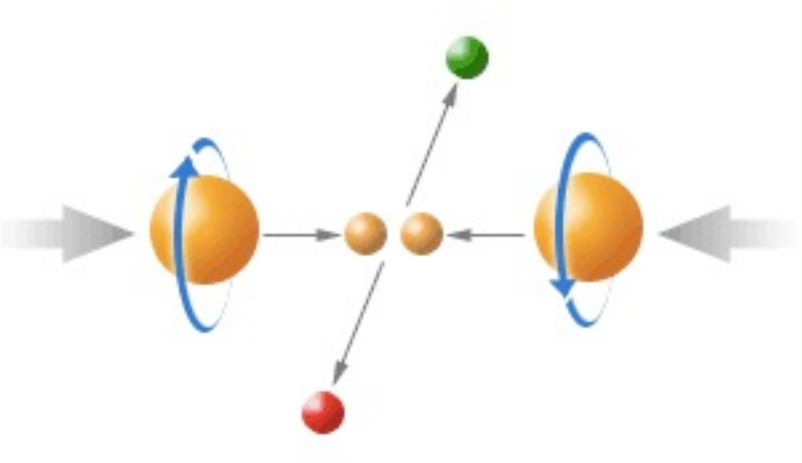
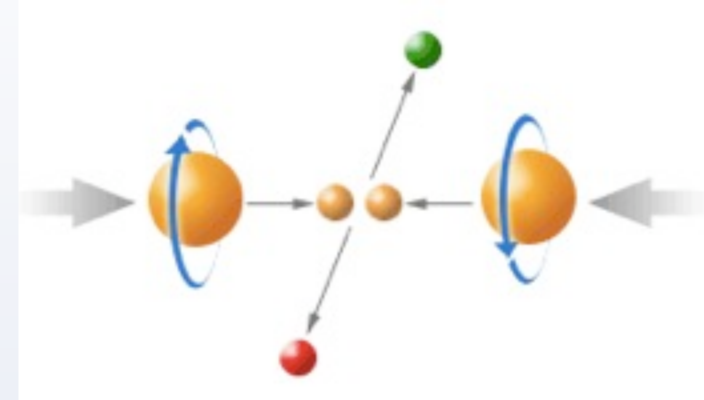
# *Bessel Weighted Asymmetries as an experimental tool for the Future/present*


***Leonard Gamberg Penn State***



Boer, LG,Musch,Prokudin JHEP 2011
Boer,LG,Musch,Prokudin in prep

# Comments Importance of TMDs in studying partonic content of the nucleon



- Single inclusive hadron production in hadronic collisions largest/ oldest observed TSSAs
- From theory view notoriously challenging from partonic picture twist-3 power suppressed in hard scale (vs. w/ SIDIS, DY, $e^+e^-$)
- Connection w/ twist 2 "TMD" approach
  - Operator level ETQS fnct 1st moment of Sivers

$$gT_F(x,x) = -\int d^2k_T \frac{|k_T^2|}{M} f_{1T}^{\perp}(x,k_T^2) + \text{"UV"} \ldots$$

$$= -2M f_{1T}^{\perp(1)}(x)$$

Boer Pijlman Mulders NPB -03

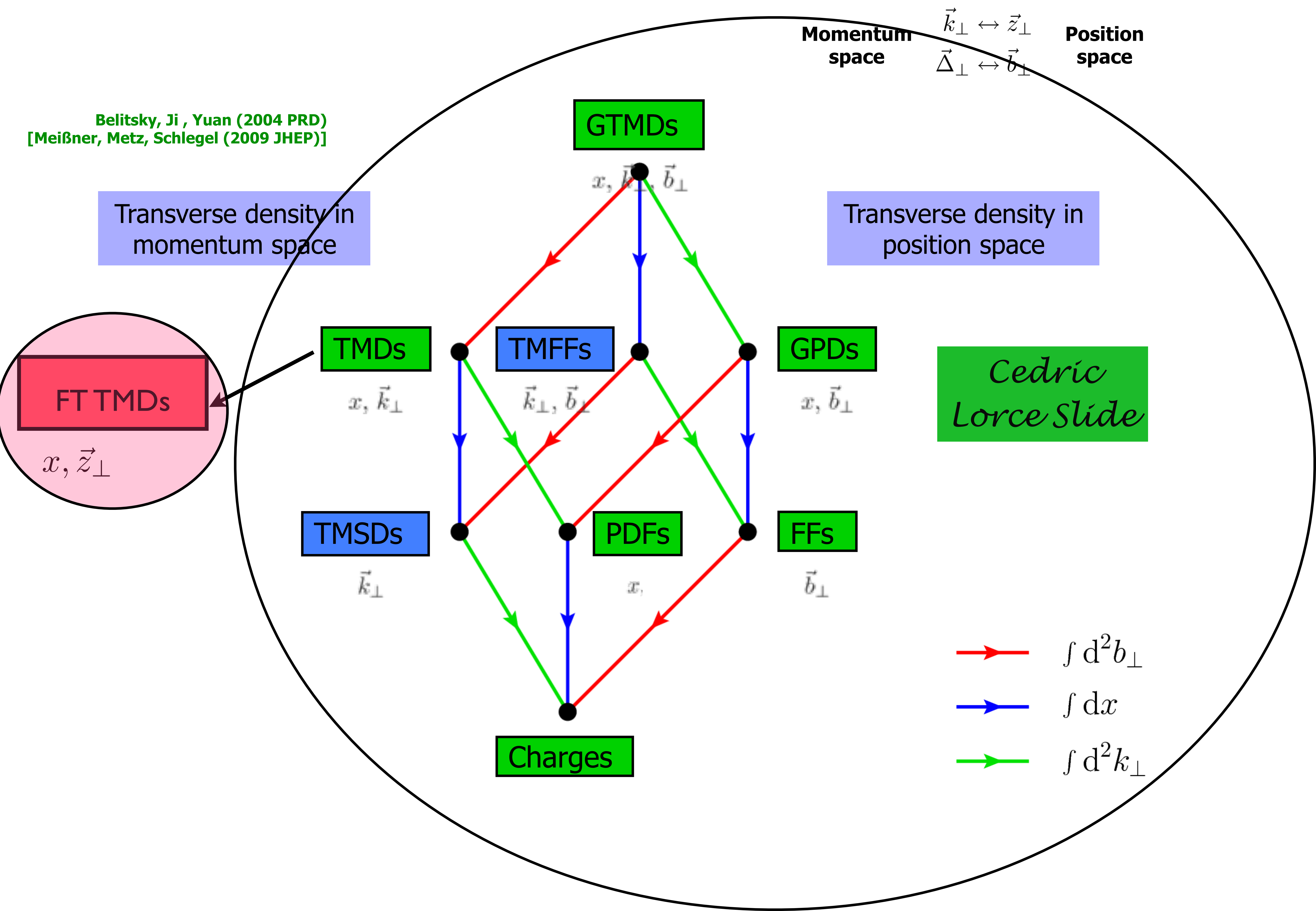
$$\tilde{f}_{1T}^{\perp(1)}(x,|\boldsymbol{b}_T|) = \int d^2p_T \frac{|p_T|}{|\boldsymbol{b}_T|M^2} J_1(|\boldsymbol{b}_T||p_T|) f_{1T}^{\perp}(x,p_T^2)$$

Boer, LG, Musch, Prokudin JHEP-2011--arXiv:1107.529

# Outline

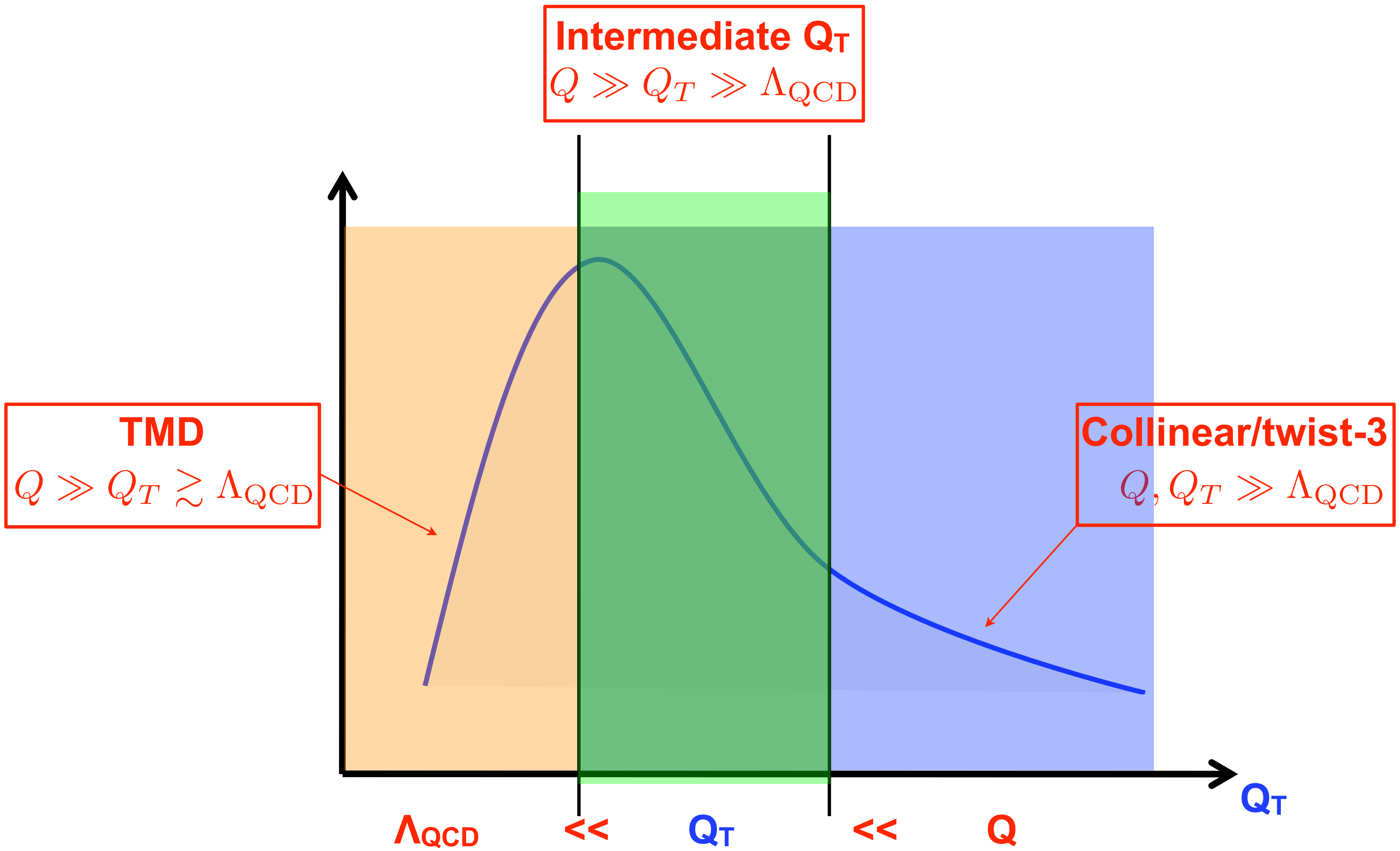
- **Review transverse spin Effects - TSSAs**
  - Transverse Spin Effects-twist 3 & TMD twist 2
- **Summary Elements Factorization-SIDIS**
- **Role color gauge Inv.- “T-odd” TMDs**
  **Gauge Links-“process dependence”**
- Merit of Bessel Weighted Asymmetries (BWA) “S/T” pic of SIDIS
- Fourier Transformed SIDIS cross section & “FT” TMDs
- Cancellation of the Soft & Sudakov hard factor from BWA
- Conclusions

# Partonic Structure of Nucleon



# Connection of twist 3 and twist 2 approach "overlap regime"

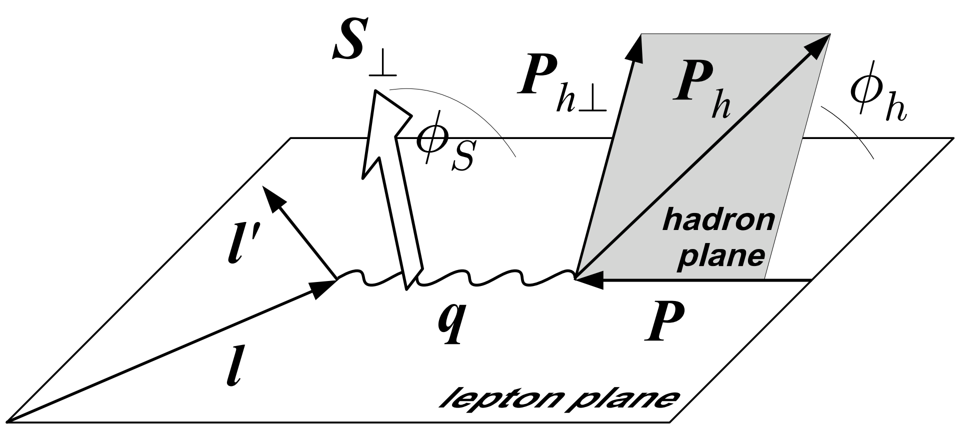
**Ji,Qiu,Vogelsang, Yuan PRL 2006 ...**
**Bacchetta, Boer, Diehl, Mulders JHEP 2008**

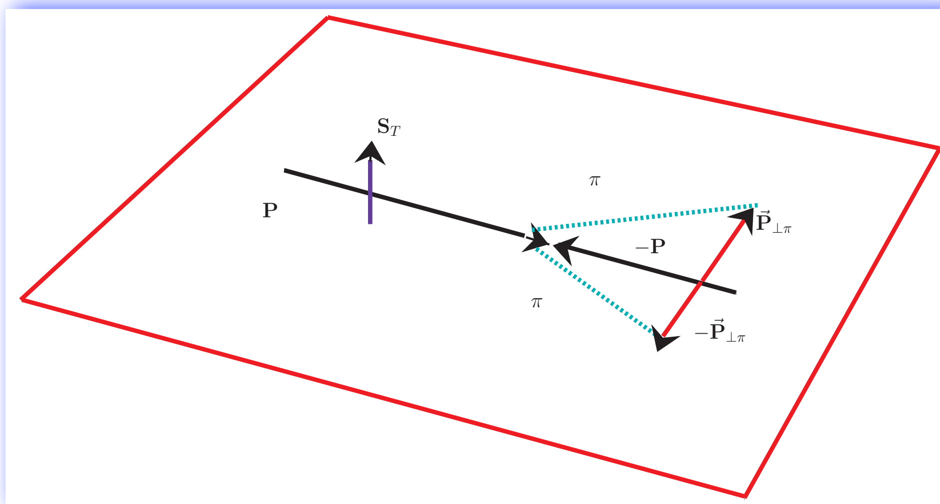



- Explore role parton model processes in twist-2&3 approaches
  LG & Z. Kang PLB 2011, D'Alesio, LG, Z. Kang, C.Pisano PLB 2011
  "exploring impact of Gauge Inv"

# Two methods to account for SSA in QCD

- Depends on momentum of probe $q^2 = -Q^2$ and momentum of produced hadron $\boldsymbol{P}_{h\perp}$ relative to hadronic scale $k_T^2 (\equiv k_\perp^2) \sim \Lambda_{\mathrm{QCD}}^2$ and




- $k_\perp^2 \sim P_{h\perp}^2 \ll Q^2$ two scales-TMDs

$$\Delta\sigma(P_h, S) \sim \Delta f_{a/A}^{\perp}(x, p_\perp) \otimes D_{h/c}(z, K_\perp) \otimes \hat{\sigma}_{\mathrm{parton}}$$

- $k_\perp^2 \ll P_{h\perp}^2 \sim Q^2$ twist 3 factorization-ETQSs

$$\Delta\sigma(P_h, S) \sim \frac{1}{Q} f_{a/A}^{\perp}(x) \otimes f_{b/B}(x) \otimes D_{h/c}(z) \otimes \hat{\sigma}_{\mathrm{parton}}$$

# Reaction Mechanism w/ Partonic Description

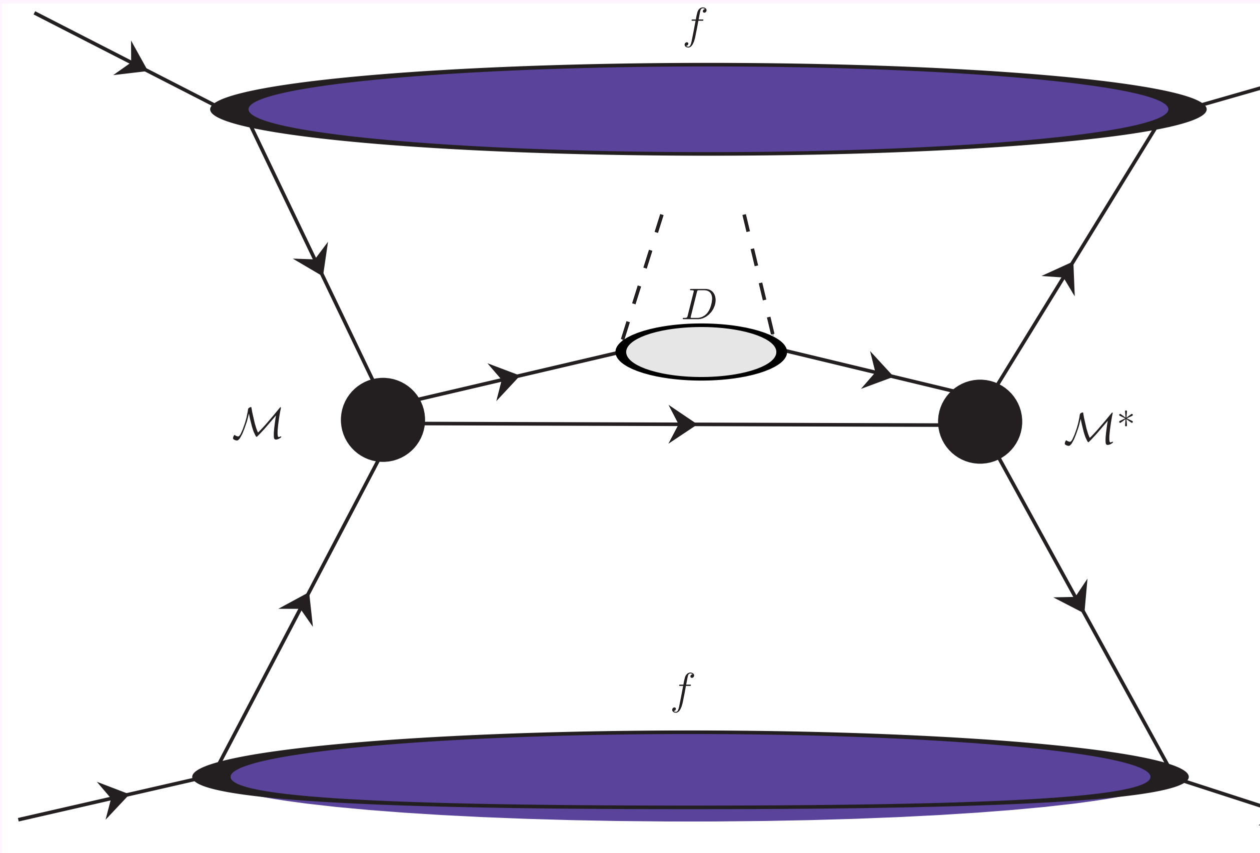
## Collinear factorized QCD parton dynamics

$$\Delta\sigma^{pp^{\uparrow}\to\pi X} \sim f_a \otimes f_b \otimes \Delta\hat{\sigma} \otimes D^{q\to\pi}$$

$$\Delta\hat{\sigma} \equiv \hat{\sigma}^{\uparrow} - \hat{\sigma}^{\downarrow}$$

$$|\uparrow / \downarrow\rangle = (|+\rangle \pm i|-\rangle)$$

$$\hat{a}_N = \frac{\hat{\sigma}^{\uparrow} - \hat{\sigma}^{\downarrow}}{\hat{\sigma}^{\uparrow} + \hat{\sigma}^{\downarrow}} \sim \frac{\mathrm{Im}\left(\mathcal{M}^{+*}\mathcal{M}^{-}\right)}{|\mathcal{M}^{+}|^2 + |\mathcal{M}^{-}|^2}$$



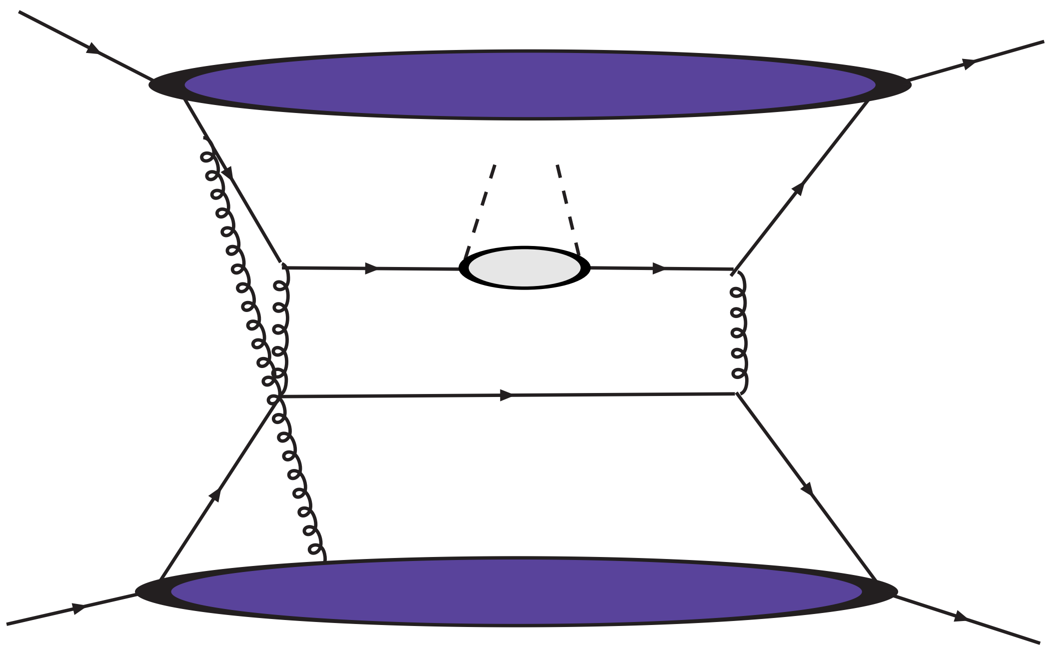

**Interference of helicity flip and non-flip amps**
**1) requires breaking of chiral symmetry $m_q/E$**
**2) relative phases require higher order corrections**

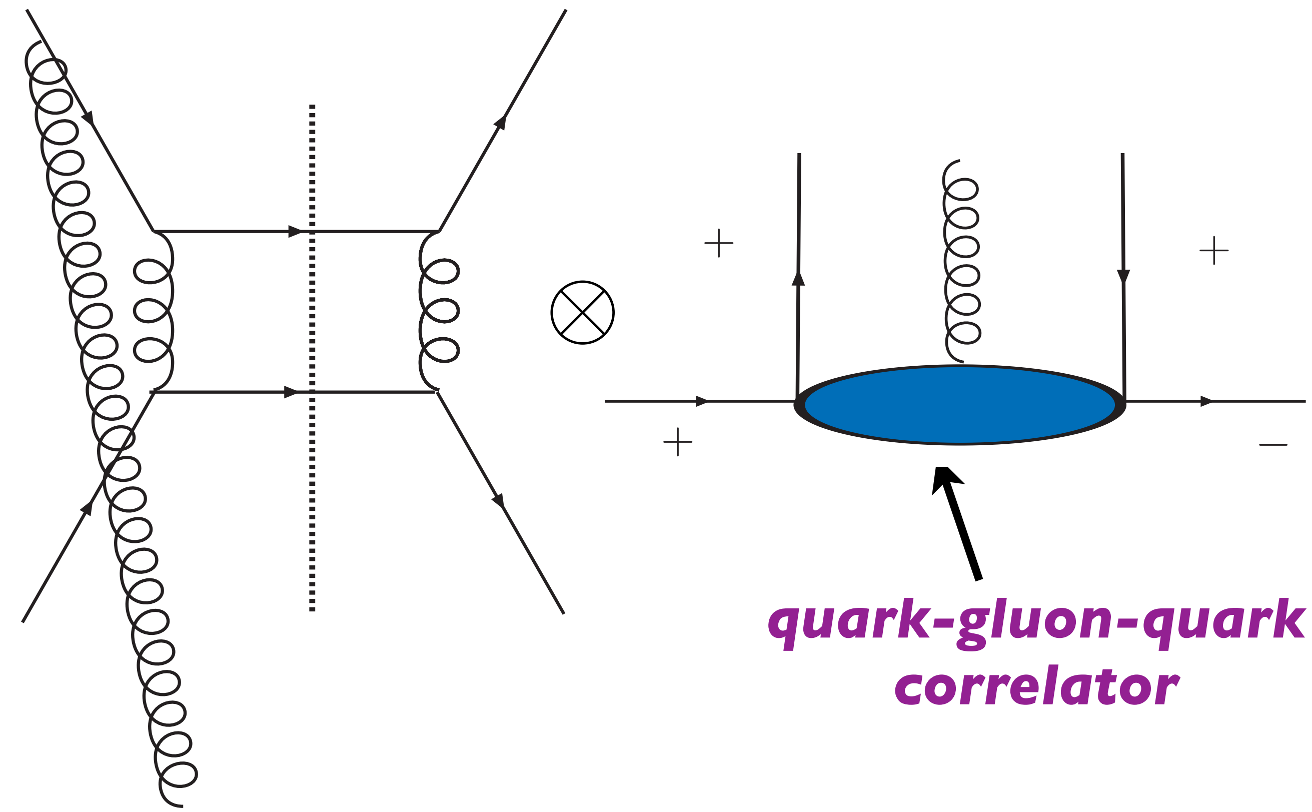
# Twist 3 ETQS approach-"Partonic Picture"

## $Q \sim P_T >> \Lambda_{\rm qcd}$ One scale Collinear fact Twist 3

## Phases in soft poles of prop hard processes

Efremov & Teryaev PLB 1982






$$\Delta\sigma \sim f_a \otimes T_F \otimes H_{ETQS} \otimes D^{q \to h}$$

$$\frac{1}{xs + i\epsilon} = \mathcal{P}\left(\frac{1}{xs}\right) \pm i\pi\delta(xs)$$

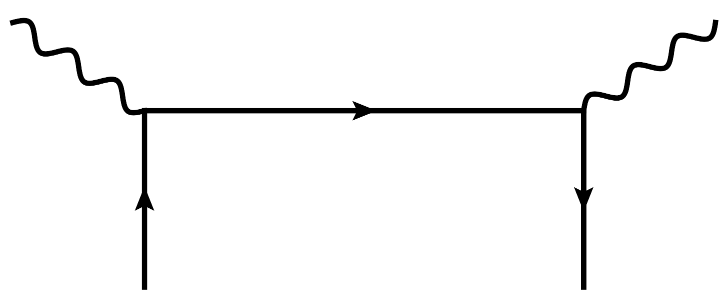
- Phases from interference of two-parton three-parton scattering amplitudes

- Factorization and Pheno: Qiu, Sterman 1991,1999..., Koike et al, 2000, ... 2010, Ji, Qiu, Vogelsang, Yuan, 2005 ... 2008 ..., Yuan, Zhou 2008, 2009, Kang, Qiu, 2008, 2009 ... Kouvaris Ji, Qiu, Vogelsang! 2006, Vogelsang and Yuan PRD 2007

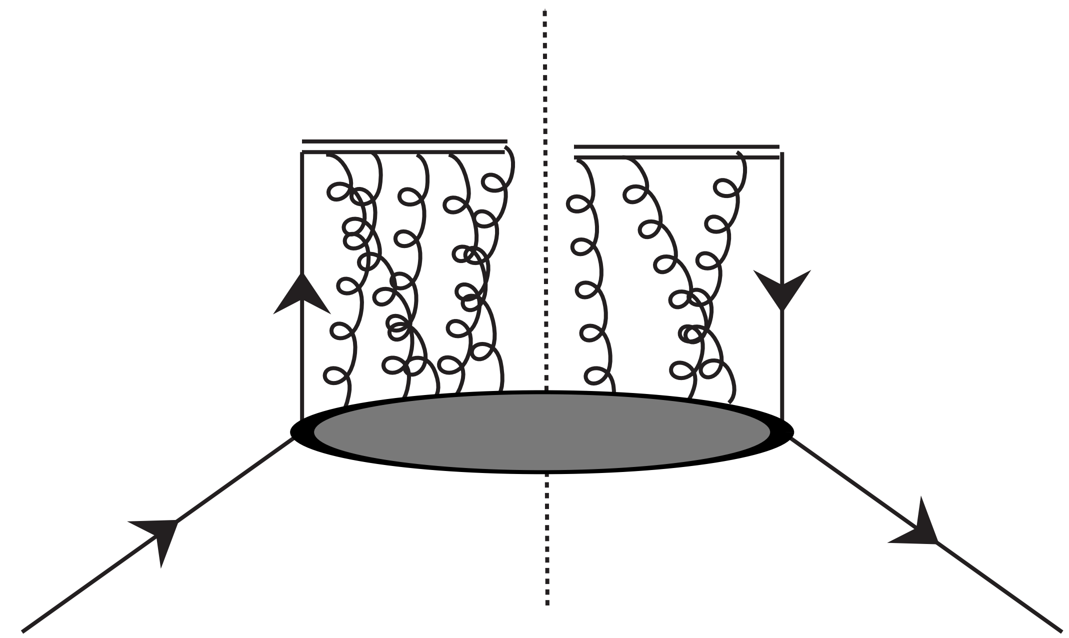
# *TSSAs thru "T-odd" non-pertb.* **spin-orbit** *correlations….*

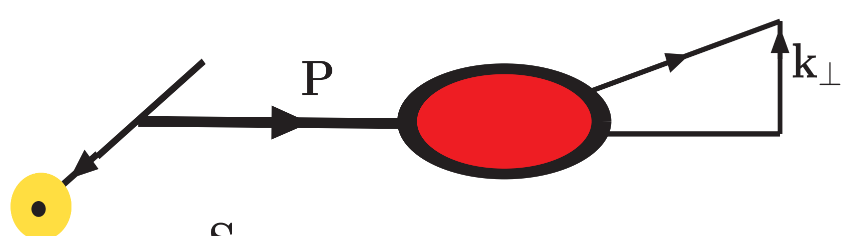
## Sensitivity to $p_T \sim \mathbf{k}_T << \sqrt{Q^2}$

- Sivers PRD: 1990 **TSSA is associated w/ correlation** $transverse$ **spin and momenta in initial state hadron**



$$\otimes$$



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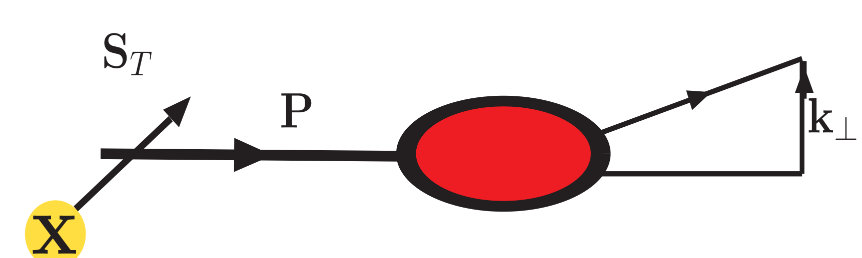
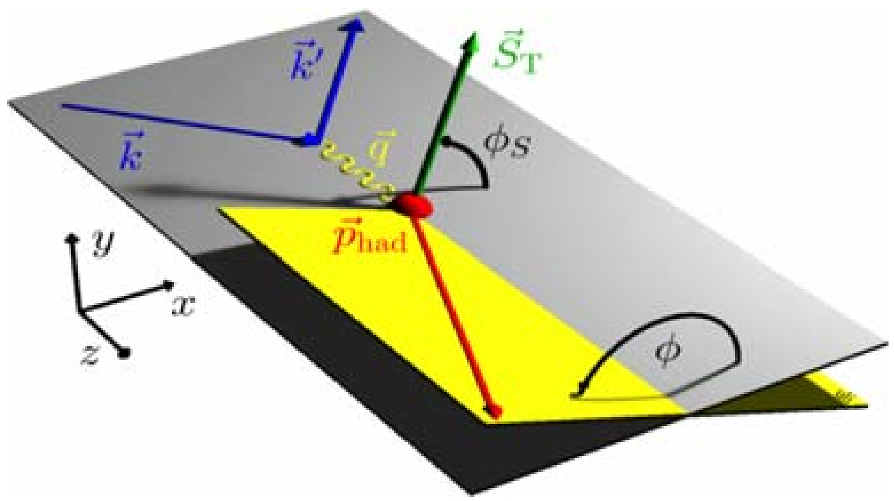



$$\Delta\sigma^{pp^{\uparrow}\to\pi X} \sim D \otimes f \otimes \Delta f^{\perp} \otimes \hat{\sigma}_{Born} \Rightarrow$$

$$\boldsymbol{\Delta f^{\perp}(x, k_{\perp}) = i S_T \cdot (P \times k_{\perp})}\; f_{1T}^{\perp}(x, \boldsymbol{k}_{\perp})$$

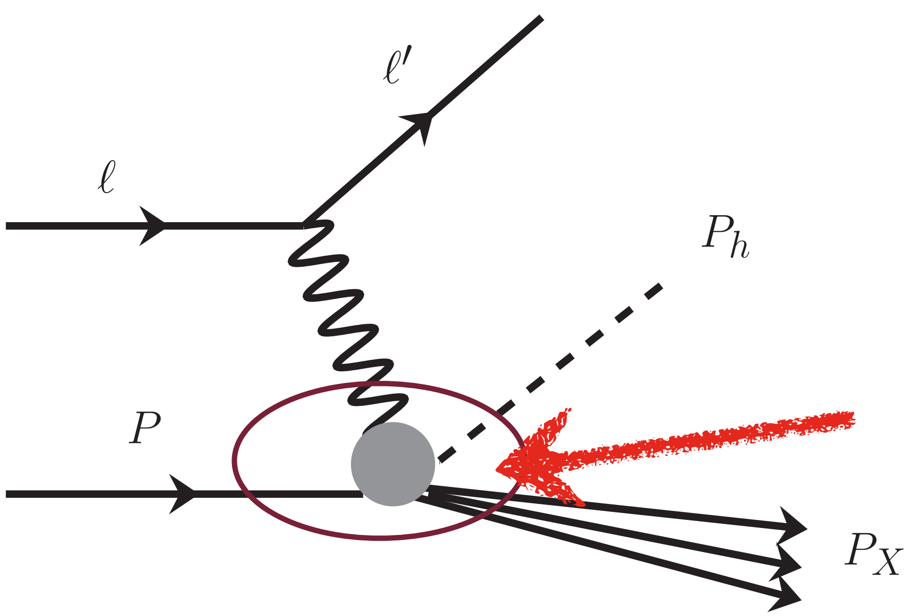
# Factorization in Parton Model



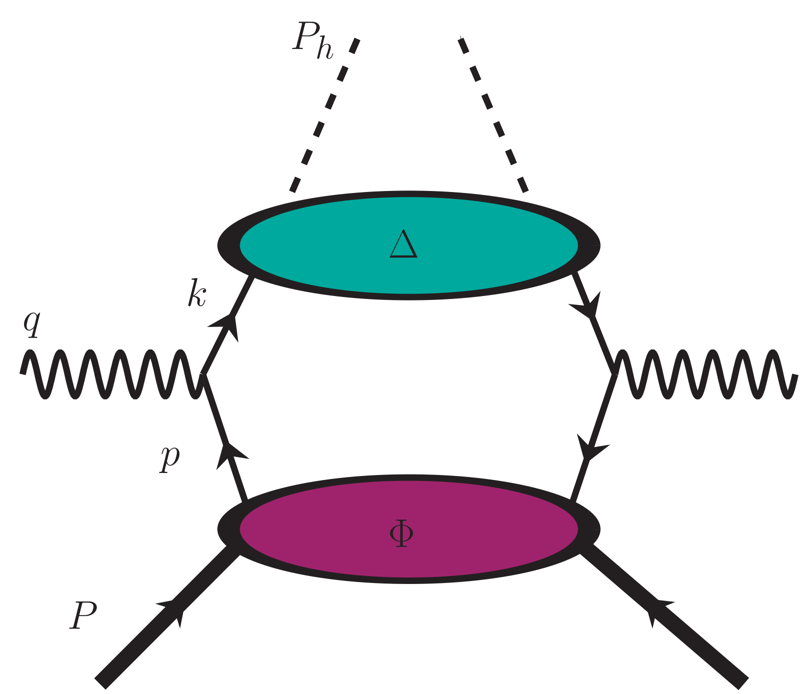

$$x_B = \frac{Q^2}{2P \cdot q}$$

$$z_h = \frac{P \cdot P_h}{P \cdot q} \approx \frac{P_h^-}{q^-}$$

Parton model & DIS kinematics




Factorize



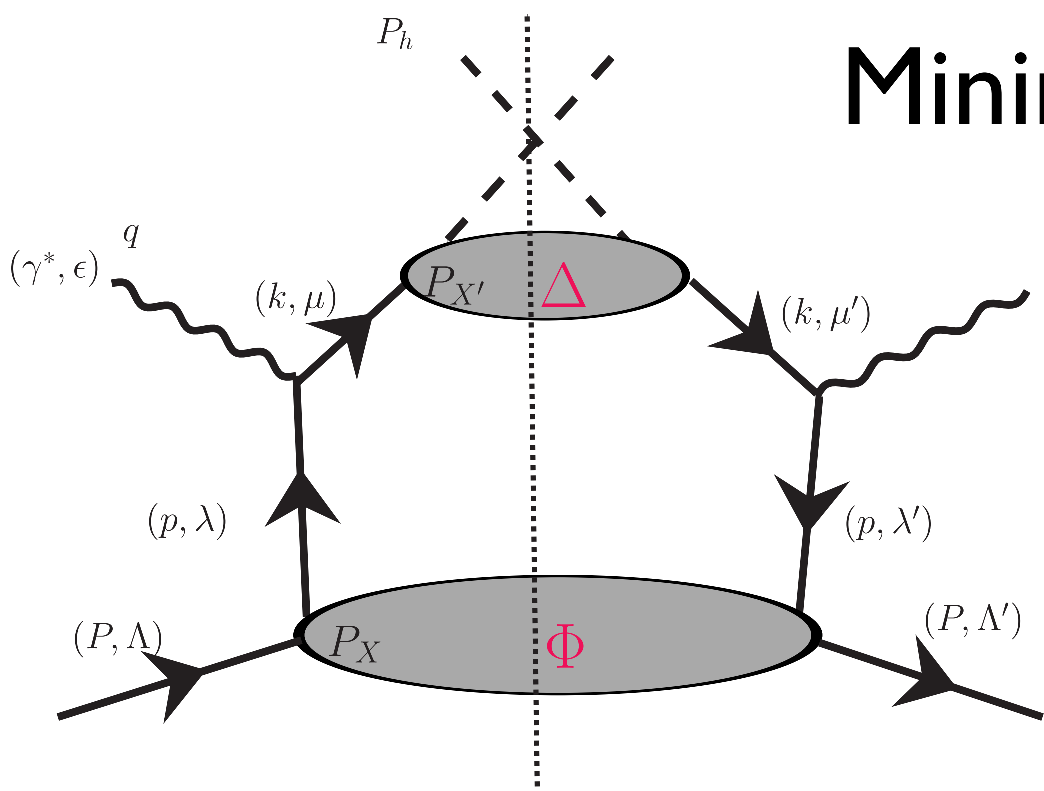

$$\frac{d\sigma}{dx_B\, dy\, d\psi\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}|\, d|\boldsymbol{P}_{h\perp}|} = \frac{\alpha^2}{x_B y Q^2} \frac{y^2}{(1-\varepsilon)} \left(1 + \frac{\gamma^2}{2x_B}\right) L_{\mu\nu} W^{\mu\nu},$$

# Factorization $P_T$ of hadron small sensitive to intrinsic transv. momentum of partons

$$W^{\mu\nu}(q,P,S,P_h) = \int \frac{d^2\mathbf{p}_T}{(2\pi)^2} \int \frac{d^2\mathbf{k}_T}{(2\pi)^2} \delta^2(\mathbf{p}_T - \frac{\mathbf{P}_{h\perp}}{z_h} - \mathbf{k}_T) \mathrm{Tr}\left[\Phi(x,\mathbf{p}_T)\gamma^\mu \Delta(z,\mathbf{k}_T)\gamma^\nu\right]$$

$$\Phi(x,\mathbf{p}_T) = \int dp^- \Phi(p,P,S)|_{p^+=x_B P^+}\,, \qquad \Delta(z,\mathbf{k}_T) = \int dk^- \Delta(k,P_h)|_{k^-=\frac{P^-}{z_h}}$$

*Small transverse momentum !!!*



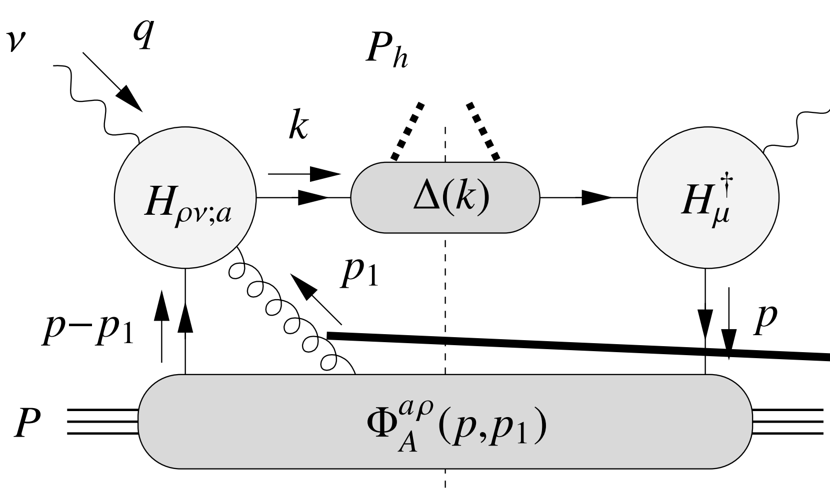

Minimal requirement satisfy color gauge invariance

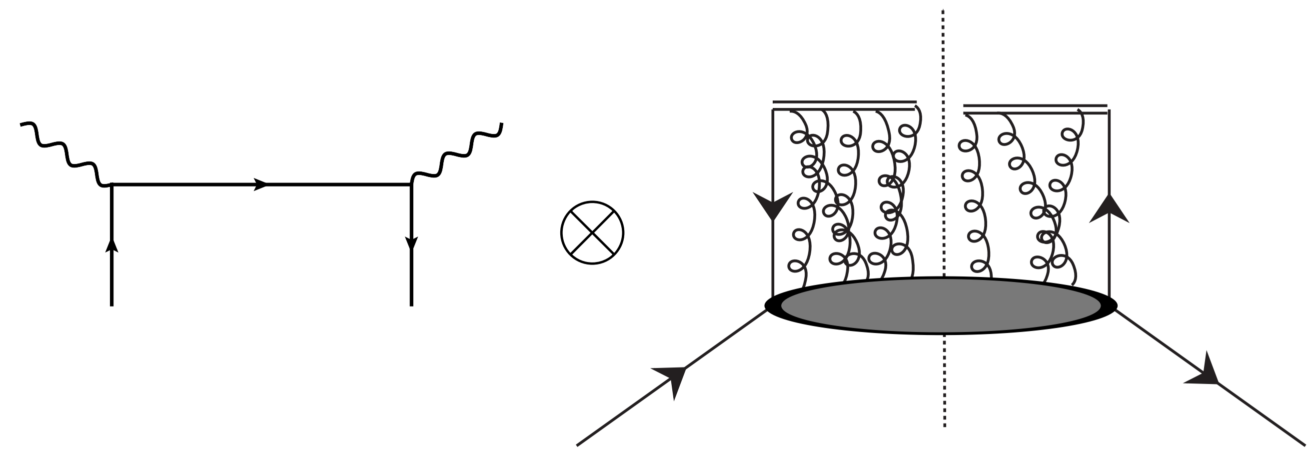
# *Minimal Requirement for PARTON MDL Factorization*

**Gauge link determined re-summing leading gluon interactions btwn soft and hard**

Efremov,Radyushkin Theor. Math. Phys. 1981,Belitsky, Ji, Yuan NPB 2003,
Boer, Bomhof, Mulders Pijlman, et al.  2003 - 2008- NPB, PLB, PRD

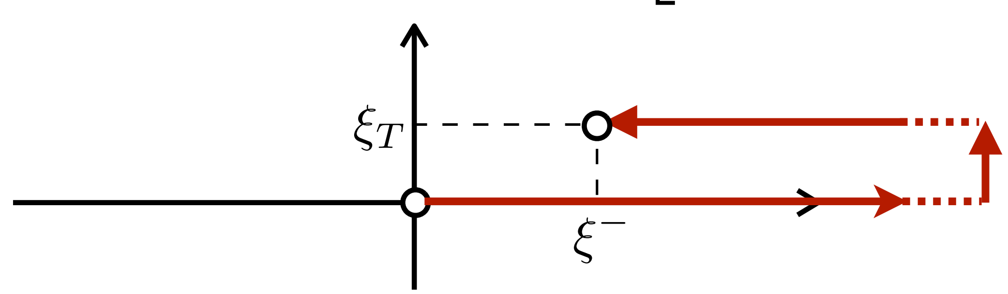
$$\Phi^{[\mathcal{U}[\mathcal{C}]]}(x,p_T) = \int \frac{d\xi^- d^2\xi_T}{2(2\pi)^3} e^{ip\cdot\xi} \langle P|\overline{\psi}(0)\mathcal{U}^{[C]}_{[0,\xi]}\psi(\xi^-,\xi_T)|P\rangle|_{\xi^+=0}$$



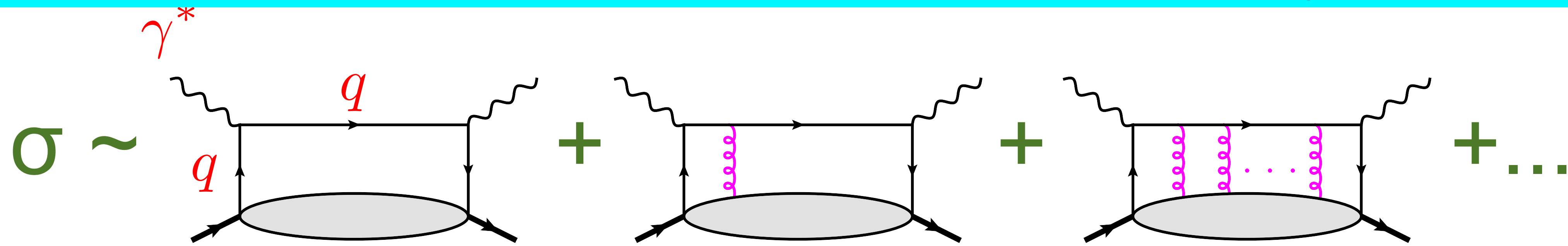
- **The path $[C]$ is fixed by hard subprocess within hadronic process.**

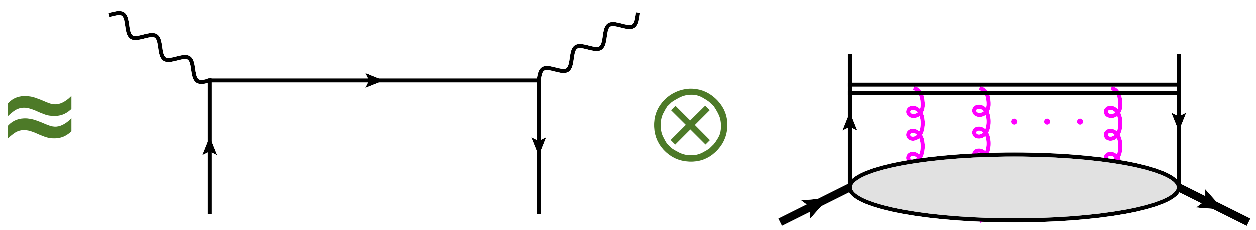
$$W_{\mu\nu}(q,P,S,P_h) = \int d^4p d^4k \delta^4(p+q-k)\mathrm{Tr}\left[\Phi^{\mathcal{U}^{[C]}_{[\infty;\xi]}}(p) H^\dagger_\mu(p,k)\Delta(k)H_\nu(p,k)\right]$$




$U_{[+]}$

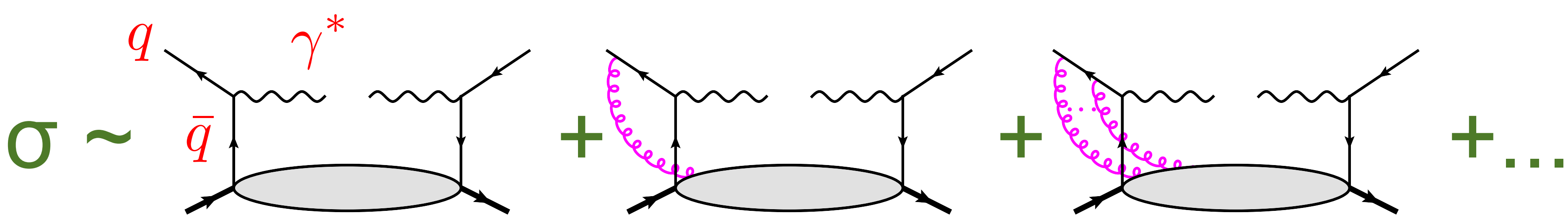
# *Gauge link determined re-summing leading gluon interactions btwn soft and hard Process Dependence break down of Universality*

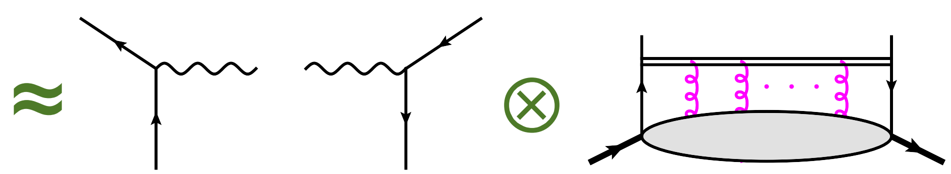



PDFs with SIDIS gauge link

$$\mathcal{P}\, e^{ig \int_y^{\infty} d\lambda \cdot A(\lambda)}$$




PDFs with DY gauge link

$$\mathcal{P}\, e^{ig \int_y^{-\infty} d\lambda \cdot A(\lambda)}$$

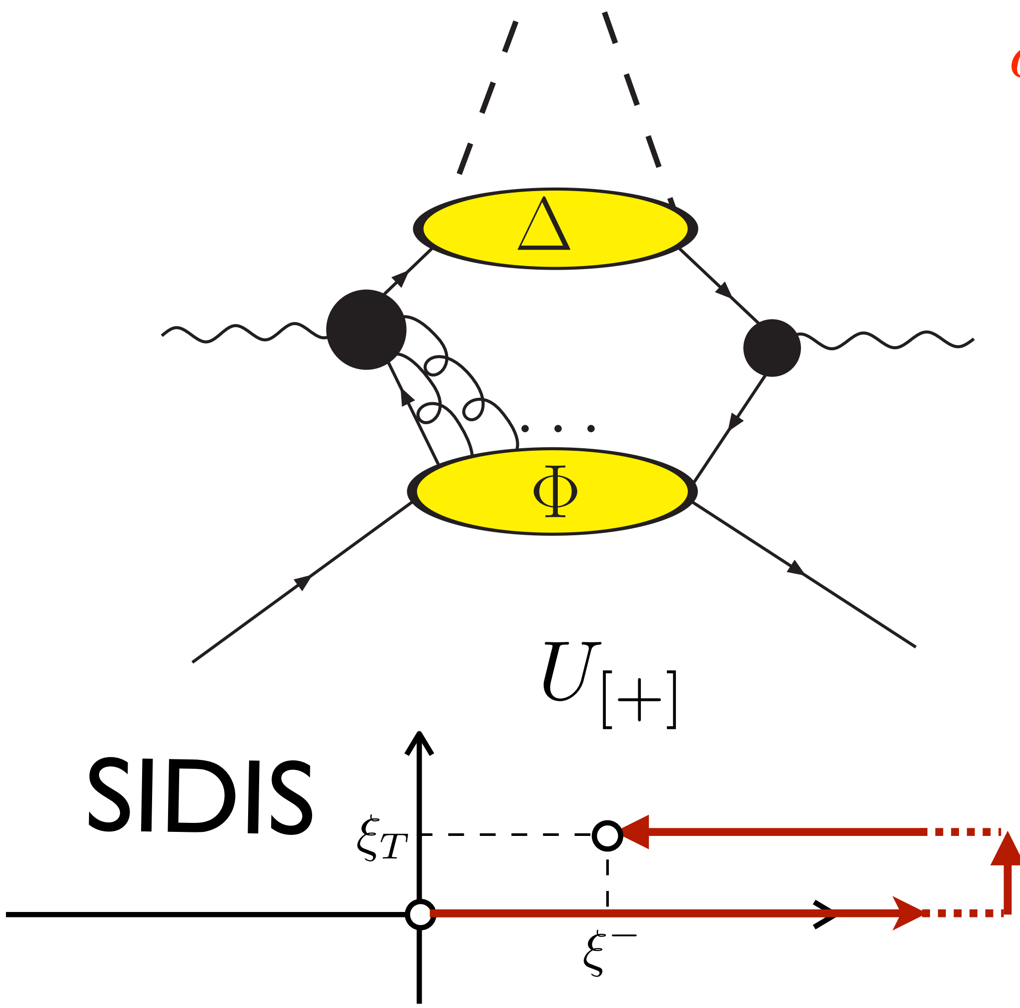
# “Generalized Universality” Fund. Prediction of QCD Factorization

$$f^{\perp}_{1T_{sidis}}(x, k_T) = -f^{\perp}_{1T_{DY}}(x, k_T) \qquad p_T \sim \mathbf{k}_T << \sqrt{Q^2}$$

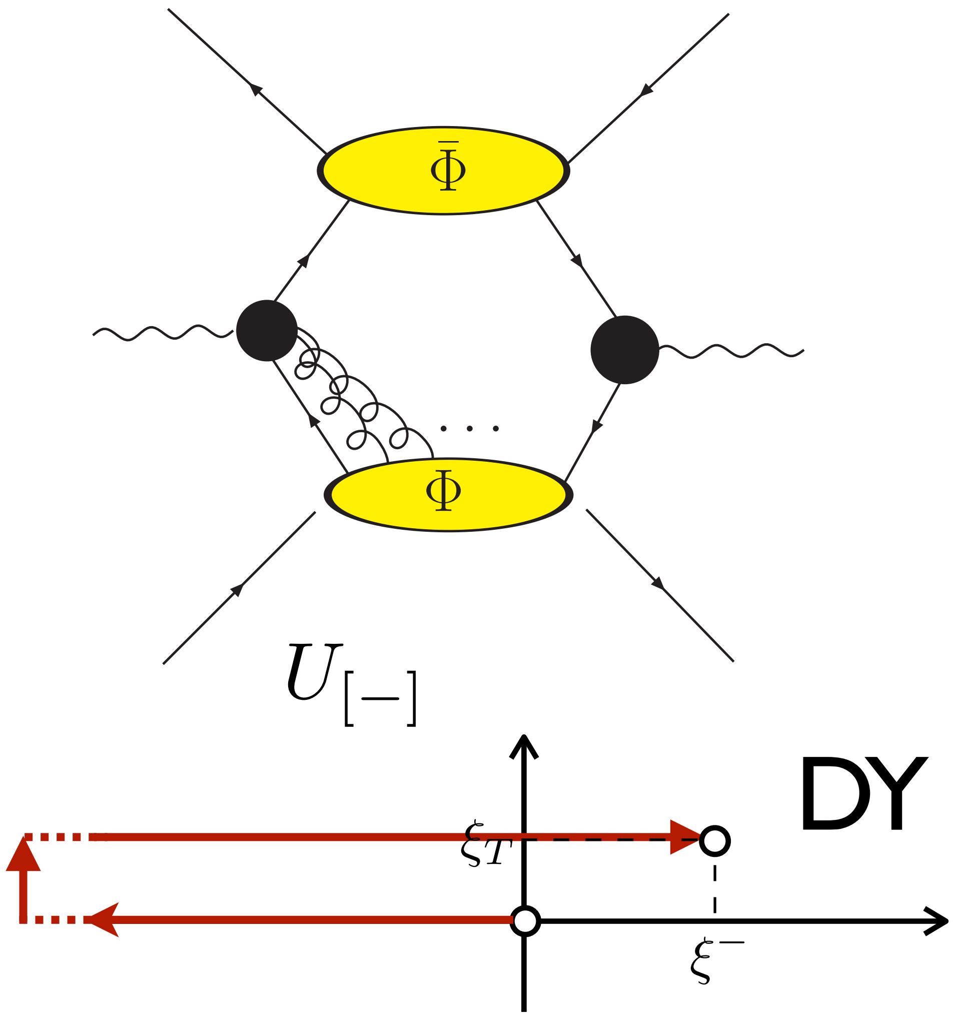
***EIC conjunction with DY exp. E906-Fermi, RHIC II, Compass, JPARC***

Process Dependence, Collins PLB 02, Brodsky et al. NPB 02, Boer Mulders Pijlman Bomhoff 03, 04 ...

$$d\sigma = L_{\mu\nu} \mathcal{W}^{\mu\nu} \Rightarrow$$








$$\Phi^{[+]*}(x, p_T) = i\gamma^1\gamma^3\Phi^{[-]}(x, p_T)i\gamma^1\gamma^3$$

# Partonic picture Structure Functions CONVOLUTION

$$\mathcal{C}\big[w\,f\,D\big] = x\sum_a e_a^2 \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,\delta^{(2)}\big(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\big)\,w(\boldsymbol{p}_T,\boldsymbol{k}_T)\,f^a(x,p_T^2)\,D^a(z,k_T^2)$$

$$F_{UU,T} = \mathcal{C}\big[f_1 D_1\big], \qquad F_{LL} = \mathcal{C}\big[g_{1L} D_1\big],$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\right], \quad F_{UT}^{\sin(\phi_h+\phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T}{M_h} h_1 H_1^{\perp}\right],$$

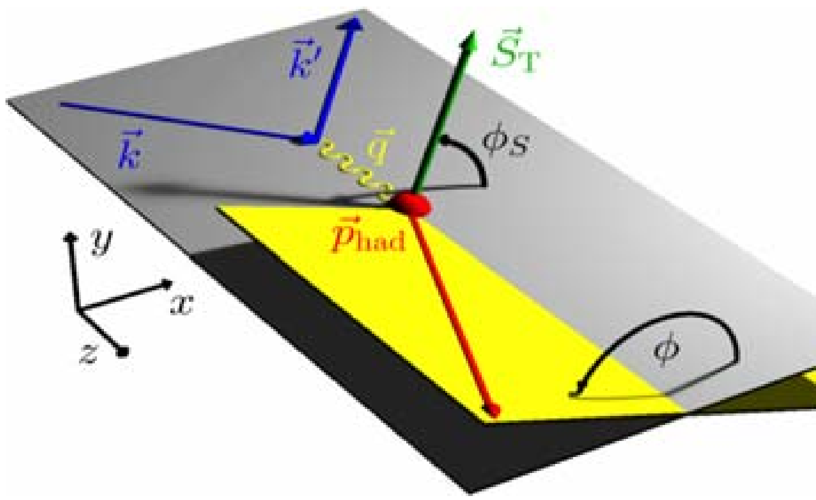
$$F_{UL}^{\sin 2\phi_h} = \mathcal{C}\left[-\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T) - \boldsymbol{k}_T\cdot\boldsymbol{p}_T}{M M_h} h_{1L}^{\perp} H_1^{\perp}\right], \quad F_{UU}^{\cos 2\phi_h} = \mathcal{C}\left[-\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T) - \boldsymbol{k}_T\cdot\boldsymbol{p}_T}{M M_h} h_1^{\perp} H_1^{\perp}\right],$$

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = \mathcal{C}\left[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\,(\boldsymbol{p}_T\cdot\boldsymbol{k}_T) + \boldsymbol{p}_T^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T) - 4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)}{2M^2 M_h}\, h_{1T}^{\perp} H_1^{\perp}\right]$$

# SIDIS- CS model indpen. thru structure functions

$$
\begin{aligned}
\frac{d\sigma}{dx_B\,dy\,d\psi\,dz_h\,d\phi_h\,dP_{h\perp}^2} &= \frac{\alpha^2}{x_B y Q^2}\,\frac{y^2}{2\,(1-\varepsilon)}\left(1+\frac{\gamma^2}{2x_B}\right)\Bigg\{F_{UU,T}+\varepsilon F_{UU,L}+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\,F_{UU}^{\cos\phi_h} \\
&+ \quad \varepsilon\cos(2\phi_h)\,F_{UU}^{\cos 2\phi_h}+\lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\,F_{LU}^{\sin\phi_h} \\
&+ \quad S_\parallel\left[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\,F_{UL}^{\sin\phi_h}+\varepsilon\sin(2\phi_h)\,F_{UL}^{\sin 2\phi_h}\right] \\
&+ \quad S_\parallel\lambda_e\left[\sqrt{1-\varepsilon^2}\,F_{LL}+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\,F_{LL}^{\cos\phi_h}\right] \\
&+ \quad |\boldsymbol{S}_\perp|\bigg[\sin(\phi_h-\phi_S)\left(F_{UT,T}^{\sin(\phi_h-\phi_S)}+\varepsilon\,F_{UT,L}^{\sin(\phi_h-\phi_S)}\right) \\
&+ \quad \varepsilon\,\sin(\phi_h+\phi_S)\,F_{UT}^{\sin(\phi_h+\phi_S)}+\varepsilon\,\sin(3\phi_h-\phi_S)\,F_{UT}^{\sin(3\phi_h-\phi_S)} \\
&+ \quad \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\,F_{UT}^{\sin\phi_S}+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin(2\phi_h-\phi_S)\,F_{UT}^{\sin(2\phi_h-\phi_S)}\bigg] \\
&+ \quad |\boldsymbol{S}_\perp|\lambda_e\bigg[\sqrt{1-\varepsilon^2}\,\cos(\phi_h-\phi_S)\,F_{LT}^{\cos(\phi_h-\phi_S)}+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\,F_{LT}^{\cos\phi_S} \\
&+ \quad \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos(2\phi_h-\phi_S)\,F_{LT}^{\cos(2\phi_h-\phi_S)}\bigg]\Bigg\},
\end{aligned}
$$

*Kotzinian NPB 95,*
*Mulders Tangermann NPB 96,*
*Bacchetta et al JHEP 08*




# Structure functions projected from cross section

$$
A_{XY}^{\mathcal{F}} \equiv 2\,\frac{\int d\phi_h\,d\phi_S\,\mathcal{F}(\phi_h,\phi_S)\;\left(d\sigma^{\uparrow}-d\sigma^{\downarrow}\right)}{\int d\phi_h d\phi_S\,\left(d\sigma^{\uparrow}+d\sigma^{\downarrow}\right)}\,,
$$

$XY$-polarization e.g.

$\mathcal{F}(\phi_h,\phi_S)=\sin(\phi_h-\phi_S).$

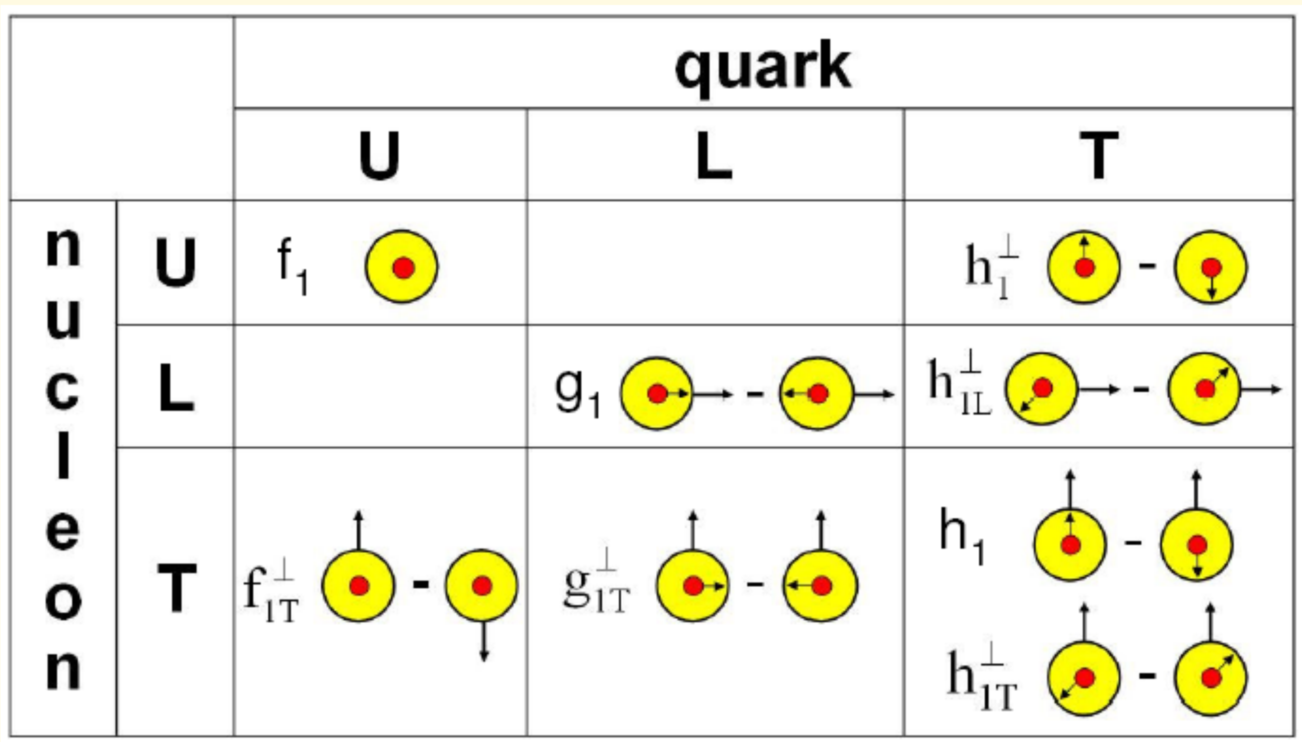
# Thus 8 "LT" TMDs: Correlation Matrix Dirac space

$$\Phi^{[\gamma^+]}(x,\boldsymbol{p}_T) \equiv f_1(x,\boldsymbol{p}_T^2) + \frac{\epsilon_T^{ij} p_{Ti} S_{Tj}}{M} f_{1T}^{\perp}(x,\boldsymbol{p}_T^2)$$

$$\Phi^{[\gamma^+\gamma_5]}(x,\boldsymbol{p}_T) \equiv \lambda\, g_{1L}(x,\boldsymbol{p}_T^2) + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M} g_{1T}(x,\boldsymbol{p}_T^2)$$

$$\Phi^{[i\sigma^{i+}\gamma_5]}(x,\boldsymbol{p}_T) \equiv S_T^i\, h_{1T}(x,\boldsymbol{p}_T^2) + \frac{p_T^i}{M}\left(\lambda\, h_{1L}^{\perp}(x,\boldsymbol{p}_T^2) + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M} h_{1T}^{\perp}(x,\boldsymbol{p}_T^2)\right)$$

$$+ \frac{\epsilon_T^{ij} p_T^j}{M} h_1^{\perp}(x,\boldsymbol{p}_T^2)$$



# Weighted asymmetries Model independent Deconvoltuion of CS in terms of moments of TMDs

Kotzinian, Mulders PLB 97, Boer, Mulders PRD 98

$$A_{UT,T}^{w_1 \sin(\phi_h - \phi_S)} = 2\,\frac{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}| d\phi_h\, d\phi_S\, w_1(|\boldsymbol{P}_{h\perp}|)\ \sin(\phi_h - \phi_S)\ \{d\sigma(\phi_h, \phi_S) - d\sigma(\phi_h, \phi_S + \pi)\}}{\int d|\boldsymbol{P}_{h\perp}|\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d\phi_S\, w_0(|\boldsymbol{P}_{h\perp}|)\ \{d\sigma(\phi_h, \phi_S) + d\sigma(\phi_h, \phi_S + \pi)\}},$$

**e.g.** $$\mathcal{W}_{\text{Sivers}} = \frac{|\boldsymbol{P}_{h\perp}|}{zM} \sin(\phi_h - \phi_S)$$

$$A_{UT}^{\frac{|\boldsymbol{P}_{h\perp}|}{z_h M} \sin(\phi_h - \phi_s)} = -2\,\frac{\sum_a e_a^2\ f_{1T}^{\perp(1)}(x)\ D_1^{a(0)}(z)}{\sum_a e_a^2\ f_1^{a(0)}(x)\ D_1^{a(0)}(z)}$$

***Undefined w/o regularization to subtract infinite contribution at large transverse momentum***

Bacchetta et al. JHEP 08

# Comments

- Propose generalize Bessel Weights-"BW"
- BW procedure has advantages
  - ★ Structure functions become simple product $\mathcal{P}[\quad]$ rather than convolution $\mathcal{C}[\quad]$
  - ★ CS has simpler S/T interpretation as a $b_T\,[\mathrm{GeV}^{-1}]$ multipole expansion in terms of conjugate to $\boldsymbol{P}_{h\perp}$
  - ★ Use Fourier Bessel tranforms
  - ★ The usefulness of Fourier-Bessel transforms in studying the factorization as well as the scale dependence of transverse momentum dependent cross section has been known for quite sometime

    Collins Soper (81), Ellis,Fleishon,Stirling (81), Ji,Ma,Yuan (05),Collins, Found. of PQCD, Cambridge University Press(11), Aybat Rogers PRD (11), Abyat, Collins, Qiu, Rogers arXiv(11), Aybat, Prokudin, Rogers arXiv (11), Anselminio, Bolglione, Melis arXiv (12)

# Further Comments

- Introduces a free parameter $\mathcal{B}_T\ [\mathrm{GeV}^{-1}]$ Fourier conjugate to $\boldsymbol{P}_{h\perp}$
- Provides a regularization of infinite contributions at lg. transverse momentum when $\mathcal{B}_T^2$ is non-zero for moments
- Addtnl. bonus soft factor eliminated from weighted asymmetries
- Cancellation of perturbative Sudakov Broadening (new)-mentioned by D. Boer
- Cancellation of hard cross section-new observation (new)
- Possible to compare observable at different scales.... could be useful for an EIC

# Advantages of Bessel Weighting

1. "Deconvolution"-CS-struct fncts simple product "$\mathcal{P}$"

$$W^{\mu\nu}(\boldsymbol{P}_{h\perp}) \equiv \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2}\, e^{-i\boldsymbol{b}_T\cdot\boldsymbol{P}_{h\perp}}\, \tilde{W}^{\mu\nu}(\boldsymbol{b}_T)\,,$$

$$\tilde{\Phi}_{ij}(x, z\boldsymbol{b}_T) \equiv \int d^2\boldsymbol{p}_T\, e^{iz\boldsymbol{b}_T\cdot\boldsymbol{p}_T}\, \Phi_{ij}(x, \boldsymbol{p}_T)$$

$$\tilde{\Delta}_{ij}(z, \boldsymbol{b}_T) \equiv \int d^2\boldsymbol{K}_T\, e^{i\boldsymbol{b}_T\cdot\boldsymbol{K}_T}\, \Delta_{ij}(z, \boldsymbol{K}_T)$$

$$\frac{d\sigma}{dx_B\, dy\, d\psi\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d|\boldsymbol{P}_{h\perp}|} = \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2} e^{-i\boldsymbol{b}_T\cdot\boldsymbol{P}_{h\perp}} \left\{ \frac{\alpha^2}{x_B y Q^2} \frac{y^2}{(1-\varepsilon)} \left(1 + \frac{\gamma^2}{2x_B}\right) L_{\mu\nu}\tilde{W}^{\mu\nu} \right\}.$$

$$2M\tilde{W}^{\mu\nu} = \sum_a e_a^2\, \mathrm{Tr}\left(\tilde{\Phi}(x, z\boldsymbol{b}_T)\gamma^\mu \tilde{\Delta}(z, \boldsymbol{b}_T)\gamma^\nu\right).$$

## 1.“Deconvolution”-Sivers struct fnct simple product “$\mathcal{P}$“

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\right],$$

“dipole structure”

$$\mathcal{C}\big[w\, f\, D\big] = x \sum_a e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}\big(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\big)\, w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x, p_T^2)\, D^a(z, k_T^2)$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = -x_B \sum_a e_a^2 \int \frac{d|\boldsymbol{b}_T|}{(2\pi)} |\boldsymbol{b}_T|^2 J_1(|\boldsymbol{b}_T|\,|\boldsymbol{P}_{h\perp}|) M z\ \tilde{f}_{1T}^{\perp a(1)}(x, z^2\boldsymbol{b}_T^2)\, \tilde{D}_1^a(z, \boldsymbol{b}_T^2).$$

$\tilde{f}_1,\ \tilde{f}_{1T}^{\perp(1)}$, and $\tilde{D}_1$ are Fourier Transf. of TMDs/FFs and finite

- Transversity and Collins

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = \mathcal{C}\left[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\,(\boldsymbol{p}_T\cdot\boldsymbol{k}_T)+\boldsymbol{p}_T^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)-4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)}{2M^2M_h}\,h_{1T}^{\perp}H_1^{\perp}\right]$$

write out in cylindrical polar--
traceless tensor irreducible
tensor no mixture of Bessels "$J_3$"

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = x_B\sum_a e_a^2\int\frac{d|\boldsymbol{b}_T|}{(2\pi)}|\boldsymbol{b}_T|^4 J_3(|\boldsymbol{b}_T|\,|\boldsymbol{P}_{h\perp}|)\frac{M^2M_hz^3}{4}\,\tilde{h}_{1T}^{\perp a(2)}(x,z^2\boldsymbol{b}_T^2)\,\tilde{H}_1^{\perp a(1)}(z,\boldsymbol{b}_T^2)\,.$$

Simple product " $\mathcal{P}$ "

# TMDs in “config” space--Bessel MOMENTS

a) F.T. SIDIS cross section w/ following Bessel moments

$$
\begin{aligned}
\tilde{f}(x,\boldsymbol{b}_T^2) &\equiv \int d^2\boldsymbol{p}_T\, e^{i\boldsymbol{b}_T\cdot\boldsymbol{p}_T}\; f(x,\boldsymbol{p}_T^2) \\
&= 2\pi \int d|\boldsymbol{p}_T||\boldsymbol{p}_T|\; J_0(|\boldsymbol{b}_T||\boldsymbol{p}_T|)\; f^a(x,\boldsymbol{p}_T^2)\ , \\
\tilde{f}^{(n)}(x,\boldsymbol{b}_T^2) &\equiv n!\left(-\frac{2}{M^2}\partial_{\boldsymbol{b}_T^2}\right)^n \tilde{f}(x,\boldsymbol{b}_T^2) \\
&= \frac{2\pi\ n!}{(M^2)^n}\int d|\boldsymbol{p}_T||\boldsymbol{p}_T|\left(\frac{|\boldsymbol{p}_T|}{|\boldsymbol{b}_T|}\right)^n J_n(|\boldsymbol{b}_T||\boldsymbol{p}_T|)\; f(x,\boldsymbol{p}_T^2)\ ,
\end{aligned}
$$

b) n.b. connection to $\boldsymbol{p}_T$ moments

$$
\tilde{f}^{(n)}(x,0) = \int d^2\boldsymbol{p}_T \left(\frac{\boldsymbol{p}_T^2}{2M^2}\right)^n f(x,\boldsymbol{p}_T^2) \equiv f^{(n)}(x)
$$

# ★ CS has simpler ***S/T*** interpretation--multipole expansion in terms of $b_T\,[\mathrm{GeV}^{-1}]$ conjugate to $\boldsymbol{P}_{h\perp}$

$$
\begin{aligned}
&\frac{d\sigma}{dx_B\, dy\, d\phi_S\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d|\boldsymbol{P}_{h\perp}|} = \\
&\frac{\alpha^2}{x_B\, y\, Q^2}\, \frac{y^2}{(1-\varepsilon)} \left(1+\frac{\gamma^2}{2x_B}\right) \int \frac{d|\boldsymbol{b}_T|}{(2\pi)} |\boldsymbol{b}_T| \Big\{ J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UU,T} + \varepsilon\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UU,L} \\
&+ \quad \sqrt{2\,\varepsilon(1+\varepsilon)}\, \cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UU}^{\cos\phi_h} + \varepsilon \cos(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UU}^{\cos(2\phi_h)} \\
&+ \lambda_e\, \sqrt{2\,\varepsilon(1-\varepsilon)}\, \sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LU}^{\sin\phi_h} \\
&+ \quad S_\parallel \Big[ \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UL}^{\sin\phi_h} + \varepsilon \sin(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UL}^{\sin 2\phi_h} \Big] \\
&+ S_\parallel \lambda_e \Big[ \sqrt{1-\varepsilon^2}\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LL} + \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LL}^{\cos\phi_h} \Big] \\
&+ |\boldsymbol{S}_\perp| \Big[ \sin(\phi_h-\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|) \left( \mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon\, \mathcal{F}_{UT,L}^{\sin(\phi_h-\phi_S)} \right) \\
&\qquad + \ \varepsilon\, \sin(\phi_h+\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} \\
&\qquad + \ \varepsilon\, \sin(3\phi_h-\phi_S)\, J_3(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} \\
&\qquad + \ \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin\phi_S\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UT}^{\sin\phi_S} \\
&\qquad + \ \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin(2\phi_h-\phi_S)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UT}^{\sin(2\phi_h-\phi_S)} \Big] \\
&+ |\boldsymbol{S}_\perp| \lambda_e \Big[ \sqrt{1-\varepsilon^2}\, \cos(\phi_h-\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LT}^{\cos(\phi_h-\phi_S)} \\
&\qquad + \ \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos\phi_S\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LT}^{\cos\phi_S} \\
&\qquad + \ \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos(2\phi_h-\phi_S)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LT}^{\cos(2\phi_h-\phi_S)} \Big] \Big\}
\end{aligned}
$$

$$\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} = \frac{1}{4}\mathcal{P}[\tilde{h}_{1T}^{\perp(2)}\ \tilde{H}_1^{\perp(1)}]$$

# Structure Functions become

$$\mathcal{F}_{UU,T} = \mathcal{P}[\tilde{f}_1^{(0)}\ \tilde{D}_1^{(0)}],$$

$$\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} = -\mathcal{P}[\tilde{f}_{1T}^{\perp(1)}\ \tilde{D}_1^{(0)}],$$

$$\mathcal{F}_{LL} = \mathcal{P}[\tilde{g}_{1L}^{(0)}\ \tilde{D}_1^{(0)}],$$

$$\mathcal{F}_{LT}^{\cos(\phi_h-\phi_s)} = \mathcal{P}[\tilde{g}_{1T}^{(1)}\ \tilde{D}_1^{(0)}],$$

$$\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} = \mathcal{P}[\tilde{h}_1^{(0)}\ \tilde{H}_1^{\perp(1)}],$$

$$\mathcal{F}_{UU}^{\cos(2\phi_h)} = \mathcal{P}[\tilde{h}_1^{\perp(1)}\ \tilde{H}_1^{\perp(1)}],$$

$$\mathcal{F}_{UL}^{\sin(2\phi_h)} = \mathcal{P}[\tilde{h}_{1L}^{\perp(1)}\ \tilde{H}_1^{\perp(1)}],$$

$$\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} = \frac{1}{4}\mathcal{P}[\tilde{h}_{1T}^{\perp(2)}\ \tilde{H}_1^{\perp(1)}].$$

$$\mathcal{P}[\tilde{f}^{(n)}\,\tilde{D}^{(m)}] \equiv x_B \sum e_a^2\,(zM|\boldsymbol{b}_T|)^n\,(zM_h|\boldsymbol{b}_T|)^m\,\tilde{f}^{a(n)}(x, z^2\boldsymbol{b}_T^2)\,\tilde{D}^{a(m)}(z, \boldsymbol{b}_T^2)$$

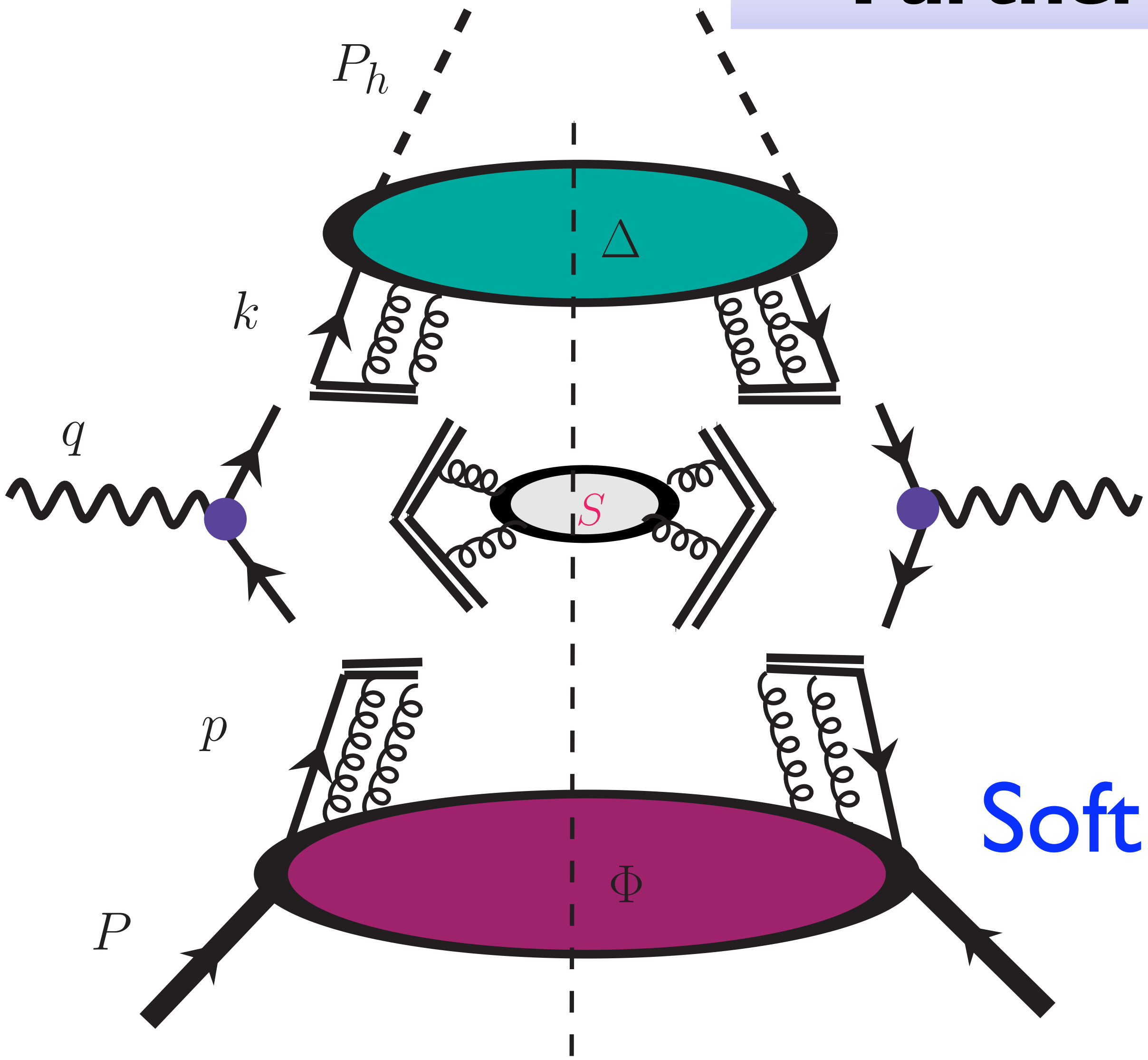
# Correlator w/ explicit *spin orbit* correlations

$$\tilde{\Phi}^{[\gamma^+]}(x,\boldsymbol{b}_T) = \tilde{f}_1(x,\boldsymbol{b}_T^2) - i\,\epsilon_T^{\rho\sigma} b_{T\rho} S_{T\sigma}\, M\, \tilde{f}_{1T}^{\perp(1)}(x,\boldsymbol{b}_T^2)\,,$$

$$\tilde{\Phi}^{[\gamma^+\gamma^5]}(x,\boldsymbol{b}_T) = S_L\,\tilde{g}_{1L}(x,\boldsymbol{b}_T^2) + i\,\boldsymbol{b}_T\cdot\boldsymbol{S}_T M\,\tilde{g}_{1T}^{(1)}(x,\boldsymbol{b}_T^2)\,,$$

$$\begin{aligned}\tilde{\Phi}^{[i\sigma^{\alpha+}\gamma^5]}(x,\boldsymbol{b}_T) &= S_T^\alpha\,\tilde{h}_1(x,\boldsymbol{b}_T^2) + i\,S_L\,b_T^\alpha M\,\tilde{h}_{1L}^{\perp(1)}(x,\boldsymbol{b}_T^2)\\ &+\frac{1}{2}\left(b_T^\alpha b_T^\rho + \frac{1}{2}\,\boldsymbol{b}_T^2\,g_T^{\alpha\rho}\right)M^2\,S_{T\rho}\tilde{h}_{1T}^{\perp(2)}(x,\boldsymbol{b}_T^2)\\ &-i\,\epsilon_T^{\alpha\rho} b_{T\rho} M\tilde{h}_1^{\perp(1)}(x,\boldsymbol{b}_T^2)\,,\end{aligned}$$

# Further Beyond "tree level" factorization




CS NPB 81,CSS NPB 1985 Collins, Hautman PLB 00, Adilbi, Ji, Ma, Yuan PRD 05, Cherednikov, Karanikas, Stefanis NPB 10, Collins Oxford Press 2011, Abyat, Rogers PRD 2011, Abyat, Collins, Qiu, Rogers arXiv 2011 ... Echevarria,Idilbi, Scimemi arXiv2012

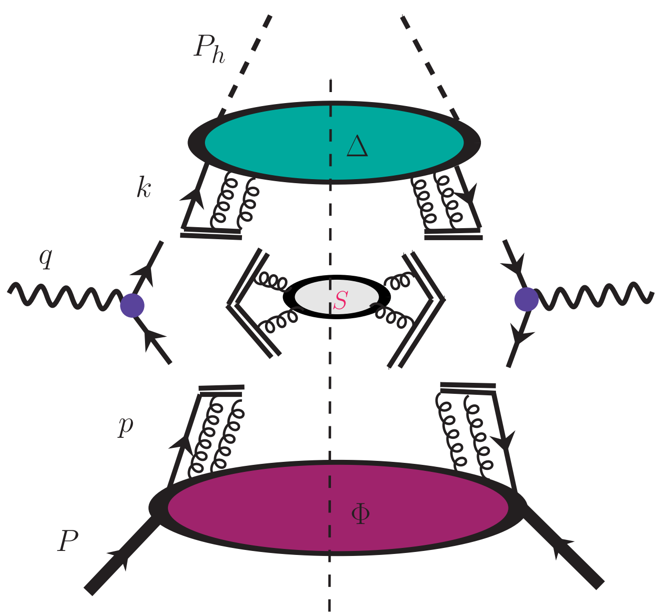
Soft factor

- Extra divergences at one loop and higher
- Various strategies to address them
- Extra variables needed to regulate divergences
- Modifies convolution integral by introduction soft factor
- Will show cancels in Bessel weighted asymmetries

Collins Soper NPB 1981, Collins Metz PRL 2004, Ji, Ma, Yuan PRD 2005, also Bacchetta Boer Diehl Mulders JHEP 2008

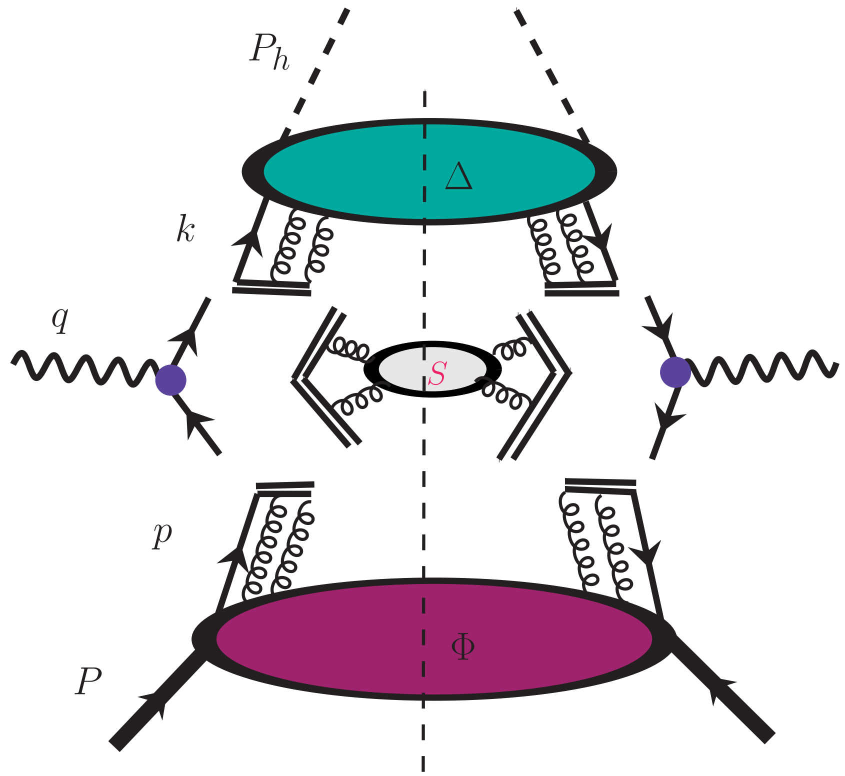
# Comments on Soft factor

- Collective effect soft gluons not associated with distribution frag function-factorizes into a matrix of Wilson lines in QCD vacuum
- Subtracts soft divergences from TMD pdf and FF
- Considered to be universal in hard processes
(Collins & Metz PRL 04, Ji, Ma, Yuan PRD 05)
- At tree level (zeroth order $\alpha_s$ ) unity-parton model
- Absent tree level pheno analyses of experimental data
(e.g. Anselmino et al PRD 05 & 07, Efremov et al PRD 07)




- Potentially, results of analyses can be difficult to compare at different energies **issue for EIC**
- Correct description of energy scale dependence of cross section and asymmetries in TMD picture, soft factor must be included
( Ji, Ma, Yuan 2005, Collins Oxford Press 2011, Abyat, Collins, Rogers PRD 2011)
- However, possible to consider observables where it cancels e.g. weighted asymmetries Boer, LG, Musch, Prokudin JHEP 2011

# Momentum space convolution




Adilbi, Ji, Ma, Yuan PRD 05 ....

Hard

$$\mathcal{C}\left[H; wfSD\right] \equiv x_B H(Q^2, \mu^2, \rho) \sum_a e_a^2 \int d^2p_T\, d^2K_T\, d^2\ell_T\, \delta^{(2)}(zp_T + K_T + \ell_T - P_{h\perp})\, w\left(p_T, -\frac{K_T}{z}\right)$$
$$\times f^a(x, p_T^2, \mu^2, x\zeta, \rho)\, \underline{S(\ell_T^2, \mu^2, \rho)}\, D^a(z, K_T^2, \mu^2, \hat{\zeta}/z, \rho)$$

TMD

Soft

FF

# Crucial property of Soft Factor-SIDIS

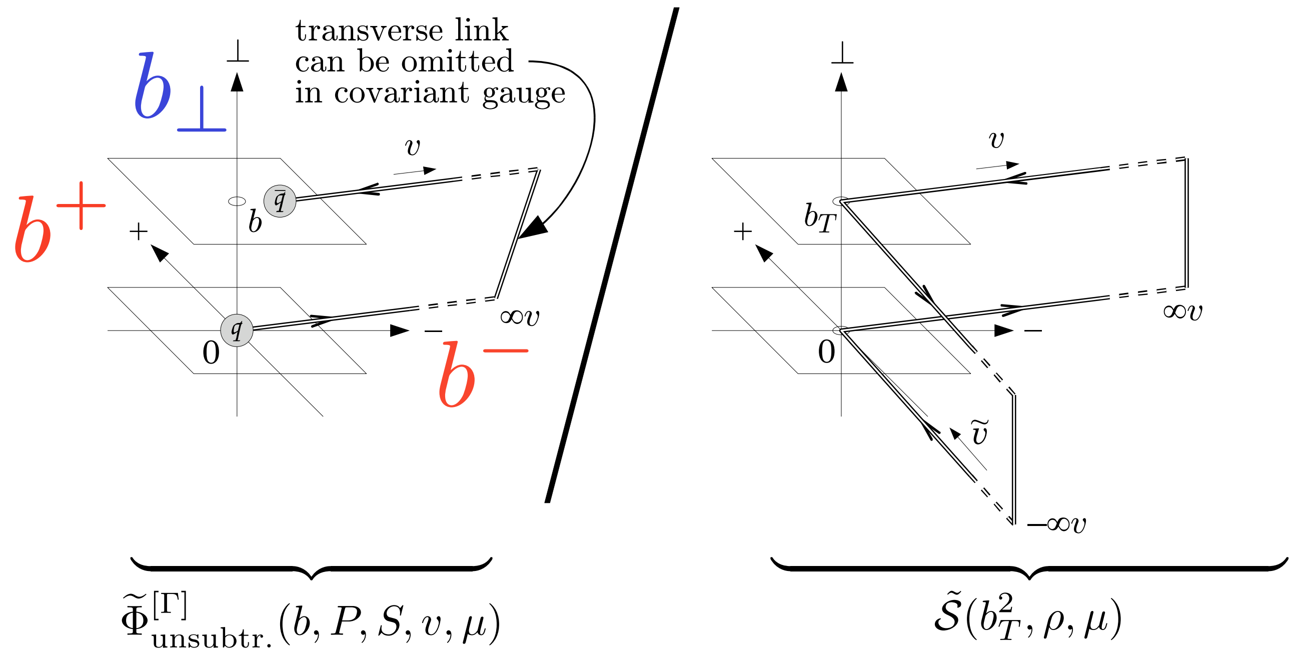
Soft factor formed from vacuum expt. value of Wilson lines involving both $v$ and $\tilde{v}$ thus depends on relative orientation of directions $\rho = \sqrt{v^- \tilde{v}^+ / v^+ \tilde{v}^-}$

$\tilde{S}^+(\boldsymbol{b}_T, \rho, \mu)$ is invariant under rotations of the $\boldsymbol{b}_T$-vector (provided $b \cdot v = 0$).

Since for TMDs we always consider the case $b^+ = 0$, we have $\boldsymbol{b}_T^2 = -b^2$,

$$\longrightarrow \quad \tilde{S}^+(b^2, \rho, \mu)$$

# Subtracted correlator off light cone




$$b \cdot v = 0$$

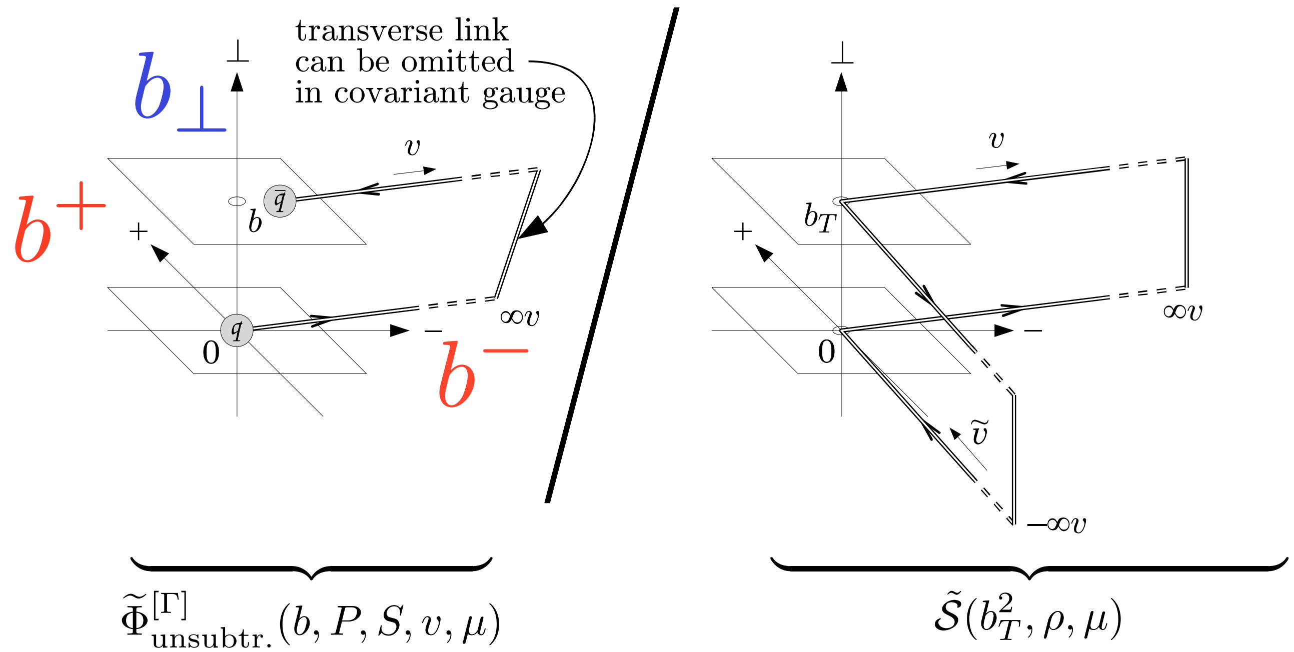
$$v = (v^-, v^+, 0)$$ **w/** lightlike directions $n = (1,0,0)$, $\bar{n} = (0,1,0)$.

# Again consider JMY framework

$$\Phi^{(+)[\Gamma]}(x, \boldsymbol{p}_T, P, S, \mu^2, \zeta, \rho) = \int \frac{db^-}{(2\pi)} \, e^{ixb^-P^+} \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2} \, e^{-i\boldsymbol{p}_T\cdot\boldsymbol{b}_T} \qquad .5$$

$$\times \underbrace{\frac{1}{2} \langle P, S| \; \bar{\psi}(0)\, \mathcal{U}[\mathcal{C}_b]\, \Gamma\, \psi(b) \; |P, S\rangle}_{\widetilde{\Phi}^{[\Gamma]}_{\text{unsub}}(b, P, S; v, \mu^2)} \Big/ \widetilde{S}^{(+)}(\boldsymbol{b}_T^2, \mu^2, \rho)\Big|_{b^+=0},$$

# First summarize what we know about correlator off light cone




$v = (v^-, v^+, 0)$ **w/** lightlike directions $n = (1,0,0)$, $\bar{n} = (0,1,0)$.

is slightly off light-cone direction $n$ & $b \cdot v = 0$

Wilson lines starting at infinity running along a direction given by the four-vector $v$ to an endpoint $a$ are denoted $\mathcal{L}_v(\infty; a)$

Direction defined in LI way $\zeta^2 = (2P \cdot v)^2 / v^2$

Direction defined in LI way $\hat{\zeta}^2 = (2P_h \cdot \tilde{v})^2 / \tilde{v}^2$

angle between $v$ and $\tilde{v}$ $\rho = \sqrt{v^- \tilde{v}^+ / v^+ \tilde{v}^-}$

**scales arising from regulating LC div gluon rap. cutoff**

# Soft factor in deconvoluted Fourier Bessel rep of CS

## $\mathcal{P}$ versus $\mathcal{C}$

$$\frac{d\sigma}{dx_B\, dy\, d\phi_S\, dz_h\, d\phi_h\, d|\boldsymbol{P}_{h\perp}|^2} \propto \frac{\alpha^2}{x_B Q^2} \int \frac{d|\boldsymbol{b}_T|}{(2\pi)} |\boldsymbol{b}_T|\, \tilde{S}(\boldsymbol{b}_T^2) \Big\{ \quad \ldots$$

$$+ J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{P}[\tilde{f}_1\ \tilde{D}_1]$$

Soft factor is

- spin blind
- flavor blind
- factors in $\mathcal{P}$
- Universal

Idilbi,Ji,Ma,Yuan PRD 05

$$+ |\boldsymbol{S}_\perp| \sin(\phi_h - \phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{P}[\tilde{f}_{1T}^{\perp(1)}\ \tilde{D}_1]$$

$$+ \varepsilon \cos(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{P}[\tilde{h}_1^{\perp(1)}\ \tilde{H}_1^{\perp(1)}]$$

$$+ \quad \ldots 15 \quad \text{more} \quad \text{structure} \quad \text{functions}$$

## Products in terms of " $\boldsymbol{b}_T$ moments "

$$\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} = H_{UT,T}^{\sin(\phi_h-\phi_S)}(Q^2,\mu^2,\rho)\ \tilde{S}^{(+)}(\boldsymbol{b}_T^2,\mu^2,\rho)\ \mathcal{P}[\tilde{f}_{1T}^{(1)}\,\tilde{D}_1^{(0)}] + \tilde{Y}_{UT,T}^{\sin(\phi_h-\phi_S)}(Q^2,\boldsymbol{b}_T^2)\ .$$

$$\mathcal{P}[\tilde{f}^{(n)}\tilde{D}^{(m)}] \equiv x_B \sum e_a^2 (zM|\boldsymbol{b}_T|)^n (zM_h|\boldsymbol{b}_T|)^m\, \tilde{f}^{a(n)}(x, z^2\boldsymbol{b}_T^2, \mu^2, \zeta, \rho)\tilde{D}^{a(m)}(z, \boldsymbol{b}_T^2, \mu^2, \hat{\zeta}, \rho)$$

# 2. Bessel Weighting & cancellation of soft factor

Bessel weighting-projecting out Sivers using orthogonality of Bessel Fncts.

$$\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM} = \frac{2\,J_1(|\boldsymbol{P}_{hT}|\mathcal{B}_T)}{zM\mathcal{B}_T}$$

$$A_{UT}^{\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM}\sin(\phi_h-\phi_S)}(\mathcal{B}_T) =$$

$$2\frac{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}|\,d\phi_h\,d\phi_S\,\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM}\sin(\phi_h-\phi_S)\,(d\sigma^{\uparrow}-d\sigma^{\downarrow})}{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}|\,d\phi_h\,d\phi_S\;\mathcal{J}_0^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)\;(d\sigma^{\uparrow}+d\sigma^{\downarrow})}$$

$$A_{UT}^{\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM}\sin(\phi_h-\phi_s)}(\mathcal{B}_T) =$$

$$-2\frac{\cancel{\tilde{S}(\mathcal{B}_T^2)}\,H_{UT,T}^{\sin(\phi_h-\phi_S)}(Q^2)\sum_a e_a^2\,\tilde{f}_{1T}^{\perp(1)a}(x,z^2\mathcal{B}_T^2)\,\tilde{D}_1^a(z,\mathcal{B}_T^2)}{\cancel{\tilde{S}(\mathcal{B}_T^2)}\,H_{UU,T}(Q^2)\sum_a e_a^2\;\tilde{f}_1^a(x,z^2\mathcal{B}_T^2)\,\tilde{D}_1^a(z,\mathcal{B}_T^2)}$$

# Sivers asymmetry with full dependences

$$A_{UT}^{\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM}\sin(\phi_h-\phi_s)}(\mathcal{B}_T) \quad =$$

$$-2\frac{\tilde{S}(\mathcal{B}_T^2,\mu^2,\rho^2)\,H_{UT,T}^{\sin(\phi_h-\phi_S)}(Q^2,\mu^2,\rho)\,\sum_a e_a^2\,\tilde{f}_{1T}^{\perp(1)a}(x,z^2\mathcal{B}_T^2;\mu^2,\zeta,\rho)\,\tilde{D}_1^a(z,\mathcal{B}_T^2;\mu^2,\hat{\zeta},\rho)}{\tilde{S}(\mathcal{B}_T^2,\mu^2,\rho^2)\,H_{UU,T}(Q^2,\mu^2,\rho)\,\sum_a e_a^2\,\tilde{f}_1^a(x,z^2\mathcal{B}_T^2;\mu^2,\zeta,\rho)\,\tilde{D}_1^a(z,\mathcal{B}_T^2;\mu^2,\hat{\zeta},\rho)}$$

## 3. Circumvents the problem of ill-defined $p_T$ moments

$$A_{UT}^{\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM}\sin(\phi_h-\phi_s)}(\mathcal{B}_T) \quad =$$

$$-2\frac{\cancel{\tilde{S}(\mathcal{B}_T^2,\mu^2,\rho^2)}H_{UT,T}^{\sin(\phi_h-\phi_S)}(Q^2,\mu^2,\rho)\sum_a e_a^2\,\tilde{f}_{1T}^{\perp(1)a}(x,z^2\mathcal{B}_T^2;\mu^2,\zeta,\rho)\,\tilde{D}_1^a(z,\mathcal{B}_T^2;\mu^2,\hat{\zeta},\rho)}{\cancel{\tilde{S}(\mathcal{B}_T^2,\mu^2,\rho^2)}H_{UU,T}(Q^2,\mu^2,\rho)\sum_a e_a^2\,\tilde{f}_1^a(x,z^2\mathcal{B}_T^2;\mu^2,\zeta,\rho)\,\tilde{D}_1^a(z,\mathcal{B}_T^2;\mu^2,\hat{\zeta},\rho)}$$

**Traditional weighted asymmetry recovered but UV divergent**

$$\lim_{\mathcal{B}_T\to 0} w_1 = 2J_1(|\boldsymbol{P}_{h\perp}|\mathcal{B}_T)/zM\mathcal{B}_T \longrightarrow |\boldsymbol{P}_{h\perp}|/zM$$

$$A_{UT}^{\frac{|\boldsymbol{P}_{h\perp}|}{z_h M}\sin(\phi_h-\phi_s)} = -2\frac{\sum_a e_a^2\, f_{1T}^{\perp(1)}(x)\ D_1^{a(0)}(z)}{\sum_a e_a^2\, f_1^{a(0)}(x)\ D_1^{a(0)}(z)}$$

Bacchetta et al. JHEP 08

*undefined w/o regularization*

# CS resummation 81

When $\Lambda^2_{QCD} \ll P^2_{h\perp} \ll Q^2$ get large double logs
talks of Qiu, Kang, Idilbi, Scimemi, Guzi .....

$$\mathcal{F}_{UT}(x,z,b,Q^2) = \tilde{f}_{1T}^{\perp(1)}(x, z^2\boldsymbol{b}^2, \mu_L^2, \mu_L^2/C_2, \rho)\tilde{D}_1(z_h, \boldsymbol{b}^2, \mu_L^2, \mu_L^2/C_2, \rho)$$
$$\times\, \tilde{S}(\boldsymbol{b}_T^2, \mu_L^2, \rho)\tilde{H}_{UT}(1/C_2^2\rho, \rho)e^{-S_{\rm hard}}e^{-S_{UT}^{NP}}$$

# Evolution of TMDs

- Needs to resum double logarithms, typically it involves two steps:
  - Energy evolution of the unpolarized PDFs

Idilbi-Ji-Ma-Yuan, 2004

$$\zeta\frac{\partial}{\partial\zeta}q(x,b,\mu^2,x\zeta,\rho)=\left(K(\mu,b)+G(\mu,x\zeta)\right)q(x,b,\mu^2,x\zeta,\rho)$$

  - Since it contains double logarithms, the kernel still contains single logarithms

$$\mu\frac{d}{d\mu}K(\mu,b)=-\gamma_K=-\mu\frac{d}{d\mu}G(\mu,\zeta)$$

- Solving two equations--equivalently one resums the double logs
  - First for the evolution equation of K and G

$$K(b,\mu)+G(x\zeta,\mu)=K(b,\mu_L)+G(x\zeta,\mu_H)-\int_{\mu_L}^{\mu_H}\frac{d\tilde{\mu}}{\tilde{\mu}}\gamma_K(\alpha(\tilde{\mu}))$$

  - Then feed the solution back to the energy evolution equation

$$q(x,b,\mu,x\zeta,\rho)\;=\;\exp\left\{-\int_{\mu_L}^{C_2x\zeta}\frac{d\mu}{\mu}\left[\ln\left(\frac{C_2x\zeta}{\mu}\right)\gamma_K(\alpha(\mu))-K(b,\mu_L)-G(\mu/C_2,\mu)\right]\right\}$$
$$\times q(x,b,\mu,x\zeta_0=\mu_L/C_2,\rho)\ ,$$

# The formalism contains all the evolutions

- Similar for the unpolarized fragmentation function

$$\hat{q}(z,b,\mu,\hat{\zeta}/z,\rho) = \exp\left\{-\int_{\mu_L}^{C_2\hat{\zeta}/z}\frac{d\mu}{\mu}\left[\ln\left(\frac{C_2\hat{\zeta}}{z\mu}\right)\gamma_K(\alpha(\mu)) - K(b,\mu_L) - G(\mu/C_2,\mu)\right]\right\}$$
$$\times\hat{q}(z,b,\mu,\hat{\zeta}_0/z = \mu_L/C_2,\rho)\ .$$

- Hard function and Soft function contain only single logs

$$\mu\frac{\partial S(\vec{b}_\perp,\mu^2,\rho)}{\partial\mu} = \gamma_S(\rho)S(\vec{b}_\perp,\mu^2,\rho)$$

$$\mu\frac{dH(Q^2/\mu^2,\rho)}{d\mu} = -\left(4\gamma_F - \gamma_S(\rho)\right)H(Q^2/\mu^2,\rho)$$

- Eventually collect all the terms

$$F(x_B,z_h,b,Q^2) = q\left(x_B,z_hb,\mu_L^2,\mu_L/C_2,\rho\right)\hat{q}\left(z_h,b,\mu_L^2,\mu_L/C_2,\rho\right)S(b,\mu_L^2,\rho)H\left(1/C_2^2\rho,\rho\right)$$
$$\times\exp\left\{-2\int_{\mu_L}^{C_2Q\sqrt{\rho}}\frac{d\mu}{\mu}\left[\ln\left(\frac{C_2Q\sqrt{\rho}}{\mu}\right)\gamma_K(\alpha(\mu)) - K(b,\mu_L)\right.\right.$$
$$\left.\left.-G(\mu/C_2,\mu) - 2\gamma_F + \frac{1}{2}\gamma_S(\rho)\right]\right\}\ ,$$

# Further Cancellation of Sudakov and hard CS

When $\Lambda^2_{QCD} \ll P_h^2 \ll Q^2$ get large DL

talks of Qiu, Kang, Idilbi, and Scimemi ...

maybe
scaling effect from derivative of Bessel
function

still a scaling effect from derivative
of Bessel function $J_1(|\boldsymbol{b}||\boldsymbol{P}_{h\perp}|)$

$$\mathcal{A}_{UT}(x,z,b,Q^2)$$
$$= \frac{\tilde{f}_{1T}^{\perp(1)}(x,z^2\boldsymbol{b}^2,\mu_L^2,\mu_L^2/C_2,\rho)\tilde{D}_1(z_h,\boldsymbol{b}^2,\mu_L^2,\mu_L^2/C_2,\rho)\tilde{S}(\boldsymbol{b}_T^2,\mu_L^2,\rho)\tilde{H}_{UT}(1/C_2^2\rho,\rho)e^{-S_{\text{hard}}}e^{-S_{UT??}^{NP}}}{\tilde{f}_1(x,z^2\boldsymbol{b}^2,\mu_L^2,\mu_L^2/C_2,\rho)\tilde{D}_1(z_h,\boldsymbol{b}^2,\mu_L^2,\mu_L^2/C_2,\rho)\tilde{S}(\boldsymbol{b}_T^2,\mu_L^2,\rho)\tilde{H}_{UU}(1/C_2^2\rho,\rho)e^{-S_{\text{hard}}}e^{-S_{UU??}^{NP}}}$$

$e^{-S(b,Q)}$ = Sudakov form factor due to resummation large logs

$$\mathcal{A}_{UT}(x,z,b,Q^2) = \frac{\tilde{f}_{1T}^{\perp(1)}(x,z^2\boldsymbol{b}^2,\mu_L^2,\mu_L^2/C_2,\rho)\tilde{D}_1(z_h,\boldsymbol{b}^2,\mu_L^2,\mu_L^2/C_2,\rho)e^{-S_{UT??}^{NP}}}{\tilde{f}_1(x,z^2\boldsymbol{b}^2,\mu_L^2,\mu_L^2/C_2,\rho)\tilde{D}_1(z_h,\boldsymbol{b}^2,\mu_L^2,\mu_L^2/C_2,\rho)e^{-S_{UU??}^{NP}}}$$

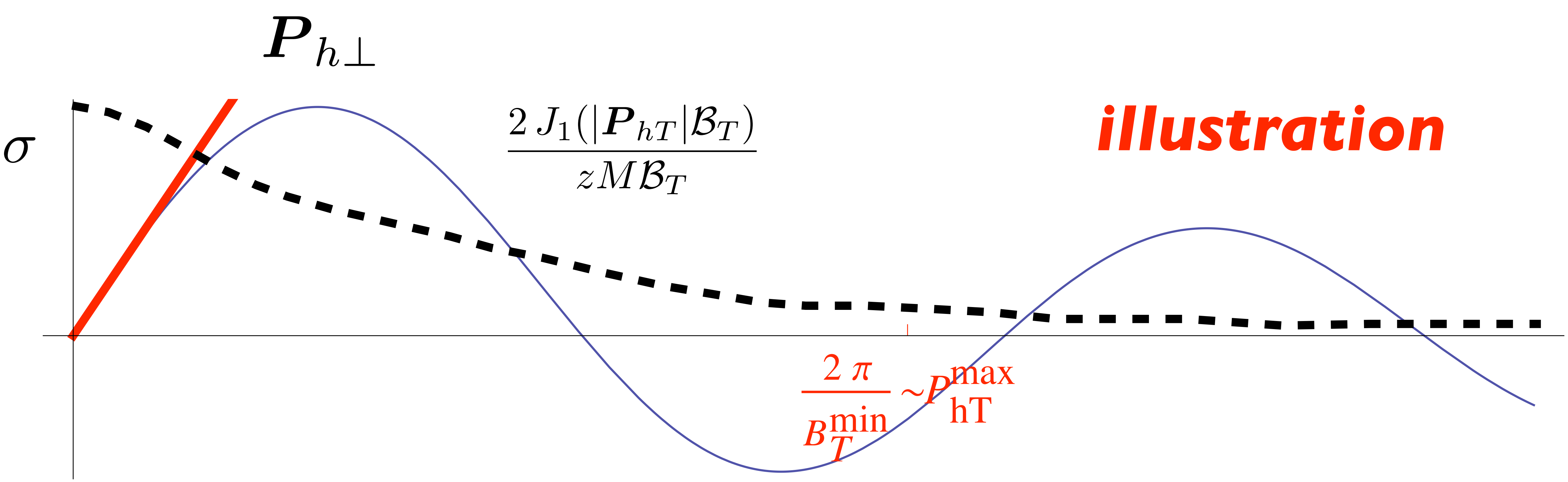
In prep. Boer, LG, Prokudin, Musch

# Conclusions

- Propose generalize Bessel Weights
- Theoretical weighting procedure- advantages
- Introduces a free parameter $\mathcal{B}_T\,[\mathrm{GeV}^{-1}]$ that is Fourier conjugate to $\boldsymbol{P}_{h\perp}$
- Provides a regularization of infinite contributions at lg. transverse momentum when $\mathcal{B}_T^2$ is non-zero
- Soft factor, pertb-Sudakov, and Hard CS eliminated from weighted asymmetries
- Possible to compare observables at different scales.... could be useful for an EIC

# Extracting TMD contribution to Asymmetries
# More sensitive to low $\boldsymbol{P}_{h\perp}$ region

$\mathcal{B}_T$ can serve as a lever arm to enhance the low $\boldsymbol{P}_{h\perp}$ description and possibly dampen lg. momentum tail of cross section. We can use it to scan the cross section



TMD frameworks have been designed to give a good description of the cross section at low transverse momentum, i.e., for $|\boldsymbol{P}_{h\perp}|/z \ll Q$. However, in weighted asymmetries we integrate over the whole range of $|\boldsymbol{P}_{h\perp}|$. The contributions from high $|\boldsymbol{P}_{h\perp}|$ thus lead to theoretical errors in the results if one does not have a description of the cross section that is valid there, even when one restricts to the region $z|\boldsymbol{b}_T| \gg 1/Q$.

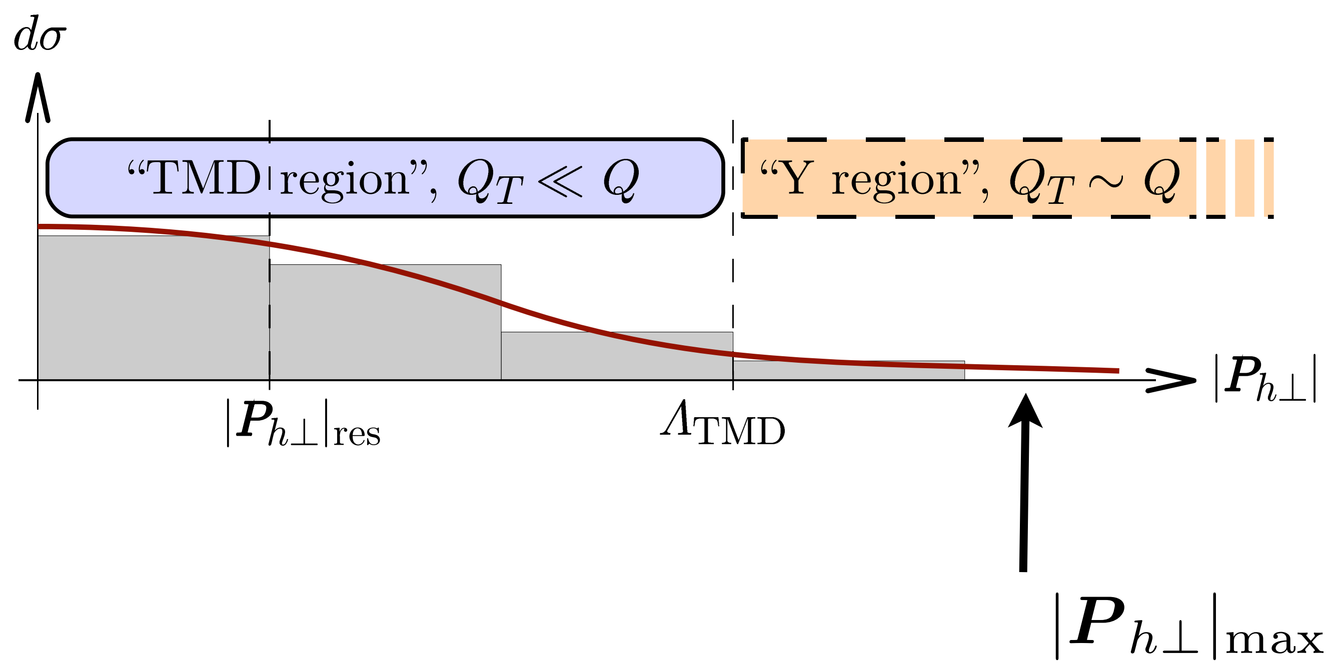
- The $Y$ term in principle included to eliminate errors but its $\mathfrak{FT}$ expected to be power suppressed in region $\boldsymbol{b}_T >> 1/Q$ since was shown to be power suppressed at small
- Thus dropping $Y$ means we approximate the full result by the large $\boldsymbol{P}_{h\perp}$ tail of the TMD expression---is this a bad approx?
- In addition extending integrals to arbitrarily large transverse momentum ignores that the physical cross section should vanish above a certain max trans. momentum

# Bound the error in neglecting $Y$ term

## $Y$ term sig btwn scale $\Lambda_{\mathrm{TMD}}$ and $|\boldsymbol{P}_{h\perp}|_{\mathrm{max}}$

$$\tilde{Y}^{\sin/\cos(N\phi_h+\ldots)}_{XY,Z}(Q^2,\boldsymbol{b}_T^2) \approx \int_{\Lambda_{\mathrm{TMD}}}^{|\boldsymbol{P}_{h\perp}|_{\mathrm{max}}} d|\boldsymbol{P}_{h\perp}|\;|\boldsymbol{P}_{h\perp}|\;2\pi J_N(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\;Y^{\sin/\cos(N\phi_h+\ldots)}_{XY,Z}(Q^2,\boldsymbol{P}_{h\perp}^2)$$

$$\lesssim \;(|\boldsymbol{P}_{h\perp}|_{\mathrm{max}}-\Lambda_{\mathrm{TMD}})\;2\sqrt{\frac{2\pi}{|\boldsymbol{b}_T|\Lambda_{\mathrm{TMD}}}}\;\left|Y^{\sin/\cos(N\phi_h+\ldots)}_{XY,Z}\right|_{\mathrm{max}}.$$

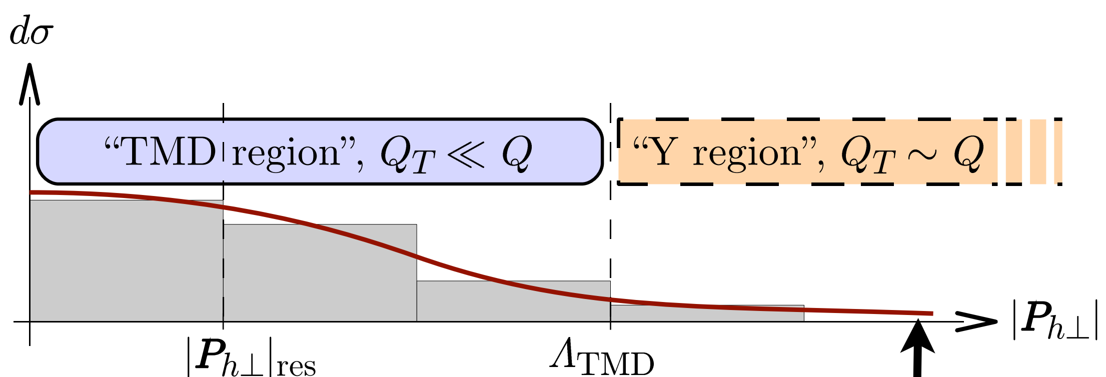


# Error in extending TMD expression into perturbative regime

$$\delta\,\mathcal{F}_{XY,Z}^{\sin/\cos(N\phi_h+\dots)}(x,\boldsymbol{b}_T^2)$$

$$\approx \int_{|\boldsymbol{P}_{h\perp}|_{\max}}^{\infty} d|\boldsymbol{P}_{h\perp}|\;|\boldsymbol{P}_{h\perp}|\;2\pi J_N(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\;[F_{XY,Z}^{\sin/\cos(N\phi_h+\dots)}]_{\mathrm{TMD}}(Q^2,\boldsymbol{P}_{h\perp}^2)$$

$$\lesssim\; 4\sqrt{\frac{2\pi|\boldsymbol{P}_{h\perp}|_{\max}}{|\boldsymbol{b}_T|^3}}\;\left|[F_{XY,Z}^{\sin/\cos(N\phi_h+\dots)}]_{\mathrm{TMD}}(Q^2,|\boldsymbol{P}_{h\perp}|^2_{\max})\right|\,,$$




$\dots$

$|\boldsymbol{P}_{h\perp}|_{\max}$

- Studies on Bessel Weighting being performed by H. Avakian M. A Aghasyan , LG,  Prokudin ...

## Cancellation of Soft Factor on level of the Matrix elements *(summarize)*

- So far we get ratios of moments of TMDs and FFs that are free of soft factor
- It was not necessary to specify explicit def. of TMDs and FFs
- We also analyze ratio of moments of TMDs directly on level of matrix elements of TMDs & FFs
- Again we find cancellation of soft factors in ratio
- Impact for Lattice calculation of moments of TMDS, Musch, Ph. Hagler, M. Engelhardt, J.W. Negele, A. Schafer arXiv 2011

# Generalized av. quark trans. momentum shift
# Soft Factor cancels

$\langle \boldsymbol{p}_y \rangle_{TU}$ := average quark momentum in transverse $y$-direction measured in a proton polarized in transverse $x$-direction.

"dipole moment", "shift"

attention divergences from high-$\boldsymbol{p}_T$-tails!

$$\langle p_y(x) \rangle_{TU}^{\mathcal{B}_T} \equiv \left. \frac{\int d|\boldsymbol{p}_T|\, |\boldsymbol{p}_T| \int d\phi_p \frac{2\, J_1(|\boldsymbol{p}_T|\mathcal{B}_T)}{\mathcal{B}_T} \sin(\phi_p - \phi_S)\ \Phi^{(+)[\gamma^+]}(x, \boldsymbol{p}_T, P, S, \mu^2, \zeta, \rho)}{\int d|\boldsymbol{p}_T|\, |\boldsymbol{p}_T| \int d\phi_p J_0(|\boldsymbol{p}_T|\mathcal{B}_T) \qquad \Phi^{(+)[\gamma^+]}(x, \boldsymbol{p}_T, P, S, \mu^2, \zeta, \rho)} \right|_{|\boldsymbol{S}_T|=1}$$

$$\langle \boldsymbol{p}_y \rangle_{TU}(\mathcal{B}_T) \equiv M \frac{\int dx\, \tilde{f}_{1T}^{\perp(1)}(x, \mathcal{B}_T^2)}{\int dx\, \tilde{f}_1^{(0)}(x, \mathcal{B}_T^2)} = \frac{\cancel{\tilde{S}(\mathcal{B}_T^2, \ldots)}\, \tilde{A}_{12B}(\mathcal{B}_T^2, 0, 0, \tilde{\zeta}, \mu)}{\cancel{\tilde{S}(\mathcal{B}_T^2, \ldots)}\, \tilde{A}_{2B}(\mathcal{B}_T^2, 0, 0, \tilde{\zeta}, \mu)}$$