Analytic resummation for TMD observables

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Outline

1. Factorization in SCET for Gauge boson transverse spectra
2. Resummation schemes
3. Scale Choice in momentum space
4. Analytical expression
5. Numerical results
6. Summary
Factorization in SCET

- $P+P \rightarrow H+X, \ P+P \rightarrow l^+ + l^- + X.$

Motivation

- Resum large logs of $qT/Q$ by setting renormalization scales in momentum space
- Obtain, for the first time, an analytic expression for resummed cross section.

\[ p \sim Q(1, 1, \lambda) \]

\[ p_c \sim Q(1, \lambda^2, \lambda), \]
\[ p_{\bar{c}} \sim Q(\lambda^2, 1, \lambda), \]
\[ p_s \sim Q(\lambda, \lambda, \lambda), \]
\[ \lambda = qT/Q \]
Factorization in SCET

**Figure:** IR modes have the same virtuality.

- Need a regulator $\nu$ that breaks boost invariance to factorize the Soft from the collinear sector

\[
p_c \sim Q (1, \lambda^2, \lambda), \quad p_{\bar{c}} \sim Q (\lambda^2, 1, \lambda), \quad p_s \sim Q (\lambda, \lambda, \lambda), \quad \lambda = q_T / Q
\]
## Transverse momentum cross section

\[
\frac{d\sigma}{dq_T^2 dy} \propto H\left(\frac{\mu}{Q}\right) \times \int d^2 \vec{q}_S d^2 \vec{q}_1 d^2 \vec{q}_2 S(\vec{q}_T, \mu, \nu) \times 
\]

\[
f^\perp_1(x_1, q_T^1, \mu, \nu, Q) f^\perp_2(x_2, q_T^2, \mu, \nu, Q) \delta^2(\vec{q}_T - \vec{q}_S - \vec{q}_1 - \vec{q}_2)
\]

- The function \( f^\perp_i \) along with the soft function \( S \) forms the **TMDPDF**.
- \( \mu, \nu \): RG equations in two scales.
- \( \nu \frac{d}{d\nu} G_i(\vec{q}_T, \nu) = \gamma^i(\vec{q}_T) \otimes G_i(\vec{q}_T, \nu) \)
Factorization

**b space formulation**

\[
\frac{d\sigma}{dq_T^2 dy} \propto H\left(\frac{\mu}{Q}\right) \int bdbJ_0(bq_T)S(b, \mu, \nu)f_1^\perp(x_1, b, \mu, \nu, Q)f_2^\perp(x_2, b, \mu, \nu, Q)
\]

**RG equations in b space are simple**

\[
\mu \frac{d}{d\mu} F_i(\mu, \nu, b) = \gamma^i_\mu F_i(\mu, \nu, b), \quad F_i \in (H, S, f^\perp_i)
\]

\[
\nu \frac{d}{d\nu} G_i(\mu, \nu, b) = \gamma^i_\nu G_i(\mu, \nu, b), \quad G_i \in (S, f^\perp_i)
\]

\[
\sum_{F_i} \gamma^i_\mu = \sum_{G_i} \gamma^i_\nu = 0
\]
One loop fixed order

\[ H(\mu, Q) = 1 + \frac{\alpha_s(\mu)}{8\pi} \Gamma_0(-4 \ln^2 \frac{\mu}{Q}) \]

\[ S(b, \mu, \nu) = \frac{1}{2\pi} \left( 1 + \frac{\alpha_s(\mu)}{8\pi} \Gamma_0(4 \ln^2 \mu b_0 + 8 \ln \mu b_0 \ln \nu b_0) \right) \]

\[ f^{\mu\nu}(z, b, Q, \mu, \nu) = 1 + \frac{\alpha_s}{8\pi} \Gamma_0 \left( \frac{4 \ln \nu}{Q} \ln \mu b_0 f_g(z, \mu) \right) \]

\[ + \int \frac{dz'}{z'} \left\{ \sum_i \left( -2 p_{g_i}^*(z/z') \ln \mu b_0 \right) f_i(z', \mu) \right\} \]

\[ b_0 = be^{-\gamma_E}/2 \]
Resummation schemes

- **b space resummation**: Default choice of $\mu = \nu = 1/b_0$, 1007.2351 De Florian et.al., 1503:00005 V. Vaidya et. al.
- **Momentum space resummation**: Both $\mu, \nu$ in momentum space, distributional scale setting, 1611.08610 Tackmann et.al., 1604.02191 P. Monni et. al.
- **Hybrid**: $\mu$ in momentum space (1007.4005, 1109.6027 Becher et al.)

**Figure**: Choice of resummation path.
Can we choose a particular physical scales in momentum space for $\mu$ and $\nu$?

**Running the Hard function in $\mu$**

- Assume a power counting $\alpha_s \log(Q/\mu_L), \log(\mu_L b_0) \sim 1$

\[
\frac{d\sigma}{dq_t^2} \propto U_{H}^{LL}(H, \mu_L) (\alpha_s(\mu_L) \log(Q/q_T))
\]

- The leading large logarithm $\alpha_s(\mu_L) \log(Q/q_T)$ goes unresummed.
- Resummation of $\log(Q/q_T)$ happens entirely in the IR sector

\[
\gamma^{\nu}_S = \Gamma_0 \frac{\alpha_s}{\pi} \log(\mu b_0)
\]
Scale choice in momentum space

Attempt at NLL $\rightarrow$ running Soft function in $\nu$

\[
\frac{d\sigma}{dq_t^2} \propto U_H^{NLL}(H, \mu_L) \int d bb J_0(bq_t) U_S(\nu_H, \nu_L, \mu_L)
\]

\[
= \ U_H^{NLL}(H, \mu_L) \int d bb J_0(bq_t) (\mu_L^2 b^2) - \Gamma_0 \frac{\alpha(\mu_L)}{2\pi} \log(\nu_H^\nu_L)
\]

\[
= \ 2 U_H^{NLL}(H, \mu_L) e^{-2\omega_s \gamma_E} \frac{\Gamma[1 - \omega_s]}{\Gamma[\omega_s]} \frac{1}{\mu_L^2} \left( \frac{\mu_L^2}{q_T^2} \right)^{1 - \omega_s}
\]

\[
\omega_s = \Gamma_0 \frac{\alpha(\mu_L)}{2\pi} \log(\nu_H^\nu_L)
\]

- Still does not work, singular at $\omega_s \sim 1$
Scale choice in momentum space

- Divergence caused due to single log structure ($\log(\mu_L b)$) in the Soft exponent
- Contribution from highly energetic soft contributions.
- Need damping at low $b$ to stabilize $b$ space exponent.
**Scale choice in momentum space**

- Resum all logarithms of the form $\alpha_s \log^2(\mu_L b_0)$

**A choice for $\nu$ in b space $\rightarrow$ include sub-leading terms**

$$\nu_L = \mu_L^n b_0^{1-n}, \quad n = \frac{1}{2} \left( 1 - \alpha(\mu_L) \frac{\beta_0}{2\pi} \log\left(\frac{\nu_H}{\mu_L}\right) \right)$$

**Soft exponent at NLL $\rightarrow$ Quadratic in $\log(\mu_L b_0)$**

$$\log(U_{S^{NLL}}(\nu_H, \nu_L, \mu_L)) = -2\Gamma_0 \frac{\alpha(\mu_L)}{2\pi} \times$$

$$\left( \log\left(\frac{\nu_H}{\mu_L}\right) \log(\mu_L b_0) + \frac{1}{2} \log^2(\mu_L b_0) + \alpha(\mu_L) \frac{\beta_0}{4\pi} \log^2(\mu_L b_0) \log\left(\frac{\nu_H}{\mu_L}\right) \right)$$
Scale choice in momentum space

A choice for $\mu_L$ in momentum space

- A choice that justifies the power counting $\log(\mu_L b_0) \sim 1$
- A choice that will minimize contributions from residual fixed order logs $\log^n(\mu_L b_0)$.
- Scale shifted away from $q_T$ due to the scale $Q$ in $b$ space exponent.

Figure: Percentage contribution of the fixed order logs as a function of the scale.
\( \mu_L \sim 1/b^* \), \( b^* \) is the value at which \( b \) space integrand peaks.

\[
\frac{d}{db} (bJ_0(bq_T)U_{soft}(b)) \bigg|_{(b=1/\mu_L)} = 0
\]
Analytical expression for cross section

Mellin-Barnes representation of Bessel function

- Polynomial integral representation for Bessel function is needed

\[ J_0(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{\Gamma[-t]}{\Gamma[1+t]} \left(\frac{1}{2z}\right)^{2t} \]

b space integral

\[ U_S = C_1 \exp[-A \log^2(Ub)] \]
\[ l_b = \int_0^\infty db \int_0^\infty b \int_0^\infty db \int_0^{i\infty} dt \frac{\Gamma[-t]}{\Gamma[1+t]} \left(\frac{bq_T}{2}\right)^{2t} \exp[-A \log^2(Ub)] \]

No Landau pole
Analytical expression for cross section

\[ I = \frac{2C_1}{iq_T^2} \frac{1}{\sqrt{4\pi A}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma[-t]}{\Gamma[1 + t]} \exp\left[\frac{1}{A}(t - t_0)^2\right] dt \]

\[ t_0 = -1 + A \log\left(\frac{2U}{q_T}\right) \to \text{saddle point} \]

- Path of steepest descent is parallel to the imaginary axis
- Suppression controlled by \( 1/A \sim \frac{4\pi}{\alpha_s} 1/\Gamma_{cusp}^{(0)} \)
- \( t = c + ix \)

\[ I = \frac{2C_1}{q_T^2} \frac{1}{\sqrt{4\pi A}} \int_{-\infty}^{\infty} dx \frac{\Gamma[-c - ix]}{\Gamma[1 + c + ix]} \exp\left[-\frac{1}{A}(x - i(c - t_0))^2\right] \]

- Saddle point weak in resummation region; very strong in Fixed order limit \( A \to 0 \).
What choice do we make for $c$? Obvious choice $c = t_0$? $c$ depends on $A$ and hence on the details of the process.

For percent level accuracy, we need info about $F(x) = \frac{\Gamma[-c-ix]}{\Gamma[1+c+ix]}$ out to $x_l \sim \sqrt{2A \log(10)}$

Worst case scenario $A \sim 0.5 \implies x_l \sim 1.5$

A Taylor series expansion around the saddle point is not enough.

Choose $c = -1$, the saddle point in the limit $A \to 0$ for all observables and use a more suitable basis for expanding $F(x)$
Analytical expression for cross section

Guidelines for choosing a basis for expansion

1. Fixed order cross section

\[ I_{\text{exact}}^{O(\alpha_s)} = -2\Gamma_{\text{cusp}}^{(0)} \frac{\alpha(\mu_L)}{4\pi} \frac{2}{q_T^2} \left( F'[0] \log \left( \frac{\mu_L e^{-\gamma_E}}{q_T} \right) + \frac{F''[0]}{4} \right) \]

2. To correctly reproduce the fixed order cross section up to \( \alpha_s^n \), we need the \( 2n^{th} \) derivative of the expansion to match the exact function \( F(x) \).

3. We need the expansion in a basis to be accurate up to \( x \sim 1.5 \).

4. The basis functions for the expansion should be chosen so as to yield a rapidly converging and analytical result.

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Analytical expression for cross section

An expansion for $F(x) = \Gamma[-1 - ix]/\Gamma[ix]$

A general basis $x^n e^{\alpha x^2 + \beta x}$ for expansion

$\hat{F}_R(x) = g_1(\text{Exp}[-g_2x^2] - \cos[g_3x])$

$\hat{F}_I(x) = f_1 \sin[f_2x] + f_3 \sinh(f_4x)$

**Figure:** Expansion for real and imaginary parts of $f(t)$, $c$ is chosen to be $-1$
Analytical expression for cross section

Expression for resummed Soft function

\[ I_b(A, L) = \frac{2C_1}{q_T^2} e^{-AL^2} \left[ -g_1 e^{-\frac{Ag_2^2}{4}} \cosh[ALg_3] + \frac{g_1 e^{\frac{A^2L^2g_2}{1+Ag_2}}}{\sqrt{1+Ag_2}} \right] \\
+ f_1 e^{-\frac{Af_2^2}{4}} \sinh[ALf_2] + f_3 e^{\frac{Af_4^2}{4}} \sin[ALf_4] \]

Parameters at NLL

\[ A = -2\Gamma_{\text{cusp}}^{(0)} \frac{\alpha(\mu_L)}{4\pi} \left( 1 + \frac{\alpha(\mu_L)\beta_0}{2\pi} \log\left( \frac{\nu_H}{\mu_L} \right) \right) \]

\[ C_1 = \text{Exp}[A \log^2(\eta)], \quad U = \mu_L \eta e^{-\gamma_E}/2 \]

\[ \eta = \text{Exp}\left[ \log\left( \frac{\nu_H}{\mu_L} \right) \left( 1 + \frac{\alpha(\mu_L)\beta_0}{2\pi} \log\left( \frac{\nu_H}{\mu_L} \right) \right) \right], \quad L = \log\left( \frac{2U}{q_T} \right) \]
**Fixed order terms**

- $I_b$ acts as a generating function for residual fixed order logs

\[
I_{even} = C_1 \int bdbJ_0(bq_T) \log^{2n}(Ub) \exp[-A \log^2(Ub)]
\]

\[
= (-1)^n \frac{d^n}{dA^n} I_b(A, L)
\]

\[
I_{odd} = C_1 \int bdbJ_0(bq_T) \log^{2n+1}(Ub) \exp[-A \log^2(Ub)]
\]

\[
= (-1)^n \frac{d^n}{dA^n} \frac{(-1)}{2A} \frac{d}{dL} I_b(A, L)
\]
Numerical results

- Easily extended to NNLL, b space exponent kept quadratic in $\log(\mu b)$

![Figure: Resummation in momentum space.](image)

- Excellent convergence for both the Higgs and Drell-Yan spectrum
- No arbitrary b space cut-off while estimating perturbative errors.
Comparison with b space resummation

**Figure:** comparison of nnll cross section in two schemes

- Difference of the order of sub-leading terms.
- More reliable perturbative error estimation in the absence of Landau pole.
Matching to fixed order

- Implement profiles in $\mu$ and $\nu$ to turn off resummation
  $$S = S_L^{(1-z(q_T))} Q^z(q_T) \quad S \in \mu, \nu$$

- Soft exponent scales as $(1 - z(q_T))$
  $$U_S = \exp[(1 - z)\gamma_S \log \left(\frac{Q}{\nu_L}\right)]$$

- This is equivalent to $A \to A(1-z)$ in $I_b(A,L)$

$$z(q_T) = \frac{1}{2} \left(1 + \tanh \left[r \left(\frac{q_T}{t} - 1\right)\right]\right)$$
Implementation momentum space resummation for transverse spectra of gauge bosons

Rapidity choice in impact parameter space

Virtuality choice in momentum space.

Analytical expression for cross section across the entire range of $q_T$ obtained for the first time.

Numerical accuracy controlled by the accuracy of the expansion for process independent function $\frac{\Gamma[-t]}{\Gamma[1+t]}$.

Outlook

Promising approach for other observables with similar factorization structure.

Non-perturbative effects need to be included for low Q as well as the low $q_T$ regime.
(A more accurate) Analytical expression for cross section

**An expansion for** $F(x) = \frac{\Gamma[-1 - ix]}{\Gamma[ix]}$

A general basis $x^n e^{\alpha x^2 + \beta x}$ for expansion

\[
\hat{F}_R(x) = g_1(\text{Exp}[-g_2 x^2] - \cos[g_3 x]) + g_4 x^2 \text{Exp}[-g_5 x^2]
\]

\[
\hat{F}_I(x) = f_1 \sin[f_2 x^2] + f_3 \sinh(f_4 x)
\]

**Figure:** (Expansion for real and imaginary parts of $f(t)$, $c$ is chosen to be -1)