3-Loop Corrections to Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering

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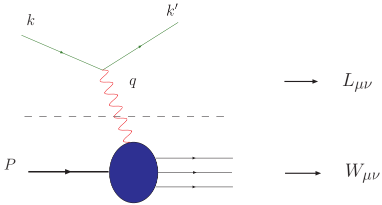
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s \rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\ .$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions.

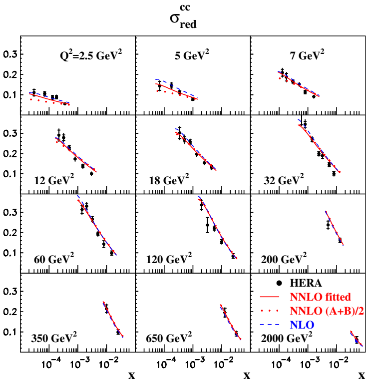
$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses

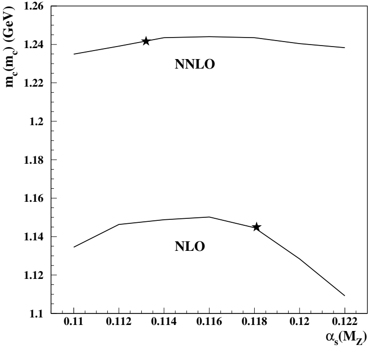
	$\alpha_s(M_Z^2)$	
BBG	$0.1134\ ^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1162 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	
Thorne	0.1136	[DIS+DY+HT*] (2014)
ABM11$_J$	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
ABM16	0.1147 ± 0.0008	+ new HERA, + $t\bar{t}$
CTEQ	$0.1159..0.1162$	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131\ ^{+\ 0.0028}_{-\ 0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
BBG	$0.1141\ ^{+0.0020}_{-0.0022}$	valence analysis, $\mathrm{N^3LO}$

$$\Delta_{\rm TH}\alpha_s = \alpha_s(\mathrm{N^3LO}) - \alpha_s(\mathrm{NNLO}) + \Delta_{\rm HQ} = +0.0009 \pm 0.0006_{\rm HQ}$$

NNLO accuracy is needed to analyze the world data. $\Longrightarrow$ NNLO HQ corrections needed.

Deep–Inelastic Scattering (DIS):





NNLO: S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172 [1212.2355]; & arxiv:1701.05838

$$m_c(m_c) = 1.252 \pm 0.018(exp) \begin{array}{c}+0.03\\-0.02\end{array} (scale) \begin{array}{c}+0.00\\-0.07\end{array} (thy)\mathrm{GeV},$$

$$m_b(m_b) = 3.84 \pm 0.12\mathrm{GeV}$$

$$m_t(m_t) = 160.9 \pm 1.1\mathrm{GeV}$$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227]].

PS corrections are exact.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x, \mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \; .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\;.$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\;.$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers, Wolfram 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments for F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009] mapping large expressions to [MATAD, Steinhauser 2000]
- Contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]
- Two masses $m_1 \neq m_2 \rightarrow$ Moments $N = 2,4,6$ [JB, Wißbrock 2011]

At 3-loop order for general values of N:

- All OMEs: terms $O(n_f T_F^2 C_{A/F})$ to F_2 [Ablinger et al. 2011, 2012]
- First contributions to $O(T_F^2 C_{A/F})$ $A_{gg,Q}$ [Ablinger et al. 2014]

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
\text{2014} \quad \boxed{L_{q,(2,L)}^{\mathsf{NS}}(N_F+1)} &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
\text{2010} \quad \boxed{L_{q,(2,L)}^{\mathsf{PS}}(N_F+1)} &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F \\
\text{2010} \quad \boxed{L_{g,(2,L)}^{\mathsf{S}}(N_F+1)} &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3\Big[\ A_{gg,Q}^{(3)}(N_F+1)\ \delta_2 \\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+ \\
&+\ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
\text{2014} \quad \boxed{H_{q,(2,L)}^{\mathsf{PS}}(N_F+1)} &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2 + \ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 \\
&+\ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1) + A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2 + \ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2 \\
&+\ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big] + \ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2 + \ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+ \\
&+\ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\} + \ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + \ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1
\end{aligned}
$$

All logarithmic corrections are known.

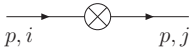
[Ablinger et al. 2010, Ablinger et al., 2014a, Ablinger et al., 2014b]

Variable Flavor Number Scheme

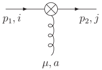
$$
\begin{aligned}
f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2) &= A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\bar{k}}(n_f,\mu^2)\right] \\
&\quad + \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\
G(n_f+1,\mu^2) &= A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2)\right] \\
&= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) + n_f\,\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \\
&\quad \otimes\Sigma(n_f,\mu^2) \\
&\quad + \left[n_f\,\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \otimes G(n_f,\mu^2)
\end{aligned}
$$

All master integrals for $A_{gg}^{(3)}$ have been completed (June 2015).

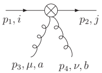
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



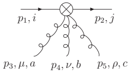
$$\delta^{ij}\Delta\!\!\!/\,\gamma_\pm(\Delta\cdot p)^{N-1}\,, \quad N\geq 1$$



$$g t^a_{ji}\Delta^\mu\Delta\!\!\!/\,\gamma_\pm\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\,, \quad N\geq 2$$



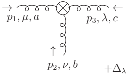
$$g^2\Delta^\mu\Delta^\nu\Delta\!\!\!/\,\gamma_\pm\sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^b t^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\right]\,,$$
$$N\geq 3$$



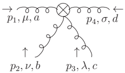
$$g^3\Delta_\mu\Delta_\nu\Delta_\rho\Delta\!\!\!/\,\gamma_\pm\sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-m-2}$$
$$\Big[\quad(t^a t^b t^c)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^a t^c t^b)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^b t^a t^c)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^b t^c t^a)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}$$
$$+(t^c t^a t^b)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^c t^b t^a)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\,,$$
$$N\geq 4$$

$$\gamma_+=1\,,\quad \gamma_-=\gamma_5\,.$$

$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}$$
$$\Big[g_{\mu\nu}(\Delta\cdot p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta\cdot p+p^2\Delta_\mu\Delta_\nu\Big]\,,\quad N\geq 2$$



$$-ig\frac{1+(-1)^N}{2}f^{abc}\Bigg($$
$$\Big[(\Delta_\nu g_{\lambda\mu}-\Delta_\lambda g_{\mu\nu})\Delta\cdot p_1+\Delta_\mu(p_{1,\nu}\Delta_\lambda-p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2}$$
$$+\Delta_\lambda\Big[\Delta\cdot p_1 p_{2,\mu}\Delta_\nu+\Delta\cdot p_2 p_{1,\nu}\Delta_\mu-\Delta\cdot p_1\Delta\cdot p_2 g_{\mu\nu}-p_1\cdot p_2\Delta_\mu\Delta_\nu\Big]$$
$$\times\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j}$$
$$+\left\{\begin{smallmatrix}p_1\to p_2\to p_3\to p_1\\ \mu\to\nu\to\lambda\to\mu\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\to p_3\to p_2\to p_1\\ \mu\to\lambda\to\nu\to\mu\end{smallmatrix}\right\}\Bigg)\,,\quad N\geq 2$$



$$g^2\frac{1+(-1)^N}{2}\Bigg(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)$$
$$+f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4)+f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Bigg)\,,$$
$$O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)=\Delta_\nu\Delta_\lambda\Bigg\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2}$$
$$+[p_{4,\mu}\Delta_\sigma-\Delta\cdot p_4 g_{\mu\sigma}]\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i}$$
$$-[p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 g_{\mu\sigma}]\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i}$$
$$+[\Delta\cdot p_1\Delta\cdot p_4 g_{\mu\sigma}+p_1\cdot p_4\Delta_\mu\Delta_\sigma-\Delta\cdot p_4 p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 p_{4,\mu}\Delta_\sigma]$$
$$\times\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Bigg\}$$
$$-\left\{\begin{smallmatrix}p_1\leftrightarrow p_2\\ \mu\leftrightarrow\nu\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}p_3\leftrightarrow p_4\\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4\\ \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}\,,\quad N\geq 2$$

The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	125	642	1233

A FORM [Vermaseren 2000] program was written in order to perform the γ-matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$\begin{aligned} A_{qq,Q}^{(3),\mathrm{NS}} &\rightarrow \text{7426 scalar integrals.} \\ A_{gq,Q}^{(3)} &\rightarrow \text{12529 scalar integrals.} \\ A_{Qq}^{(3),\mathrm{PS}} &\rightarrow \text{5470 scalar integrals.} \end{aligned}$$

$\Longrightarrow$ Need to use integration by parts identities.
$\Longrightarrow$ The reduction for all OMEs has been completed.

$\Longrightarrow$ Use special computers: 12 units with overall 3.2 TB RAM, 97 TB fast disc, hundreds of `mathematica` lic. ; IBP: several TB of final relations.

Integration by parts

We use Reduze [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a C++ program based on Laporta's algorithm.

$$(\Delta \cdot k)^N \to \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

$\Longrightarrow$ additional propagator.

Number of master integrals:

$$\begin{aligned} A_{qq,Q}^{(3),\mathrm{NS}} &\quad\to\quad 35 \ \text{ master integrals } \surd. \\ A_{gq,Q}^{(3)} &\quad\to\quad 41 \ \text{ master integrals } \surd. \\ A_{Qq}^{(3),\mathrm{PS}} &\quad\to\quad 66 \ \text{ master integrals } \surd. \\ A_{gg,Q}^{(3)} &\quad\to\quad 205 \ \text{ master integrals } \surd. \\ A_{Qg}^{(3)} &\quad\to\quad 340 \ \text{ master integrals. (224 done by June 2015.)} \end{aligned}$$

116 master integrals to be done $\Longrightarrow$ CIS-type

24 integral families are required and implemented in Reduze.

Calculation of the master integrals

For the calculation of the master integrals we use a wide variety of tools:

- Hypergeometric functions.
- Summation methods based on difference fields, implemented in the Mathematica program Sigma [C. Schneider, 2005–].
  - Reduction of the sums to a small number of key sums.
  - Expansion the summands in ε.
  - Simplification by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–].
  - Harmonic sums, polylogarithms and their various generalizations are algebraically reduced using the package HarmonicSums [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011,2013].
- Mellin-Barnes representations.
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
- Systems of Differential Equations.
- Almkvist-Zeilberger Theorem as Integration Method.

V-Topology

Emergence of new nested sums :

$$\sum_{i=1}^{N} \binom{2i}{i} (-2)^i \sum_{j=1}^{i} \frac{1}{j\binom{2j}{j}} S_{1,2}\left(\tfrac{1}{2}, -1; j\right)$$
$$= \int_0^1 dx \frac{x^N - 1}{x-1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{w_{17},-1,0}(x) - 2\mathrm{H}^*_{w_{18},-1,0}(x)\right]$$
$$+\frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^N - 1}{x+1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{12}(x) - 2\mathrm{H}^*_{13}(x)\right]$$
$$+c_3 \int_0^1 dx \frac{(-8x)^N - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1-x}} ,$$

$$w_{12} = \frac{1}{\sqrt{x(8-x)}}, \qquad w_{13} = \frac{1}{(2-x)\sqrt{x(8-x)}},$$
$$w_{17} = \frac{1}{\sqrt{x(8+x)}}, \qquad w_{18} = \frac{1}{(2+x)\sqrt{x(8+x)}} .$$

Non-iterative Iterative Integrals

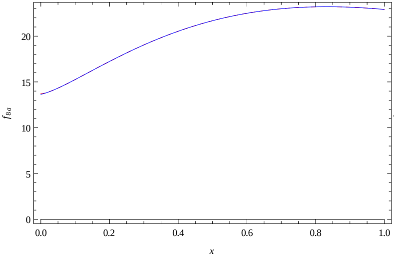
The live after iterative integrals and/or differential equations factorizing completely to 1st order:

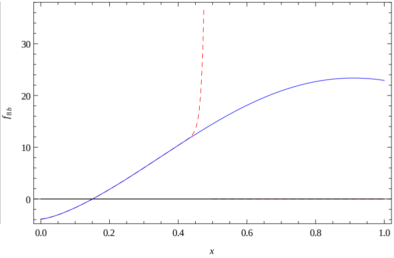
- Iterative integrals/nested sums in QFT have been very well understood during the last 19 years since 1998. [J. Vermaseren, E. Remiddi, JB];
- Now even general alphabets (including up to root valued letters).
- Even single-scale Feynman integrals lead beyond that [Sabry's kite, 1962]
- Currently worked out by the community. [Ablinger, Adams, Ananthanarayan, Behring, Bijnes, JB, Bloch, Bogner, Brown, DeFreitas, Gangl, Ghosh, Hebbar, Hoeij, Imamoglu, Laporta, Levin, Müller-Stach, Remiddi, Schneider, Schweitzer, Tancredi, Vidunnas, Weinzierl, Zagier, Zayadeh, ...]

$$
\begin{aligned}
\mathbb{H}_{a_1,\dots,a_{m-1};\{a_m;F_m(r(y_m))\},a_{m+1},\dots,a_q}(x) &= \int_0^x dy_1 f_{a_1}(y_1)\int_0^{y_1} dy_2 \dots \int_0^{y_{m-1}} dy_m f_{a_m}(y_m) F_m[r(y_m)] \\
&\quad \times H_{a_{m+1},\dots,a_q}(y_{m+1}), \\
F[r(y)] &= \int_0^1 dz g(z, r(y)), \quad r(y) \in \mathbb{Q}[y], \\
\psi_{1a}^{(0)}(x) &= \sqrt{2\sqrt{3}\pi}\,\frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4}\, {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; z\right] \\
z &= \frac{x^2(x^2-9)^2}{(x^2+3)^3}.
\end{aligned}
$$

CIS-series; In some cases: complete elliptic integrals at very special rational arguments. Highly precise numerical representations already available. On the structural side: Relations to elliptic polylogarithms [in the elliptic case].

Non-iterative Iterative Integrals





- Have to handle branch-points in case.
- Relations due to shuffle algebras.
- Further relations due to triangle group; Important relations between different solutions of the homogeneous equations.

Most of the master integrals infected by the new CIS solutions are iterated integrals over a few of the former ones.

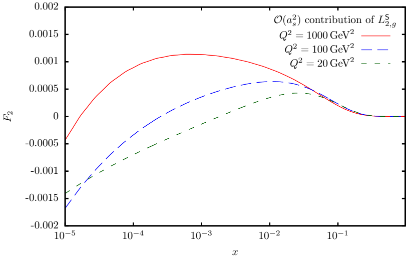
- We have identified the whole respective tree in case of our project.
- It would be interesting to view the corresponding situation in case of $\sigma(pp \to t\bar{t})(\hat{s})$.

Spill-Off:
New Mathematical Function Classes and Algebras

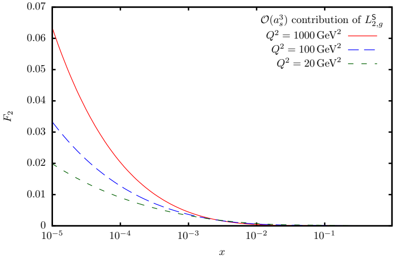
- 1998: Harmonic Sums [Vermaseren; JB]
- 1999: Harmonic Polylogarithms [Remiddi, Vermaseren]
- 2001: Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- 2004: Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2014: Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [Ablinger, JB, Raab, Radu, Schneider]
- 2016: Elliptic integrals with (involved) rational arguments appear in part of the functions of our project already as base cases. They stem from Heun equations. [since April 2016.] [Ablinger, Behring, JB, De Freitas, van Hoeij, Raab, Schneider, DESY16-147].

Particle Physics Generates NEW Mathematics.

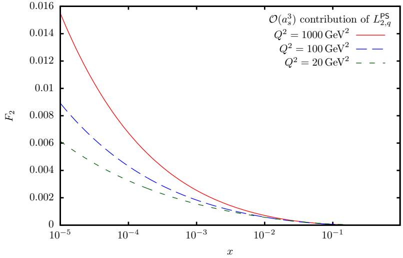
Numerical Results : $L_{g,2}^{\mathrm{S}}$ and $L_{q,2}^{\mathrm{PS}}$



$O(a_s^2)$ $L_{2,g}^{\mathrm{S}}$

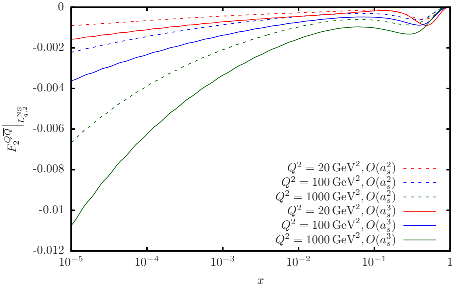


$O(a_s^3)$ $L_{2,g}^{\mathrm{S}}$

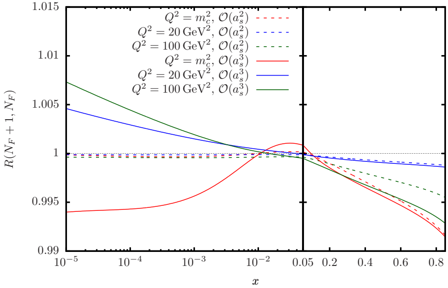


$L_{q,2}^{\mathrm{PS}}$

$L_{q,2}^{\mathrm{NS}}$

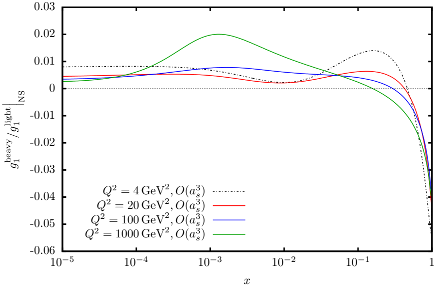


Contribution to $F_2(x, Q^2)$

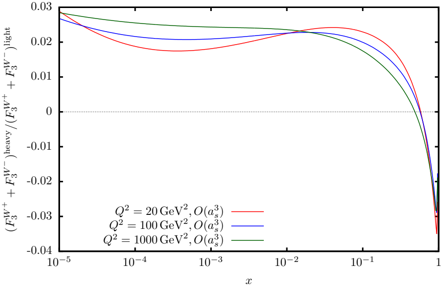


VFNS matching

NS corrections to $g_{1(2)}(x, Q^2)$ and $xF_3^{W^+ + W^-}$



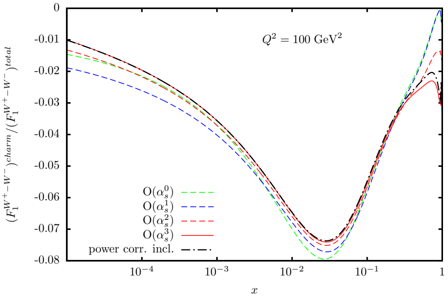
$g_1(x, Q^2)$



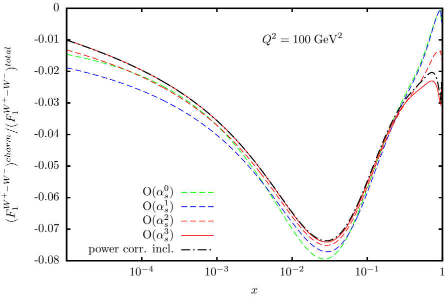
$xF_3^{W^+ + W^-}(x, Q^2)$

The corrections to $g_2(x, Q^2)$ are obtained using the Wandzura-Wilczek relation.

NS corrections to $F_1^{W^+-W^-}$ and $F_2^{W^+-W^-}$



$F_1^{W^+-W^-}(x, Q^2)$

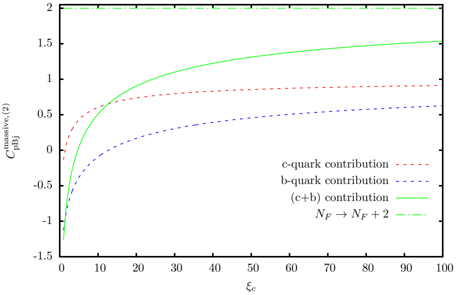


$F_2^{W^+-W^-}(x, Q^2)$

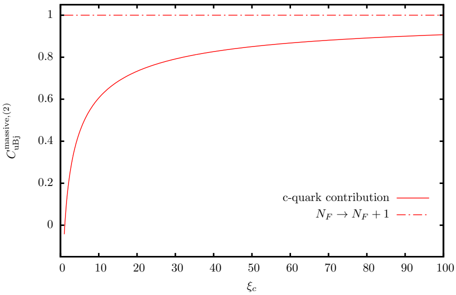
The massless corrections are due to Davies, Vogt, Moch, Vermaseren, LT-1084.

from: A. Behring et al., Phys.Rev. D94 (2016)114006

$O(\alpha_s^2)$ Complete NS corrections



pol BJ sum rule



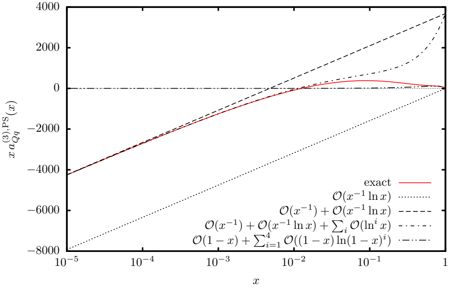
unp. BJ sum rule

Note the negative corrections at low Q^2!

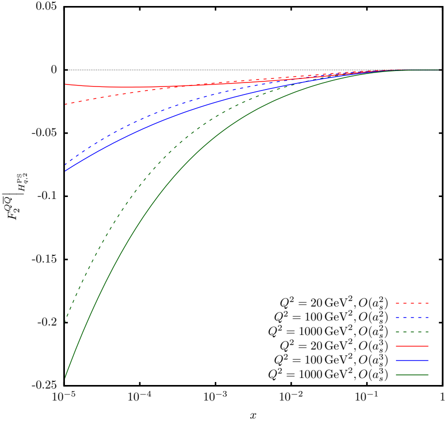
Already for charm it takes quite a while to become massless.

JB, G. Falcioni, A. De Freitas, Nucl. Phys. B910 (2016) 568.

$H_{q,2}^{\mathrm{PS}}$



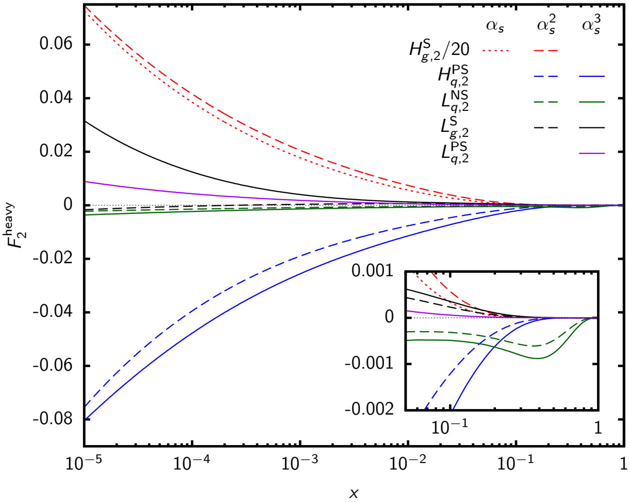
$a_{Qq}^{(3),\mathrm{PS}}$



Contribution to $F_2(x, Q^2)$

The leading small x approximation corresponding to CCH, 1991, departs from the physical result everywhere except for $x = 1$.

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100\mathrm{GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

3-Loop OME: $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = {} & \frac{1+(-1)^N}{2}\Bigg\{C_F^2 T_F\Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} - \frac{4P_{69}S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& +\tilde{\gamma}_{gq}^{(0)}\left(\frac{128\left(S_{-4}-S_{-3}S_1+S_{-3,1}+2S_{-2,2}\right)}{3N(N+1)(N+2)}+\frac{4(5N^2+5N-22)S_1^2S_2}{3N(N+1)(N+2)}+\cdots\right)+\cdots\Bigg] \\
& +C_A C_F T_F\Bigg[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2}+\frac{32P_2S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& -\frac{64P_{14}S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)}-\frac{16P_{23}S_{-4}}{3(N-1)N^2(N+1)^2(N+2)}+\frac{4P_{63}S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)}+\cdots\Bigg] \\
& +C_A^2 T_F\Bigg[-\frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2}+\frac{256P_5S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& +\frac{32P_{30}S_{-2,1,1}+16P_{35}S_{-3,1}+16P_{44}S_{-4}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{16P_{52}S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)}+\frac{8P_{36}S_2^2}{9(N-1)N^2(N+1)^2}+\cdots\Bigg] \\
& +C_F T_F^2\Bigg[-\frac{16P_{48}\binom{2N}{N}4^{-N}\left(\sum_{i=1}^{N}\frac{4^i S_1(i-1)}{\binom{2i}{i}i^2}-7\zeta_3\right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}-\frac{32P_{86}S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& +\frac{16P_{45}S_1^2}{27(N-1)N^3(N+1)^3(N+2)}-\frac{16P_{45}S_2}{9(N-1)N^3(N+1)^3(N+2)}+\cdots\Bigg]+\cdots\Bigg\}
\end{aligned}
\tag{1}
$$

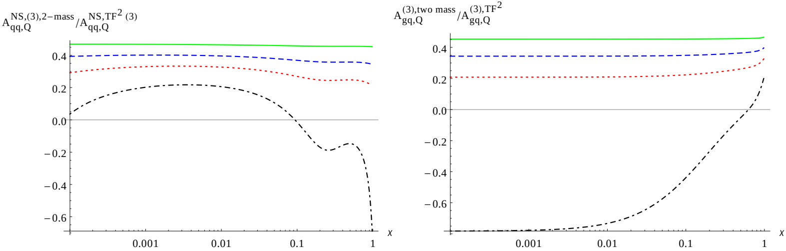
Also, with this calculation we were able to re-derive the three loop anomalous dimension $\gamma_{gg}^{(3)}$ for the terms $\propto T_F$, and obtained agreement with the literature.

Moments for graphs with two massive lines ($m_1 \neq m_2$)

$$
\begin{aligned}
a_{Qg}^{(3)}(N=6) \;=\;& \frac{1}{2}\Bigg\{T_F^2 C_A \Bigg\{\frac{69882273800453}{367569090000} - \frac{395296}{19845}\zeta_3 + \frac{1316809}{39690}\zeta_2 + \frac{832369820129}{14586075000}x + \frac{1511074426112}{624023544375}x^2 - \frac{84840004938801319}{690973782403905000}x^3 \\
&+ \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{11771644229}{194481000} + \frac{78496}{2205}\zeta_2 - \frac{1406143531}{69457500}x - \frac{105157957}{180093375}x^2 + \frac{2287164970759}{7669816654500}x^3\right] \\
&+ \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{324148}{19845} + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{156992}{6615} \\
&+ \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{128234}{3969} - \frac{112669}{330750}x + \frac{98746}{51975}x^2 + \frac{31340489}{17027010}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{68332}{6615} \\
&+ \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{83755534727}{583443000} + \frac{78496}{2205}\zeta_2 + \frac{1406143531}{69457500}x + \frac{105157957}{180093375}x^2 - \frac{2287164970759}{7669816654500}x^3\right] \\
&+ \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{412808}{19845}\Bigg\} \\
&+ T_F^2 C_F \Bigg\{-\frac{3161811182177}{71471767500} + \frac{447392}{19845}\zeta_3 + \frac{9568018}{4862025}\zeta_2 - \frac{64855635472}{2552563125}x + \frac{1048702178522}{97070329125}x^2 + \frac{1980566069882672}{2467763508585375}x^3 \\
&+ \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{1786067629}{204205050} - \frac{111848}{15435}\zeta_2 - \frac{128543024}{24310125}x - \frac{22957168}{3361743}x^2 - \frac{2511536080}{2191376187}x^3\right] \\
&+ \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{111848}{19845} - \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{223696}{46305} \\
&+ \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{22238456}{4862025} - \frac{1504864}{231525}x - \frac{355888}{40425}x^2 - \frac{255717856}{42567525}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{223696}{46305} \\
&+ \ln\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{24797875607}{1021025250} - \frac{111848}{15435}\zeta_2 + \frac{128543024}{24310125}x + \frac{22957168}{3361743}x^2 + \frac{2511536080}{2191376187}x^3\right] \\
&+ \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{1230328}{138915}\Bigg\}\Bigg\} + O\left(x^4\ln^3(x)\right)
\end{aligned}
$$

$\rightarrow$ `q2e/exp` [Harlander, Seidensticker, Steinhauser 1999] $x = m_1^2/m_2^2$

Moments for graphs with two massive lines ($m_1 \neq m_2$)



dash-dotted: $Q^2 = 30\text{GeV}^2$, dotted: $Q^2 = 50\text{GeV}^2$, dashed: $Q^2 = 100\text{GeV}$, full: $Q^2 = 1000\text{GeV}^2$

Analytic general N results are available for $A_{qq,Q}^{\rm NS}$, $A_{Qq}^{\rm PS}$ and the scalar integrals of $A_{gg,Q}$.

Conclusions

- 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, WC.
  2010: Wilson Coefficients $L_q^{(3),\mathrm{PS}}(N)$, $L_g^{(3),\mathrm{S}}(N)$.
- 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated for general N.
- Here new functions occur (including a larger number of root-letters in iterated integrals).
- 2014 $L_q^{\mathrm{NS},(3)}$, $A_{gq,Q}^{\mathrm{S},(3)}$, $A_{qq,Q}^{\mathrm{NS,TR}(3)}$, $H_{2,q}^{\mathrm{PS}(3)}$ and $A_{Qq}^{\mathrm{PS}(3)}$ were completed.
- A method for the calculation of graphs with two massive lines of equal masses and operator insertions has been developed and applied $A_{gg,Q}^{(3)}$.
- The method can be generalized to the case of unequal masses. Here the moments for $N = 2, 4, 6$ for all graphs with two quark lines of unequal masses are now known [$\longrightarrow$ extended renormalization]; for some OMEs the complete 2-mass structure has been computed. New VFNS relations !
- The $O(\alpha_s^2)$ charged current Wilson coefficients have been completed.

Conclusions

- All corresponding 3-loop anomalous dimensions were computed, those for transversity for the first time ab initio; those for the PS- and the qg-case independently for the first time.
- In all NS-cases [NC and CC] we also computed all power corrections at $O(a_s^2)$ and the associated sum rules in the inclusive case improving an earlier result by JB & W. van Neerven.
- All master integrals based on iterative integrals over whatsoever alphabet for $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ have been computed and $A_{gg,Q}^{(3)}$ is known for any even integer moment $N \geq 2$. Here all the topologies, including the ladder- and V-topologies have been solved.
- We have all the principal means to reconstruct $A_{Qg}^{(3)}$ systematically at very high accuracy. The full analytic solution will request more mathematical efforts.
- Different new computer-algebra and mathematical technologies were developed. These efforts will continue. The technologies are certainly useful for various present and upcoming calculations for the LHC and ILC.

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