigma x_B s-p_\perp^2+i\epsilon)}|y_\perp) \\
& +(x_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_l(x_B)(i\overleftarrow{\partial}_k+U_k)(2\delta_i^k\delta_m^l-g_{im}g^{kl})U^\dagger\frac{1}{\sigma x_B s-p_\perp^2-i\epsilon}U \\
&\qquad\qquad +\mathcal{F}_i(x_B)\frac{\sigma x_B s}{p_\perp^2(\sigma x_B s-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}|p\rangle\;+\;O(\alpha_s^2)
\end{aligned}
$$

Conclusions and outlook

1. Conclusions
   - The evolution equation for gluon TMD at any x_B and transverse momenta.
   - Interpolates between linear DGLAP and Sudakov limits and the non-linear low-x BK regime
2. Outlook
   - Conformal invariance (for N=4 SYM)?
   - Transition between collinear factorization and k_T factorization.

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Thank you for attention!