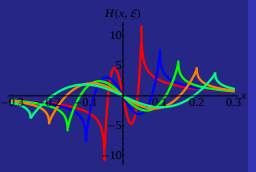

Singularities of Generalized Parton Distributions

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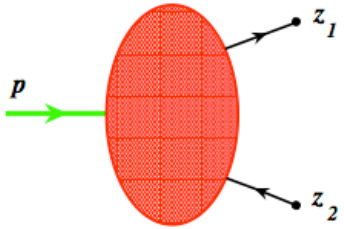
Hadrons in Terms of Quarks and Gluons

How to relate hadronic states $|p, s\rangle$

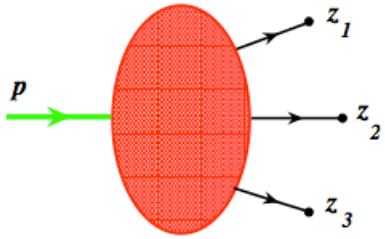
to quark and gluon fields $q(z_1)\ ,\ q(z_2)\ ,\ \ldots$?

Standard way: use matrix elements

$$\langle\, 0 \,|\, \bar{q}_\alpha(z_1)\, q_\beta(z_2) \,|\, M(p), s \,\rangle \ ,\ \langle\, 0 \,|\, q_\alpha(z_1)\, q_\beta(z_2)\, q_\gamma(z_3) \,|\, B(p), s \,\rangle$$



Meson–quark matrix element



Baryon–quark matrix element

- Can be interpreted as hadronic wave functions

Light-cone formalism

- Describe hadron by Fock components in infinite-momentum frame

For nucleon

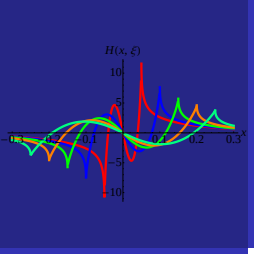
$$|P\rangle = |q(x_1P, k_{1\perp})\, q(x_2P, k_{2\perp})\, q(x_3P, k_{3\perp})\rangle + |qqqG\rangle + |qqq\bar{q}q\rangle + |qqqGG\rangle + \ldots$$

- x_i : momentum fractions

$$\sum_i x_i = 1$$

- $k_{i\perp}$: transverse momenta

$$\sum_i k_{i\perp} = 0$$

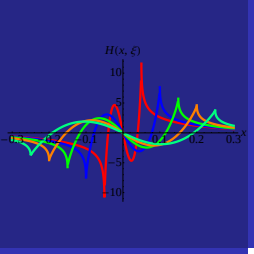


Problems of LC Formalism

- In principle: Solving bound-state equation

$$H|P\rangle = E|P\rangle$$

  one gets $|P\rangle$ which gives complete information about hadron structure
- In practice: Equation (involving infinite number of Fock components) has not been solved and is unlikely to be solved in near future
- Experimentally: LC wave functions are not directly accessible
- Way out: Description of hadron structure in terms of phenomenological functions



Phenomenological Functions

"Old" functions:

- Form Factors
- Usual Parton Densities
- Distribution Amplitudes

"New" functions:

Generalized Parton Distributions (GPDs)

GPDs = Hybrids of

Form Factors, Parton Densities and Distribution Amplitudes

"Old" functions

are limiting cases of "new" functions

Form Factors

Form factors are defined through matrix elements

of electromagnetic and weak currents between hadronic states

Nucleon EM form factors:

$$\langle\, p',\, s' \,|\, J^\mu(0) \,|\, p,\, s \,\rangle = \bar{u}(p', s') \left[\gamma^\mu F_1(t) + \frac{\Delta^\nu \sigma^{\mu\nu}}{2m_N} F_2(t) \right] u(p, s)$$

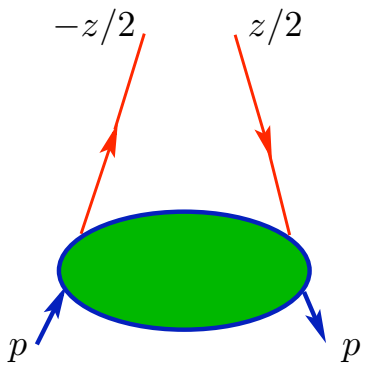
$(\Delta = p - p', t = \Delta^2)$

- Electromagnetic current
$$J^\mu(z) = \sum_{f\,\mathrm{flavor}} e_f \bar{\psi}_f(z) \gamma^\mu \psi_f(z)$$
- Helicity non-flip form factor
$$F_1(t) = \sum_f e_f F_{1f}(t)$$
- Helicity flip form factor
$$F_2(t) = \sum_f e_f F_{2f}(t)$$

Usual Parton Densities

Parton Densities are defined through forward matrix elements

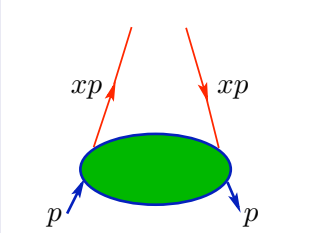
of quark/gluon fields separated by lightlike distances



Unpolarized quarks case:

$$\langle\, p \,|\, \bar{\psi}_a(-z/2)\gamma^\mu \psi_a(z/2) \,|\, p \,\rangle\big|_{z^2=0}$$
$$= 2p^\mu \int_0^1 \left[e^{-ix(pz)} f_a(x) - e^{ix(pz)} f_{\bar{a}}(x) \right] dx$$

Momentum space interpretation



$f_{a(\bar{a})}(x)$ is probability

to find a ($\bar{a}$) quark with momentum xp

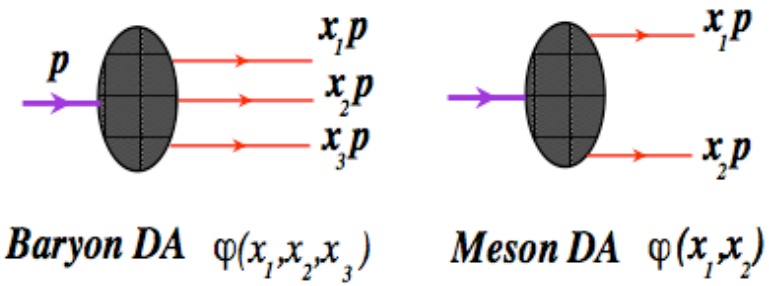
Local limit $z = 0$

$\Rightarrow$ sum rule

$\int_0^1 [f_a(x) - f_{\bar{a}}(x)]\, dx = N_a$

for valence quark numbers

Distribution Amplitudes



DAs may be interpreted as

- LC wave functions integrated over transverse momentum
- Matrix elements $\langle 0|\mathcal{O}|p\rangle$ of LC operators

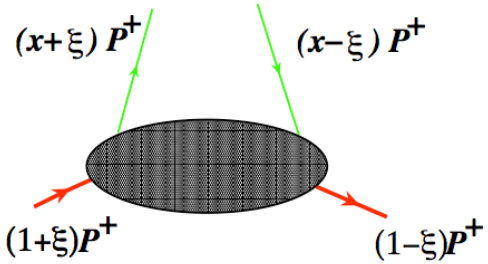
For pion (π^+):

$$\langle\, 0\,|\,\bar{\psi}_d(-z/2)\gamma_5\gamma^\mu\psi_u(z/2)\,|\,\pi^+(p)\,\rangle\big|_{z^2=0}$$
$$= ip^\mu f_\pi \int_{-1}^{1} e^{-i\alpha(pz)/2}\varphi_\pi(\alpha)\,d\alpha$$

with $\alpha = x_1 - x_2$ or $x_1 = (1+\alpha)/2,\ x_2 = (1-\alpha)/2$

Generalized Parton Distributions

Momentum fractions taken wrt average momentum $P = (p + p')/2$



4 functions of x, ξ, t:

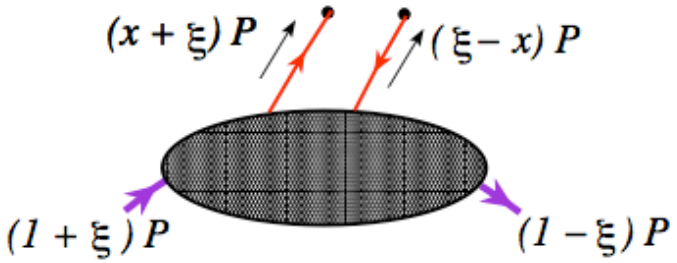
$H, E, \widetilde{H}, \widetilde{E}$

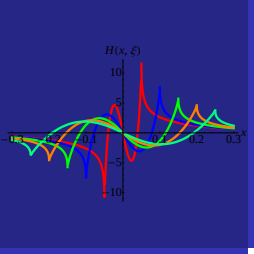
wrt hadron/parton helicity flip

$+/+, -/+, +/-, -/-$

- Skeweness $\xi \equiv \Delta^+/2P^+$ is $\xi = x_{Bj}/(2 - x_{Bj})$
- 3 regions:

$\xi < x < 1$	$\sim$	quark distribution
$-1 < x < -\xi$	$\sim$	antiquark distribution
$-\xi < x < \xi$	$\sim$	distribution amplitude for $N \to \bar{q}qN'$





Definition of GPDs

- In scalar case, define GPD by

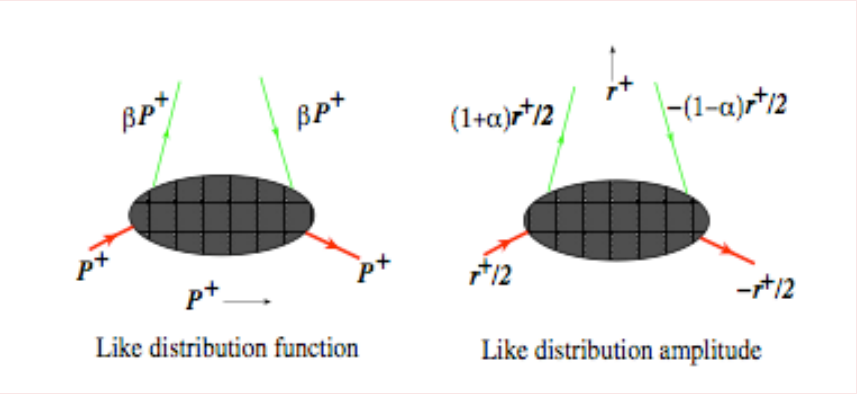
$$\langle P + r/2|\psi(-z/2)\psi(z/2)|P - r/2\rangle|_{z^2=0}$$
$$= \int_{-1}^{1} e^{-ix(Pz)} H(x, \xi; t)\, dx$$

- Invariant momentum transfer $t = r^2$
- Skeweness $\xi = r^+/2P^+$
- $r = 0 \Rightarrow$ usual (forward) distribution

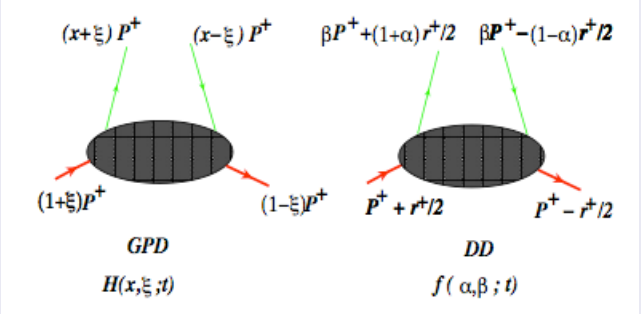
$$f(x) = H(x, \xi = 0; t = 0)$$

Double Distributions

"Superposition" of P^+ and r^+ momentum fluxes

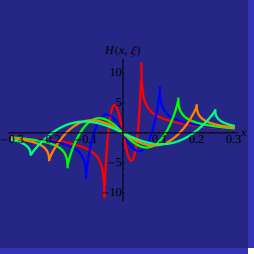


Connection with GPDs



Basic relation between fractions

$$x = \beta + \xi\alpha$$



Parton distributions and matrix elements

- For a scalar target, one may write

$$\langle P + r/2|\psi(0)\{\overleftrightarrow{\partial}_{\mu_1} \dots \overleftrightarrow{\partial}_{\mu_n}\}\psi(0)|P - r/2\rangle$$
$$= A_{n0}\{P_{\mu_1} \dots P_{\mu_n}\} + A_{nn}\{r_{\mu_1} \dots r_{\mu_n}\}$$
$$+ \sum_{l=1}^{n-1} A_{nl}\{P_{\mu_1} \dots P_{\mu_{n-l}} r_{\mu_{n-l+1}} \dots r_{\mu_n}\}$$

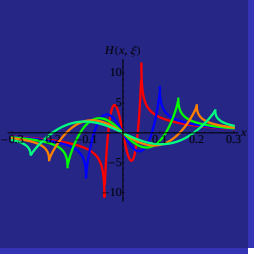
- $r = 0 \Rightarrow$ usual (forward) distribution $f(\beta)$ related to $l = 0$ moments

$$\int_{-1}^{1} f(\beta)\beta^n \, d\beta = A_{n0} \tag{1}$$

- $P = 0 \Rightarrow D$-term $D(\alpha)$ related to $l = n$ moments

$$\int_{-1}^{1} D(\alpha)(\alpha/2)^n \, d\alpha = A_{nn} \tag{2}$$

- D comes with r_{μ_i} factors: it is invisible in DIS (then $r = 0$)



Definition of DDs

- Define Double Distribution (DD)

$$\frac{n!}{(n-l)!\,l!\,2^l}\int_\Omega F(\beta,\alpha)\beta^{n-l}\alpha^l\,d\beta\,d\alpha = A_{nl}$$

- Support region Ω is given by rhombus $|\alpha|+|\beta|\leq 1$
- "DD parameterization" of the matrix element

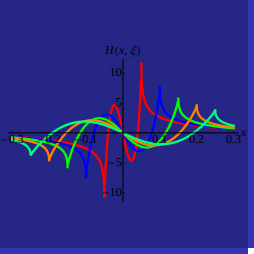
$$\langle P-r/2|\psi(-z/2)\psi(z/2)|P+r/2\rangle|_{z^2=0}$$
$$=\int_\Omega F(\beta,\alpha)\,e^{-i\beta(Pz)-i\alpha(rz)/2}\,d\beta\,d\alpha$$

- Usual (forward) distribution

$$f(\beta)=\int_{-1+|\beta|}^{1-|\beta|}F(\beta,\alpha)\,d\alpha$$

- D-term

$$D(\alpha)=\int_{-1+|\alpha|}^{1-|\alpha|}F(\beta,\alpha)\,d\beta$$



Isolating D-term

- Using $e^{-i\beta(Pz)} = [e^{-i\beta(Pz)} - 1] + 1$
- split DD-integral into "plus" part

$$\int_\Omega [F(\beta,\alpha)]_+ \, e^{-i\beta(Pz)-i\alpha(rz)/2} \, d\beta \, d\alpha$$

- and D-term part

$$\int_{-1}^{1} D(\alpha) \, e^{-i\alpha(rz)/2} \, d\alpha$$

- with

$$[F(\beta,\alpha)]_+ = F(\beta,\alpha) - \delta(\beta) \int_{-1+|\alpha|}^{1-|\alpha|} F(\gamma,\alpha) \, d\gamma$$

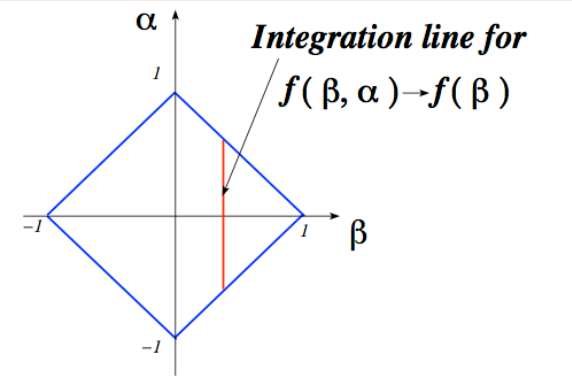
- "Plus" "+" D representation:

$$F(\beta,\alpha) = [F(\beta,\alpha)]_+ + \delta(\beta) D(\alpha)$$

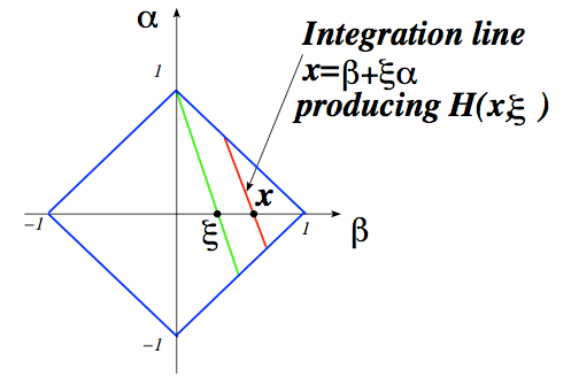
Getting GPDs from DDs

$$H(x,\xi) = \int_\Omega F(\beta,\alpha)\,\delta(x-\beta-\xi\alpha)\,d\beta\,d\alpha$$

DDs live on rhombus $|\alpha| + |\beta| \leq 1$



Converting DDs into GPDs



"Munich" symmetry:

$$f_a(\beta,\alpha;t) = f_a(\beta,-\alpha;t)$$

GPDs $H(x,\xi)$ are obtained from DDs $f(\beta,\alpha)$

by scanning DDs
at ξ-dependent angles

$\Rightarrow$ DD-tomography

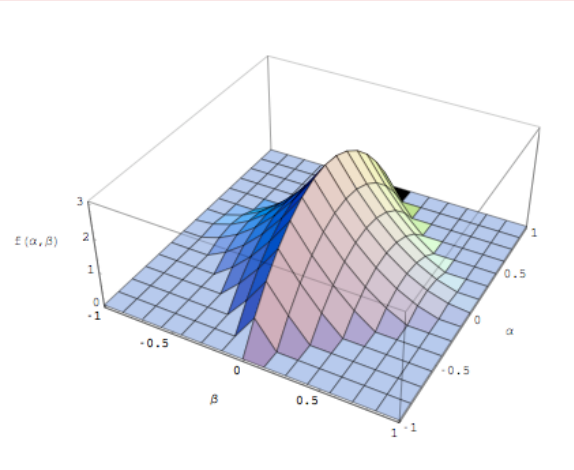
Illustration of DD→GPD conversion

Factorized model for DDs:

($\sim$ usual parton density in β-direction) $\otimes$
($\sim$ distribution amplitude in α-direction)

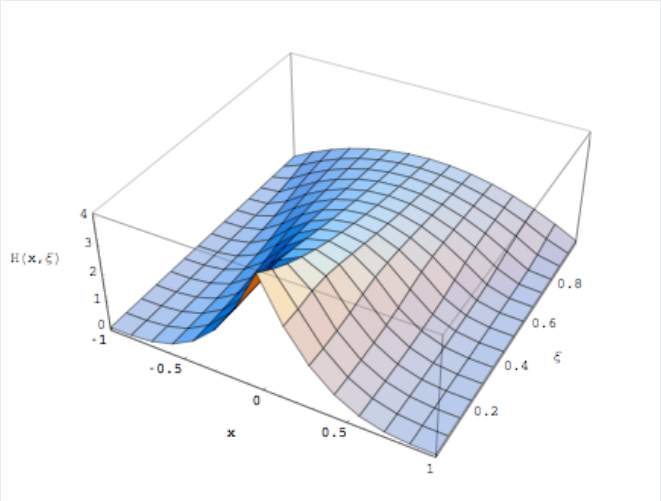
Toy model for double distribution

$$f(\beta,\alpha) = 3[(1-|\beta|)^2 - \alpha^2]\,\theta(|\alpha|+|\beta|\leq 1)$$



- Corresponds to toy "forward" distribution $f(\beta) = (1-|\beta|)^3$

GPD $H(x,\xi)$ resulting from toy DD



- For $\xi = 0$ reduces to usual parton density
- For $\xi = 1$ has shape like meson distribution amplitude

"DD plus D" Model for GPDs

- Factorized Ansatz for DDs:

$$F(\beta,\alpha) = f(\beta) h_a(\beta,\alpha)$$

Normalization

$$\int_{-1}^{1} d\alpha\, h(\beta,\alpha) = 1$$

Guarantees forward limit

$$\int_{-1}^{1} d\alpha\, f(\beta,\alpha) = f(\beta)$$

- DD modeling misses terms invisible in the forward limit:
  - Meson exchange contributions
  - D-term, which can be interpreted as σ exchange
- Inclusion of D-term induces contribution confined to $|x| < \xi$ region

$$H_D(x,\xi) = \frac{1}{|\xi|} D(x/\xi)$$

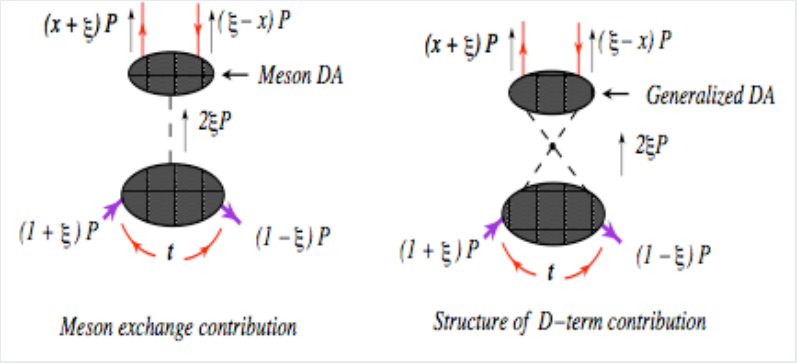
Model for GPDs based on DDs

- DD+D Ansatz: $F(\beta,\alpha) = f(\beta)h_a(\beta,\alpha) + \delta(\beta)D(\alpha)$
- General form of model profile

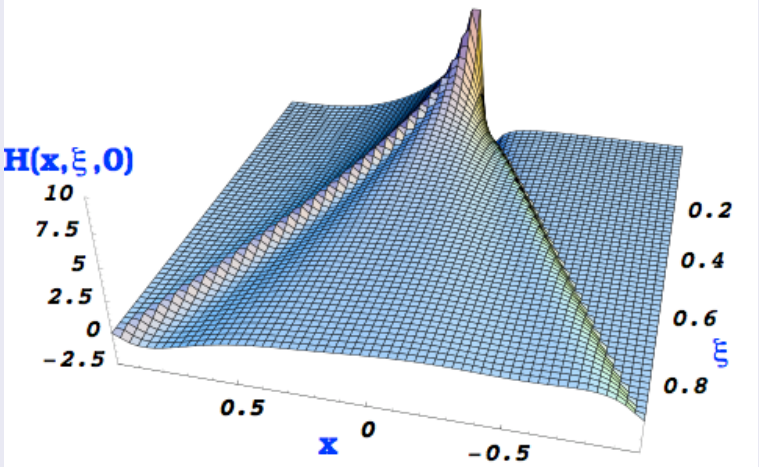
$$h(\beta,\alpha) = \frac{\Gamma(2+2b)}{2^{2b+1}\Gamma^2(1+b)} \frac{[(1-|\beta|)^2 - \alpha^2]^b}{(1-|\beta|)^{2b+1}}$$

- Power b is parameter of the model
- $b = \infty$ gives $h(\beta,\alpha) = \delta(\alpha)$ and $H(x,\xi) = f(x) + D(x/\xi)/|\xi|$

Meson and D-term terms



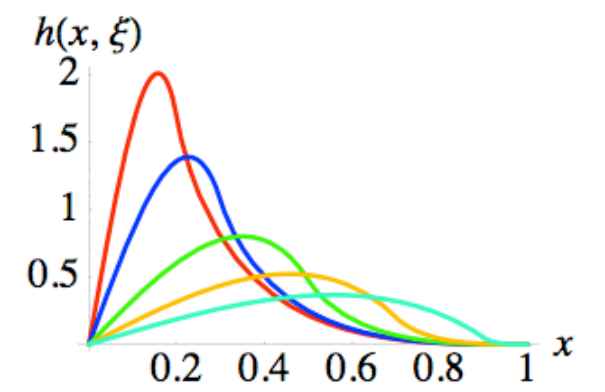
DD + D-term model



Model with Regge behavior of $f(\beta)$

- PDFs $f(\beta)$ are known to be singular for small β
- $f(\beta) \sim \beta^{-a}(1-\beta)^3$
- $x_+ = (x+\xi)/(1+\xi)$
- $x_- = (x-\xi)/(1-\xi)$
- $\sim |x-\xi|^{2-a}$+const behavior for $x \sim \xi$

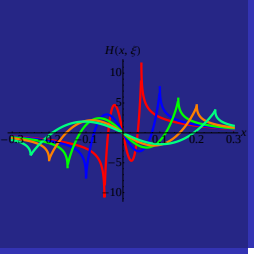
b=1 DD with Regge PDFs



$\xi = 0.2, 0.3, 0.5, 0.7, 0.9$

- Model $H(x,\xi) = \int_\Omega d\beta\, f(\beta)\, h_b(\beta,\alpha)\, \delta(x-\beta-\xi\alpha)$ with $b=1$

$$
\begin{aligned}
H(x,\xi)|_{|x|\geq\xi} &= \frac{1}{\xi^3}\left(1-\frac{a}{4}\right)\Big\{\big[(2-a)\xi(1-x)(x_+^{2-a}+x_-^{2-a}) \\
&+ (\xi^2-x)(x_+^{2-a}-x_-^{2-a})\big]\,\theta(x) - (x\to -x)\Big\} \\
H(x,\xi)|_{|x|\leq\xi} &= \frac{1}{\xi^3}\left(1-\frac{a}{4}\right)\Big\{x_+^{2-a}[(2-a)\xi(1-x)+(\xi^2-x)] \\
&- (x\to -x)\Big\}
\end{aligned}
$$



Spin-1/2 quarks: two-DD representation

- For a (pseudo)scalar target

$$\langle P - r/2|\bar{\psi}(-z/2)\gamma_\mu\psi(z/2)|P + r/2\rangle|_{\text{twist}-2}$$
$$= 2P_\mu f\big((Pz),(rz),z^2\big) + r_\mu g\big((Pz),(rz),z^2\big)$$

- Two-DD parametrization

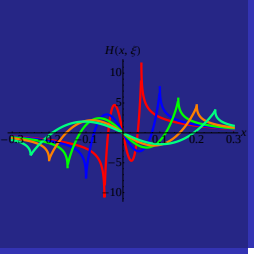
$$z^\mu \langle P - r/2|\bar{\psi}(-z/2)\gamma_\mu\psi(z/2)|P + r/2\rangle\big|_{z^2=0}$$
$$= \int_\Omega e^{-i\beta(Pz)-i\alpha(rz)/2}\Big[2(Pz)F(\beta,\alpha) + (rz)G(\beta,\alpha)\Big]\,d\beta\,d\alpha$$

- Not unique: invariant under transformation

$$F(\beta,\alpha) \to F(\beta,\alpha) + \partial\chi(\beta,\alpha)/\partial\alpha\ ,$$
$$G(\beta,\alpha) \to G(\beta,\alpha) - \partial\chi(\beta,\alpha)/\partial\beta\ ,$$

- “DD+D” form corresponds to “gauge” in which one has

$$2(Pz)F_D(\beta,\alpha) + (rz)\delta(\beta)D(\alpha)$$



Spin-1/2 quarks: single-DD representation

- Note: in local twist-2 operators $\bar{\psi}\{\gamma_\mu \overleftrightarrow{\partial}_{\mu_1} \dots \overleftrightarrow{\partial}_{\mu_n}\}\psi$ index μ is symmetrized with μ_i indices that produce $\beta P_{\mu_i} + \alpha r_{\mu_i}/2$
- $\Rightarrow$ μ also produces $\beta P_\mu + \alpha r_\mu/2$, i.e.

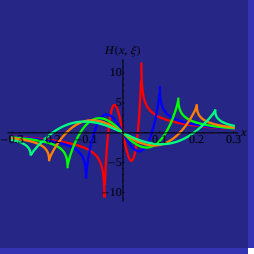
$$2(Pz)F(\beta,\alpha) + (rz)G(\beta,\alpha) = [2\beta(Pz) + \alpha(rz)]f(\beta,\alpha)$$

- Or $F(\beta,\alpha) = \beta f(\beta,\alpha)$ and $G(\beta,\alpha) = \alpha f(\beta,\alpha)$
- GPD in two-DD parametrization

$$H(x,\xi) = \int_\Omega \left[F(\beta,\alpha) + \xi G(\beta,\alpha)\right] \delta(x-\beta-\xi\alpha)\, d\beta\, d\alpha$$

- GPD in single-DD formulation

$$H(x,\xi) = \int_\Omega (\beta+\xi\alpha) f(\beta,\alpha)\, \delta(x-\beta-\xi\alpha)\, d\beta\, d\alpha$$
$$= x \int_\Omega f(\beta,\alpha)\, \delta(x-\beta-\xi\alpha)\, d\beta\, d\alpha$$



Single-DD formulation

- D-term in the single-DD case

$$D(\alpha) = \alpha \int_{-1+|\alpha|}^{1-|\alpha|} f(\beta, \alpha)\, d\beta$$

- Separating D-term

$$f(\beta, \alpha) = [f(\beta, \alpha)]_+ + \delta(\beta) \frac{D(\alpha)}{\alpha} \tag{3}$$

- Forward distribution

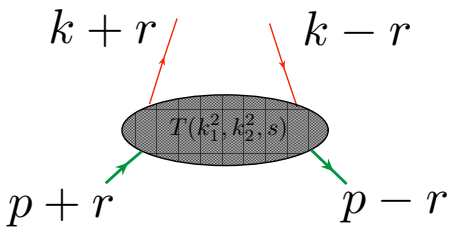
$$f(x) = \int_{-1+|x|}^{1-|x|} F(x, \alpha)\, d\alpha = x \int_{-1+|x|}^{1-|x|} f(x, \alpha)\, d\alpha$$

- Suggests factorized model

$$f(\beta, \alpha) = \frac{f(\beta)}{\beta} h(\beta, \alpha)$$

- $\Rightarrow$ Reconstructing DDs/GPDs from $f(x)/x$: very singular $\sim x^{-\alpha(0)-1}$ for small x !

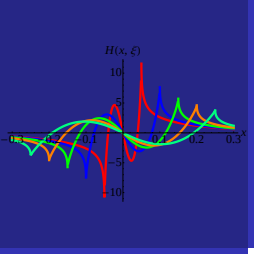
Diagrammatic model



- Quark-hadron scattering amplitude is modeled by

$$\gamma_\mu k^\mu \frac{1}{(m_1^2-(k+r)^2)^{n_1+1}} \frac{1}{(m_2^2-(k-r)^2)^{n_2+1}} T((p-k)^2)$$

- Dirac structure $\gamma_\mu k^\mu$ is necessary to provide EM gauge invariance of DVCS amplitude
- Modified propagators soften quark-hadron vertices

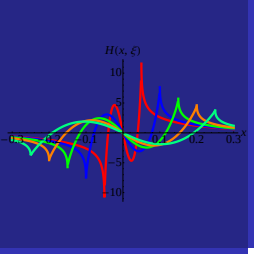


Implanting Regge behavior of $f(\beta)$

- Model is based on

$$H(x,\xi)\,P^+ \sim \int k^+ \frac{\delta(x-k^+/P^+)\,d^4k}{[m_1^2-(k+r)^2]^{n+1}[m_2^2-(k-r)^2]^{n+1}}$$
$$\times \int_0^\infty d\sigma\rho(\sigma)\left\{\frac{1}{\sigma-(P-k)^2}-\frac{1}{\sigma}\right\}$$

- First line: modified propagators providing softer quark-hadron vertices (eventually $N_1 = N_2 \equiv N$) can be obtained by $(d/dm_i^2)^{N_i}$
- Second line: quark-hadron scattering amplitude in dispersion representation subtracted at $(P-k)^2 = 0$
- Choosing $\rho(\sigma)$ to get Regge $\sim s^\alpha$ behavior in $s = (P-k)^2$
- NOTE: quark-parton amplitude $T(p,k)$ is subtracted at $(P-k)^2 = 0$, i.e. $T((P-k)^2 = 0) = 0$
- In general, one can take $T((P-k)^2 = 0) = T_0$
  $\Rightarrow$ additional contribution of D-term type



GPD in softened model

- In GPD variables $\beta P^+ + \alpha r^+ = x P^+$

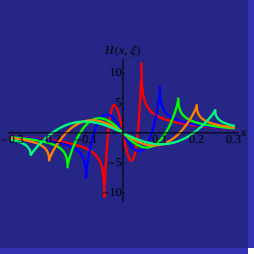
$$H(x,\xi) \sim \frac{x}{2^{2n+1}} \int_0^\infty d\sigma\, \rho(\sigma) \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha\, \frac{[(1-\beta)^2 - \alpha^2]^n}{(\beta\sigma + (1-\beta)m^2)^{2n+1}} \left\{ \delta\left(x - \beta - \alpha\xi\right) - \frac{\delta\left(x - \alpha\xi\right)}{(1-\beta)^2} \right\}$$

- Usual (forward) parton distribution corresponds to $\xi = 0$

$$H(x,\xi=0) = \frac{x}{2^{2n+1}} \int_0^\infty d\sigma\, \rho(\sigma) \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} \frac{[(1-\beta)^2 - \alpha^2]^n\, d\alpha}{(\beta\sigma + (1-\beta)m^2)^{2n+1}} \times \left\{ \delta\left(x - \beta\right) - \frac{\delta\left(x\right)}{(1-\beta)^2} \right\}$$

- Note: $x\delta(x) = 0$, thus

$$f(x) = \frac{(n!)^2}{(2n+1)!}\, x\,(1-x)^{(2n+1} \int_0^\infty \frac{d\sigma\, \rho(\sigma)}{(x\sigma + (1-x)m^2)^{2n+1}}$$



GPD in softened model, contd.

- Substituting σ-integral by forward distribution gives for GPD

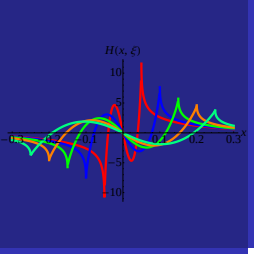
$$H(x,\xi) = \frac{x}{2^{2n+1}} \frac{(2n+1)!}{(n!)^2} \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha \, \frac{[(1-\beta)^2 - \alpha^2]^n}{(1-\beta)^{2n+1}} \, \frac{f(\beta)}{\beta} \times \left\{ \delta\left(x - \beta - \alpha\xi\right) - \frac{\delta\left(x - \alpha\xi\right)}{(1-\beta)^2} \right\}$$

- Normalized profile function:

$$h_n(\beta,\alpha) \equiv \frac{1}{2^{2n+1}} \frac{(2n+1)!}{(n!)^2} \frac{[(1-\beta)^2 - \alpha^2]^n}{(1-\beta)^{2n+1}}$$

- Result:

$$\frac{H(x,\xi)}{x} = \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha \, \frac{f(\beta)}{\beta} \, h_n(\beta,\alpha) \times \left\{ \delta\left(x - \beta - \alpha\xi\right) - \frac{\delta\left(x - \alpha\xi\right)}{(1-\beta)^2} \right\}$$



DD in softened model

- Result may be written in DD representation form

$$\frac{H(x,\xi)}{x} = \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha\, \delta\left(x-\beta-\alpha\xi\right) \times \left\{ f(\beta,\alpha) - \delta(\beta) \int_0^{1-|\alpha|} d\gamma\, \frac{f(\gamma,\alpha)}{(1-\gamma)^2} \right\}$$

with

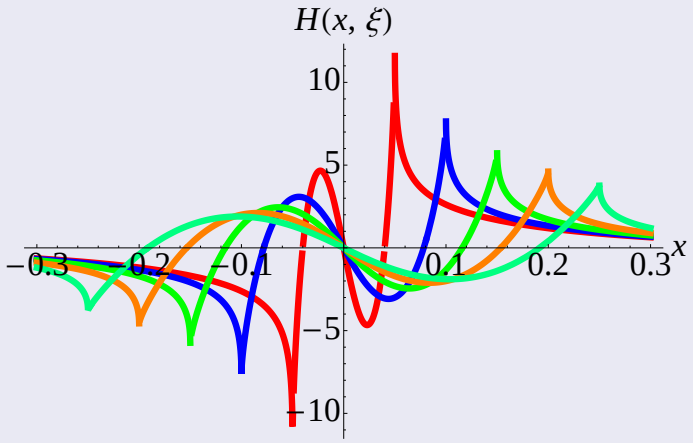
$$f(\beta,\alpha) = h_n(\beta,\alpha)\, f(\beta)\, /\beta$$

- This representation includes D-term

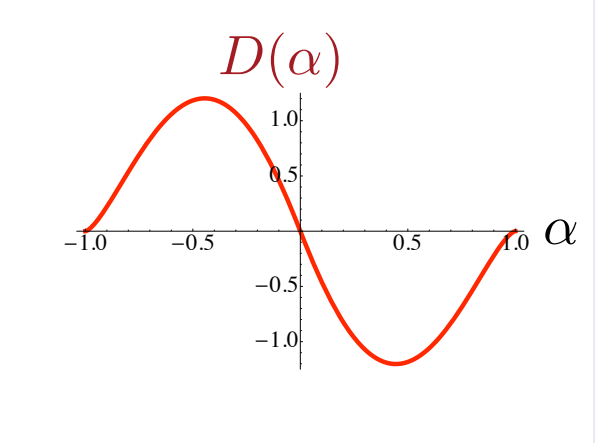
$$D(\alpha) = \alpha \int_0^{1-|\alpha|} d\beta\, \frac{f(\beta)}{\beta}\, h(\beta,\alpha) \left\{ 1 - \frac{1}{(1-\beta)^2} \right\}$$

Results for $n = 1$ profile $\sim (1-\beta)^2 - \alpha^2$

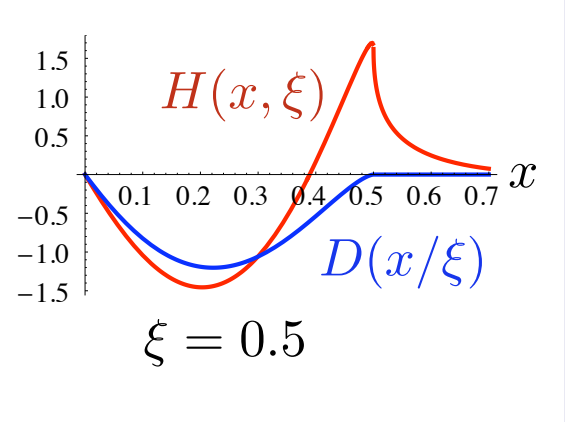
$H(x,\xi)$; $\xi = 0.05, 0.1, 0.15, 0.2, 0.25$



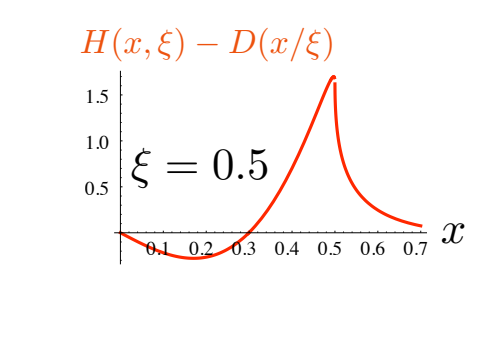
D-term

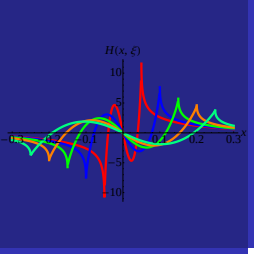


Comparison of GPD and D-term



Difference of GPD and D-term





Ambiguity in DD

- If quark-parton amplitude $T(P,k)$ equals nonzero T_0 at $(P-k)^2=0$, there is extra contribution to D-term

$$D_0(x/\xi) = \frac{T_0}{2^{2n+1}n}\left(\frac{x}{|\xi|}\right)\left(1-\frac{x^2}{\xi^2}\right)^n \theta\left(\left|\frac{x}{\xi}\right|<1\right)$$

- $\Rightarrow$ D-term contains extra information

GPD sum rules and analytic regularization

- GPD sum rule

$$\int_{-1}^{1} \frac{H(x,x) - H(x,0)}{x}\, dx = \int_{-1}^{1} \frac{D(\alpha)}{1-\alpha}\, d\alpha$$

- Note: both $H(x,0)/x$ and $H(x,x)/x$ are even functions of x
  $\Rightarrow$ cannot use principle value prescription for $x = 0$ singularity
- Analyticity assumption (D. Mueller et al.): Mellin moments

$$\Phi(j) \equiv \int_{-1}^{1} x^j \left[H(x,x) - H(x,0)\right] dx \tag{4}$$

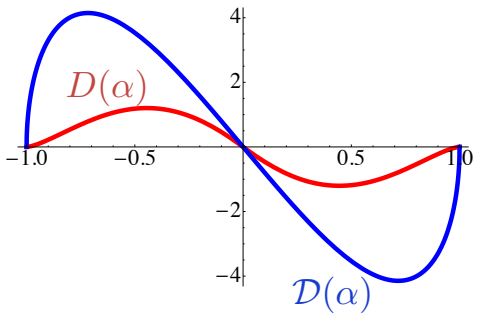
  can be analytically continued to the point $j = -1$
- Equivalent to regularization prescription

$$\int_{(0)}^{y} \frac{\lambda(x)}{x^{a+1}}\, dx = \int_{0}^{y} dx \frac{\lambda(x) - \lambda(0) - x\lambda'(0) - \ldots}{x^{a+1}}$$
$$+ \lambda(0) \int_{(0)}^{y} \frac{dx}{x^{a+1}} + \lambda'(0) \int_{(0)}^{y} \frac{dx}{x^{a}} + \ldots$$

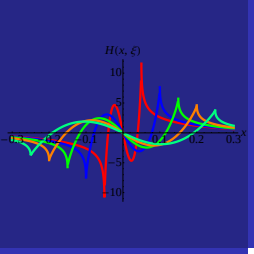
Analytic regularization

- Analytic regularization results in unambiguous D-term

$$\frac{\mathcal{D}(\alpha)}{\alpha} = 2\left[\int_0^{1-|\alpha|} \frac{\lambda(\beta,\alpha)-\lambda(0,\alpha)}{\beta^{a+1}}\, d\beta - \frac{\lambda(0,\alpha)}{a(1-|\alpha|)^a}\right]$$



- But: no first principle reason for analytic regularization to hold
- Need to extract D-term experimentally



Summary