3-Loop Corrections to Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering

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Introduction

Unpolarized Deep–Inelastic Scattering (DIS):

\[ Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x} \]

\[ \frac{d\sigma}{dQ^2 \, dx} \sim W_{\mu\nu} L^{\mu\nu} \]

\[ W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s \mid [J^{em}_\mu(\xi), J^{em}_\nu(0)] \mid P, s \rangle = \]

\[ \frac{1}{2x} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left( P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2 g_{\mu\nu}} \right) F_2(x, Q^2) . \]

Structure Functions: \( F_{2,L} \)

contain light and heavy quark contributions.
\[ \Delta_{TH} \alpha_s = \alpha_s(N^3\text{LO}) - \alpha_s(\text{NNLO}) + \Delta_{HQ} = +0.0009 \pm 0.0006_{\text{HQ}} \]

NNLO accuracy is needed to analyze the world data. \( \Rightarrow \) NNLO HQ corrections needed.
Deep–Inelastic Scattering (DIS):

\[ \sigma_{cc}^{\text{red}} \]

\[ Q^2 = 2.5 \text{ GeV}^2 \]
\[ 5 \text{ GeV}^2 \]
\[ 7 \text{ GeV}^2 \]
\[ 12 \text{ GeV}^2 \]
\[ 18 \text{ GeV}^2 \]
\[ 32 \text{ GeV}^2 \]
\[ 60 \text{ GeV}^2 \]
\[ 120 \text{ GeV}^2 \]
\[ 200 \text{ GeV}^2 \]
\[ 350 \text{ GeV}^2 \]
\[ 650 \text{ GeV}^2 \]
\[ 2000 \text{ GeV}^2 \]

\[ \alpha_s(M_Z) \]
\[ m_c(m_c) = 1.252 \pm 0.018 (\text{exp}) +0.03 -0.02 (\text{scale}) +0.00 -0.07 (\text{thy}) \text{GeV} \]
\[ m_b(m_b) = 3.84 \pm 0.12 \text{GeV} \]
\[ m_t(m_t) = 160.9 \pm 1.1 \text{GeV} \]

Yet approximate NNLO treatment [Kawamura et al. [1205.5227]].

PS corrections are exact.
Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

\[ F_{(2,L)}(x, Q^2) = \sum_j C_{j,(2,L)} \left( x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes f_j(x, \mu^2) \]

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
\( \otimes \) denotes the Mellin convolution

\[ f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz)f(y)g(z) . \]

The subsequent calculations are performed in Mellin space, where \( \otimes \) reduces to a multiplication, due to the Mellin transformation

\[ \hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) . \]
Wilson coefficients:

\[ \mathcal{C}_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = C_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2} \right) + H_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) . \]

At \( Q^2 \gg m^2 \) the heavy flavor part

\[ H_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_i C_{i,(2,L)} \left( N, \frac{Q^2}{\mu^2} \right) A_{ij} \left( \frac{m^2}{\mu^2}, N \right) \]

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B] factorizes into the light flavor Wilson coefficients \( C \) and the massive operator matrix elements (OMEs) of local operators \( O_i \) between partonic states \( j \)

\[ A_{ij} \left( \frac{m^2}{\mu^2}, N \right) = \langle j | O_i | j \rangle . \]

→ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For \( F_2(x, Q^2) : \) at \( Q^2 \gtrsim 10m^2 \) the asymptotic representation holds at the 1% level.
Status of OME calculations


**Next-to-Leading Order:**
[Laenen, van Neerven, Riemersma, Smith 1993]

\( Q^2 \gg m^2 \): via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via \( pF_q \)'s [Bierenbaum, Blümlein, Klein, 2007]

\( O(\alpha_s^2 \varepsilon) \) (for general \( N \)) [Bierenbaum, Blümlein, Klein 2008, 2009]

**Next-to-Next-to-Leading Order:** \( Q^2 \gg m^2 \)

- Moments for \( F_2 \): \( N = 2...10(14) \) [Bierenbaum, Blümlein, Klein 2009]
- mapping large expressions to [MATAD, Steinhauser 2000]
- Contributions to transversity: \( N = 1...13 \) [Blümlein, Klein, Tödtli 2009]
- Two masses \( m_1 \neq m_2 \) \( \rightarrow \) Moments \( N = 2, 4, 6 \) [JB, Wißbrock 2011]

**At 3-loop order for general values of \( N \):**

- All OMEs: terms \( O(n_f T_F^2 C_{A/F}) \) to \( F_2 \) [Ablinger et al. 2011, 2012]
- First contributions to \( O(T_F^2 C_{A/F}) A_{gg, Q} \) [Ablinger et al. 2014]
The Wilson Coefficients at large $Q^2$

\[
\begin{align*}
2014 \quad & L^\text{NS}_{q,(2,L)}(N_F+1) = a_s^2 \left[ A^{(2),\text{NS}}_{qg,Q} (N_F + 1) \delta_2 + \tilde{C}^{(2),\text{NS}}_{q,(2,L)}(N_F) \right] \\
+ & a_s^3 \left[ A^{(3),\text{NS}}_{qg,Q} (N_F + 1) \delta_2 + A^{(2),\text{NS}}_{gq,Q} (N_F + 1)C^{(1),\text{NS}}_{q,(2,L)}(N_F + 1) + \tilde{C}^{(3),\text{NS}}_{q,(2,L)}(N_F) \right] \\
2010 \quad & L^\text{PS}_{q,(2,L)}(N_F+1) = a_s^3 \left[ A^{(3),\text{PS}}_{qg,Q} (N_F + 1) \delta_2 + A^{(2),\text{PS}}_{gq,Q} (N_F + 1) F_N \tilde{c}^{(1)}_{q,(2,L)}(N_F + 1) + N_F \tilde{c}^{(3),\text{PS}}_{q,(2,L)}(N_F) \right] \\
+ & A^{(1),\text{PS}}_{qg,Q} (N_F + 1) F_N \tilde{c}^{(2)}_{g,(2,L)}(N_F + 1) + \tilde{C}^{(3),\text{PS}}_{q,(2,L)}(N_F + 1) \\
2010 \quad & L^\text{S}_{g,(2,L)}(N_F+1) = a_s^2 \left[ A^{(1)}_{gq,Q} (N_F + 1) N_F \tilde{c}^{(1)}_{g,(2,L)}(N_F + 1) + a_s^3 \left[ A^{(3),\text{PS}}_{gq,Q} (N_F + 1) \delta_2 \\
+ & A^{(1)}_{gq,Q} (N_F + 1) N_F \tilde{c}^{(2)}_{g,(2,L)}(N_F + 1) + A^{(2),\text{PS}}_{gq,Q} (N_F + 1) F_N \tilde{c}^{(1)}_{g,(2,L)}(N_F + 1) \\
+ & A^{(1),\text{PS}}_{qg,Q} (N_F + 1) \tilde{C}^{(2),\text{PS}}_{q,(2,L)}(N_F + 1) + A^{(2),\text{PS}}_{qg,Q} (N_F + 1) \tilde{C}^{(1),\text{NS}}_{g,(2,L)}(N_F + 1) \right] \\
2014 \quad & H^\text{PS}_{q,(2,L)}(N_F+1) = a_s^2 \left[ A^{(2),\text{PS}}_{qg,Q} (N_F + 1) \delta_2 + \tilde{C}^{(2),\text{PS}}_{q,(2,L)}(N_F + 1) \right] + a_s^3 \left[ A^{(3),\text{PS}}_{qg,Q} (N_F + 1) \delta_2 \\
+ & A^{(1),\text{PS}}_{qg,Q} (N_F + 1) \tilde{C}^{(1),\text{NS}}_{q,(2,L)}(N_F + 1) + A^{(1),\text{PS}}_{gq,Q} (N_F + 1) \tilde{C}^{(1),\text{NS}}_{g,(2,L)}(N_F + 1) \\
+ & A^{(2),\text{PS}}_{gq,Q} (N_F + 1) \tilde{c}^{(1)}_{g,(2,L)}(N_F + 1) + \tilde{C}^{(2),\text{PS}}_{q,(2,L)}(N_F + 1) \right] + a_s^3 \left[ A^{(3),\text{PS}}_{qg,Q} (N_F + 1) \delta_2 + A^{(2),\text{PS}}_{qg,Q} (N_F + 1) C^{(1),\text{NS}}_{q,(2,L)}(N_F + 1) \\
+ & A^{(1),\text{PS}}_{gq,Q} (N_F + 1) \tilde{C}^{(1),\text{NS}}_{g,(2,L)}(N_F + 1) + A^{(1),\text{PS}}_{qg,Q} (N_F + 1) \tilde{C}^{(2),\text{NS}}_{q,(2,L)}(N_F + 1) \\
+ & \tilde{C}^{(2),\text{PS}}_{q,(2,L)}(N_F + 1) \right] + A^{(1),\text{PS}}_{gq,Q} (N_F + 1) \tilde{C}^{(2),\text{PS}}_{g,(2,L)}(N_F + 1) + \tilde{C}^{(3),\text{PS}}_{g,(2,L)}(N_F + 1)
\end{align*}
\]

All logarithmic corrections are known.

[Ablinger et al. 2010, Ablinger et al., 2014a, Ablinger et al., 2014b]
Variable Flavor Number Scheme

\[ f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) = A^{NS}_{qq,Q}(n_f, \frac{\mu^2}{m^2}) \otimes [f_k(n_f, \mu^2) + f_{\bar{k}}(n_f, \mu^2)] + A^{PS}_{qq,Q}(n_f, \frac{\mu^2}{m^2}) \otimes \Sigma(n_f, \mu^2) + \tilde{A}^{S}_{ag,Q}(n_f, \frac{\mu^2}{m^2}) \otimes G(n_f, \mu^2) \]

\[ f_{Q+\bar{Q}}(n_f + 1, \mu^2) = \tilde{A}^{PS}_{Qq}(n_f, \frac{\mu^2}{m^2}) \otimes \Sigma(n_f, \mu^2) + \tilde{A}^{S}_{Qg}(n_f, \frac{\mu^2}{m^2}) \otimes G(n_f, \mu^2) \]

\[ G(n_f + 1, \mu^2) = A^{S}_{ag,Q}(n_f, \frac{\mu^2}{m^2}) \otimes \Sigma(n_f, \mu^2) + A^{S}_{gg,Q}(n_f, \frac{\mu^2}{m^2}) \otimes G(n_f, \mu^2) \]

\[ \Sigma(n_f + 1, \mu^2) = \sum_{k=1}^{n_f+1} [f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2)] \]

\[ = A^{NS}_{qq,Q}(n_f, \frac{\mu^2}{m^2}) + n_f A^{PS}_{qq,Q}(n_f, \frac{\mu^2}{m^2}) + \tilde{A}^{PS}_{Qq}(n_f, \frac{\mu^2}{m^2}) \otimes \Sigma(n_f, \mu^2) \]

\[ + n_f \tilde{A}^{S}_{ag,Q}(n_f, \frac{\mu^2}{m^2}) + \tilde{A}^{S}_{Qg}(n_f, \frac{\mu^2}{m^2}) \otimes G(n_f, \mu^2) \]

All master integrals for \( A^{(3)}_{gg} \) have been completed (June 2015).
Calculation of the 3-loop operator matrix elements

The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:

\[ \delta^{ij} \Delta \gamma \Delta \cdot p \cdot N \geq 1 \]

\[ g^{\alpha \beta} \Delta \gamma \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-2-j} , \ N \geq 2 \]

\[ g^{3} \Delta \gamma \Delta \gamma \sum_{j=0}^{N-3} \sum_{i=1}^{N-2-j} (\Delta \cdot p_1)^i (\Delta \cdot p_1)^{N-3-j} \]

\[ f_{abc} f_{\mu \nu \lambda} \left( p_1, p_2, p_3, p_4 \right) \]

\[ O_{\mu \nu \lambda} \left( p_1, p_2, p_3, p_4 \right) = \Delta \cdot p_1 \left\{ -g_{\mu \nu} (\Delta \cdot p_3 + \Delta \cdot p_4) \right\} \]

\[ + p_{4, \mu} g_{\mu \nu} \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \]

\[ - p_{1, \mu} g_{\mu \nu} \sum_{i=0}^{N-3} (\Delta \cdot p_1 + \Delta \cdot p_4)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \]

\[ + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu \nu} + p_1 \cdot p_4 \Delta \cdot p_3 - \Delta \cdot p_1 \Delta \cdot p_4 - \Delta \cdot p_4 \Delta \cdot p_1] \]

\[ \sum_{i=0}^{N-4} \sum_{j=0}^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-4-i} \]

\[ \left\{ p_{1, \mu} p_{2, \mu} \right\} + \left\{ p_{1, \mu} p_{3, \mu} \right\} + \left\{ p_{1, \mu} p_{4, \mu} \right\} , \ N \geq 2 \]

<table>
<thead>
<tr>
<th></th>
<th>$A^{(3)\text{NS}}_{qq,Q}$</th>
<th>$A^{(3)\text{NS}}_{gq,Q}$</th>
<th>$A^{(3)\text{PS}}_{Qq}$</th>
<th>$A^{(3)\text{NS}}_{gg,Q}$</th>
<th>$A^{(3)\text{NS}}_{Qg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. diagrams</td>
<td>110</td>
<td>86</td>
<td>125</td>
<td>642</td>
<td>1233</td>
</tr>
</tbody>
</table>

A FORM [Vermaseren 2000] program was written in order to perform the $\gamma$-matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$A^{(3)\text{NS}}_{qq,Q} \rightarrow 7426$ scalar integrals.

$A^{(3)\text{NS}}_{gq,Q} \rightarrow 12529$ scalar integrals.

$A^{(3)\text{NS}}_{Qq} \rightarrow 5470$ scalar integrals.

$\Rightarrow$ Need to use integration by parts identities.

$\Rightarrow$ The reduction for all OMEs has been completed.

$\Rightarrow$ Use special computers: 12 units with overall 3.2 TB RAM, 97 TB fast disc, hundreds of mathematica lic.; IBP: several TB of final relations.
Integration by parts

We use Reduze [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a C++ program based on Laporta’s algorithm.

\[
(\Delta \cdot k)^N \to \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}
\]

\[\implies\] additional propagator.

Number of master integrals:

\[
A^{(3),\text{NS}}_{qq,Q} \to 35 \text{ master integrals } \checkmark.
\]
\[
A^{(3)}_{gq,Q} \to 41 \text{ master integrals } \checkmark.
\]
\[
A^{(3),\text{PS}}_{Qq} \to 66 \text{ master integrals } \checkmark.
\]
\[
A^{(3)}_{gg,Q} \to 205 \text{ master integrals } \checkmark.
\]
\[
A^{(3)}_{Qg} \to 340 \text{ master integrals. (224 done by June 2015.)}
\]

116 master integrals to be done \[\implies\] CIS-type

24 integral families are required and implemented in Reduze.
Calculation of the master integrals

For the calculation of the master integrals we use a wide variety of tools:

▶ Hypergeometric functions.
▶ Summation methods based on difference fields, implemented in the Mathematica program Sigma [C. Schneider, 2005–].
  ▶ Reduction of the sums to a small number of key sums.
  ▶ Expansion the summands in $\varepsilon$.
  ▶ Simplification by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–].
  ▶ Harmonic sums, polylogarithms and their various generalizations are algebraically reduced using the package HarmonicSums [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011,2013].
▶ Mellin-Barnes representations.
▶ In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
▶ Systems of Differential Equations.
▶ Almkvist-Zeilberger Theorem as Integration Method.
Emergence of new nested sums:

\[\sum_{i=1}^{N} \binom{2i}{i} (-2)^i \sum_{j=1}^{i} \frac{1}{i(2j)} S_{1,2} \left(\frac{1}{2}, -1; j\right)\]

\[= \int_0^1 dx \frac{x^N - 1}{x - 1} \sqrt{\frac{x}{8 + x}} \left[H^*_{w_{17}}, -1, 0(x) - 2H^*_{w_{18}}, -1, 0(x)\right] + \frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^N - 1}{x + 1} \sqrt{\frac{x}{8 + x}} [H^*_{12}(x) - 2H^*_{13}(x)] + c_3 \int_0^1 dx \frac{(-8x)^N - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1 - x}},\]

\[w_{12} = \frac{1}{\sqrt{x(8 - x)}},\quad w_{13} = \frac{1}{(2 - x)\sqrt{x(8 - x)}},\]

\[w_{17} = \frac{1}{\sqrt{x(8 + x)}},\quad w_{18} = \frac{1}{(2 + x)\sqrt{x(8 + x)}}.\]
Non-iterative Iterative Integrals

The live after iterative integrals and/or differential equations factorizing completely to 1st order:

• Iterative integrals/nested sums in QFT have been very well understood during the last 19 years since 1998. [J. Vermaseren, E. Remiddi, JB];
• Now even general alphabets (including up to root valued letters).
• Even single-scale Feynman integrals lead beyond that [Sabry’s kite, 1962]
• Currently worked out by the community. [Ablinger, Adams, Ananthanarayan, Behring, Bijnes, JB, Bloch, Bogner, Brown, DeFreitas, Gangl, Ghosh, Hebbar, Hoeij, Imamoglu, Laporta, Levin, Müller-Stach, Remiddi, Schneider, Schweitzer, Tancredi, Vidunna, Weinzierl, Zagier, Zayadeh, ...]

\[ \int_a^b \cdots \int_{a_m}^{b_m} \, \cdots \int_{a_q}^{b_q} \, f(x_1, \ldots, x_q) \, dx_1 \cdots dx_q \]

CIS-series; In some cases: complete elliptic integrals at very special rational arguments. Highly precise numerical representations already available. On the structural side: Relations to elliptic polylogarithms [in the elliptic case].
• Have to handle branch-points in case.
• Relations due to shuffle algebras.
• Further relations due to triangle group; Important relations between different solutions of the homogeneous equations.

Most of the master integrals infected by the new CIS solutions are iterated integrals over a few of the former ones.

• We have identified the whole respective tree in case of our project.
• It would be interesting to view the corresponding situation in case of $\sigma(pp \rightarrow t\bar{t})(\hat{s})$. 
Spill-Off:
New Mathematical Function Classes and Algebras

- 1998: Harmonic Sums [Vermaseren; JB]
- 1999: Harmonic Polylogarithms [Remiddi, Vermaseren]
- 2001: Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- 2004: Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2016: Elliptic integrals with (involved) rational arguments appear in part of the functions of our project already as base cases. They stem from Heun equations. [since April 2016.] [Ablinger, Behring, JB, De Freitas, van Hoeij, Raab, Schneider, DESY16-147].

Particle Physics Generates NEW Mathematics.
Numerical Results: $L_{g,2}^S$ and $L_{q,2}^{PS}$
$L_{q,2}^{\text{NS}}$

Contribution to $F_2(x, Q^2)$

VFNS matching
NS corrections to $g_{1(2)}(x, Q^2)$ and $xF_3^{W^+ + W^-}$

$$g_1(x, Q^2)$$

$$xF_3^{W^+ + W^-}(x, Q^2)$$

The corrections to $g_2(x, Q^2)$ are obtained using the Wandzura-Wilczek relation.
NS corrections to $F_{1}^{W^+ - W^-}$ and $F_{2}^{W^+ - W^-}$

$F_{1}^{W^+ - W^-}(x, Q^2)$

$F_{2}^{W^+ - W^-}(x, Q^2)$

The massless corrections are due to Davies, Vogt, Moch, Vermaseren, LT-1084.

$O(\alpha_s^2)$ Complete NS corrections

Note the negative corrections at low $Q^2$!
Already for charm it takes quite a while to become massless.

The leading small $x$ approximation corresponding to CCH, 1991, departs from the physical result everywhere except for $x = 1$. 
The present NC corrections to $F_2(x, Q^2)$

$Q^2 = 100\text{GeV}^2$ [\(H_{g,2}^S\) scaled down by a factor 20.]
\[ a^{(3)}_{gg,Q} = \frac{1 + (-1)^{N}}{2} \left\{ c_F^2 T_F \left[ \frac{16(N^2 + N + 2)}{N^2(N + 1)^2} \sum_{i=1}^{N} \frac{(2i)}{4^i (i+1)^2} \left( \sum_{j=1}^{i} \frac{4^j S_1(j-1)}{(2j)!} j^2 \right) - 7\zeta_3 \right] - \frac{4P_{69} S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \right\} \\
+ 3^{(0)}_{gg} \left[ \frac{128(S_{-4} - S_{-3}S_1 + S_{-3,1} + 2S_{-2,2})}{3N(N+1)(N+2)} + \frac{4(5N^2 + 5N - 22)S_1^2 S_2}{3N(N+1)(N+2)} + \cdots \right] + \cdots \right\] \\
+ C_{A} C_F T_F \left[ \frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{(2i)}{4^i (i+1)^2} \left( \sum_{j=1}^{i} \frac{4^j S_1(j-1)}{(2j)!} j^2 \right) - 7\zeta_3 \right] + \frac{32P_{2} S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} + \cdots \right\] \\
+ C_{A}^2 T_F \left[ -\frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{(2i)}{4^i (i+1)^2} \left( \sum_{j=1}^{i} \frac{4^j S_1(j-1)}{(2j)!} j^2 \right) - 7\zeta_3 \right] + \frac{256P_{5} S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} + \cdots \right\] \\
+ C_{A}^2 T_F \left[ \frac{32P_{30} S_{-2,1,1} + 16P_{35} S_{-3,1} + 16P_{44} S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16P_{52} S_{-2}}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{36} S_{2}^2}{9(N-1)N^2(N+1)^2} + \cdots \right] \\
+ C_F T_F^2 \left[ -\frac{16P_{48}}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N+1)} \left( \sum_{i=1}^{N} \frac{4^i S_1(i-1)}{(2i)!} i^2 \right) - 7\zeta_3 \right] - \frac{32P_{86} S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N+1)} \right\] \\
+ \frac{16P_{45} S_{1}^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45} S_{2}}{9(N-1)N^3(N+1)^3(N+2)} + \cdots \right\} + \cdots \right\} \\
(1) \right\} 

Also, with this calculation we were able to re-derive the three loop anomalous dimension \( \gamma_{gg}^{(3)} \) for the terms \( T_F \), and obtained agreement with the literature.
Moments for graphs with two massive lines \((m_1 \neq m_2)\)

\[
a^{(3)}_{Q_9}(N = 6) = \frac{1}{2} \left\{ T^2_{\mu} C_A \left( \frac{69882273800453}{3675690900000} - \frac{395296}{19845} \zeta_3 + \frac{1316809}{39690} \zeta_2 + \frac{832369820129}{14586075000} x + \frac{1511074426112}{624023544375} x^2 - \frac{84840004938801319}{690973782403905000} x^3 \right) \\
+ \ln \left( \frac{m_2^2}{\mu^2} \right) \left[ \frac{11771644229}{194481000} + \frac{78496}{2205} \zeta_2 - \frac{1406143531}{69457500} x - \frac{105157957}{180093375} x^2 + \frac{2287164970759}{7669816654500} x^3 \right] \\
+ \ln^2 \left( \frac{m_2^2}{\mu^2} \right) \left[ \frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln^3 \left( \frac{m_2^2}{\mu^2} \right) \left[ \frac{324148}{19845} + \ln^2 \left( \frac{m_2^2}{\mu^2} \right) \ln \left( \frac{m_2^2}{\mu^2} \right) \frac{156992}{6615} \right] \\
+ \ln \left( \frac{m_2^2}{\mu^2} \right) \ln \left( \frac{m_2^2}{\mu^2} \right) \ln \left( \frac{m_2^2}{\mu^2} \right) \left[ \frac{128234}{3969} - \frac{112669}{330750} x + \frac{98746}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln \left( \frac{m_2^2}{\mu^2} \right) \ln \left( \frac{m_2^2}{\mu^2} \right) \frac{68332}{6615} \\
+ \ln \left( \frac{m_2^2}{\mu^2} \right) \ln \left( \frac{m_2^2}{\mu^2} \right) \ln \left( \frac{m_2^2}{\mu^2} \right) \left[ \frac{1980566069882672}{26/30} + \frac{19845}{2205} \zeta_2 - \frac{19845}{2205} \zeta_2 + \frac{19845}{2205} \zeta_2 \right] + 64855635472 + 105157957 - 39690 x \\
\right\}

\to q2e/exp \ [\text{Harlander, Seidensticker, Steinhauser 1999}] \ x = \frac{m_1^2}{m_2^2}
Moments for graphs with two massive lines \((m_1 \neq m_2)\)

Analytic general N results are available for \(A_{qq,Q}^{NS}, A_{Qq}^{PS}\) and the scalar integrals of \(A_{gg,Q}\).
Conclusions

▶ 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, WC.
2010: Wilson Coefficients $L_{q}^{(3),PS}(N)$, $L_{g}^{(3),S}(N)$.

▶ 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in $\varepsilon$ can be systematically calculated for general $N$.

▶ Here new functions occur (including a larger number of root-letters in iterated integrals).

▶ 2014 $L_{q}^{NS,(3)}$, $A_{gq,Q}^{S,(3)}$, $A_{qq,Q}^{NS,TR(3)}$, $H_{2,q}^{PS(3)}$ and $A_{Qq}^{PS(3)}$ were completed.

▶ A method for the calculation of graphs with two massive lines of equal masses and operator insertions has been developed and applied $A_{gg,Q}^{(3)}$.

▶ The method can be generalized to the case of unequal masses. Here the moments for $N = 2, 4, 6$ for all graphs with two quark lines of unequal masses are now known [→ extended renormalization]; for some OMEs the complete 2-mass structure has been computed.

New VFNS relations!

▶ The $O(\alpha_s^2)$ charged current Wilson coefficients have been completed.
Conclusions

▶ All corresponding 3-loop anomalous dimensions were computed, those for transversity for the first time ab initio; those for the PS- and the qg-case independently for the first time.

▶ In all NS-cases [NC and CC] we also computed all power corrections at $O(a_s^2)$ and the associated sum rules in the inclusive case improving an earlier result by JB & W. van Neerven.

▶ All master integrals based on iterative integrals over whatsoever alphabet for $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ have been computed and $A_{gg,Q}^{(3)}$ is known for any even integer moment $N \geq 2$. Here all the topologies, including the ladder- and V-topologies have been solved.

▶ We have all the principal means to reconstruct $A_{Qg}^{(3)}$ systematically at very high accuracy. The full analytic solution will request more mathematical efforts.

▶ Different new computer-algebra and mathematical technologies were developed. These efforts will continue. The technologies are certainly useful for various present and upcoming calculations for the LHC and ILC.
Publications: Physics

J. Ablinger et al., DESY-17-062, arXiv:1705.01508
J. Ablinger et al. arXiv:1705.07030

Publications: Mathematics

J. Ablinger, JB, 1304.7071 [Contr. to a Book: Springer, Wien]
A. Ablinger et al., DESY 16-147.