High-Energy QCD factorization from DIS to pA

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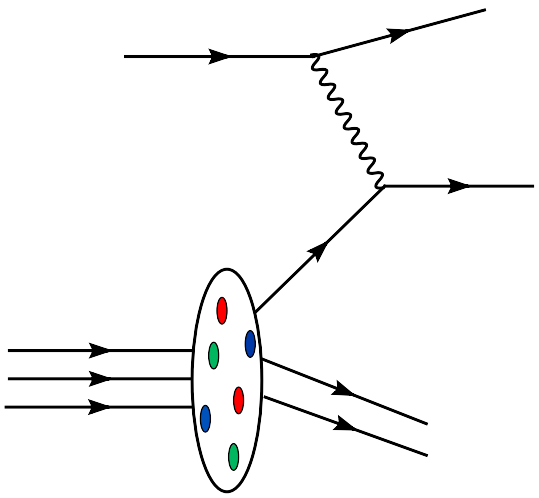
Outline

- Wilson lines formalism for scattering processes.
- Factorization in rapidity.
- Triple Pomeron vertex through Wilson line formalism: planar (leading N_c) and non-planar (next to-leading N_c) contribution.
- Analytic NLO amplitude in $\mathcal{N}$=4 SYM.
- The role of the B-Hierarchy in scattering processes.
- Factorization for Inclusive Hadron Production in pA collisions.
- Conclusions.

Incoherent-vs-Coherent

Incoherent Interactions



Bjorken Limit

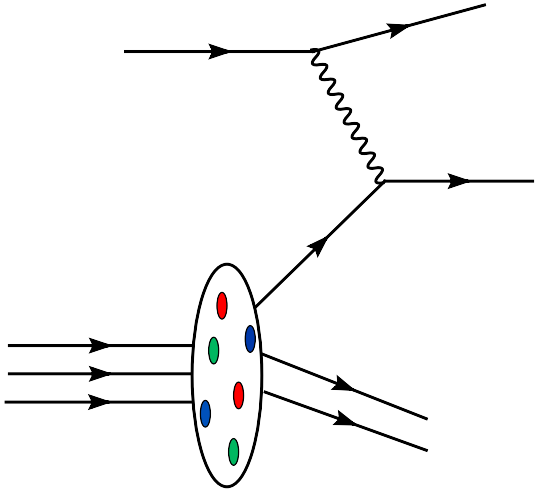
$$Q^2 \to \infty, \; s \to \infty$$

$$x_{\rm B} = \frac{Q^2}{s} \quad \text{fixed}$$

resum $\alpha_s \ln \dfrac{Q^2}{\Lambda_{\rm QCD}}$

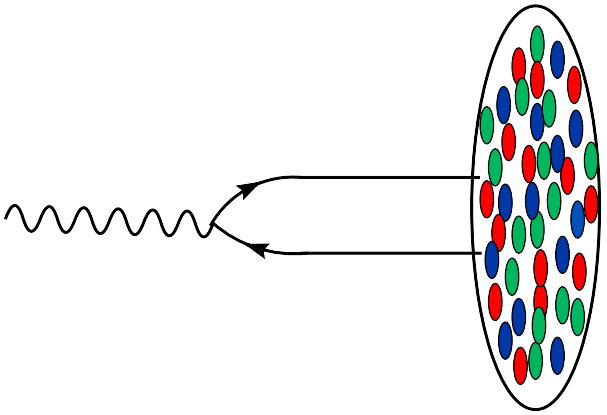
Incoherent-vs-Coherent

Incoherent Interactions



Coherent Interactions

VS.



Bjorken Limit

$$Q^2 \to \infty, \; s \to \infty$$

$$x_B = \frac{Q^2}{s} \quad \text{fixed}$$

resum $\alpha_s \ln \frac{Q^2}{\Lambda_{QCD}}$

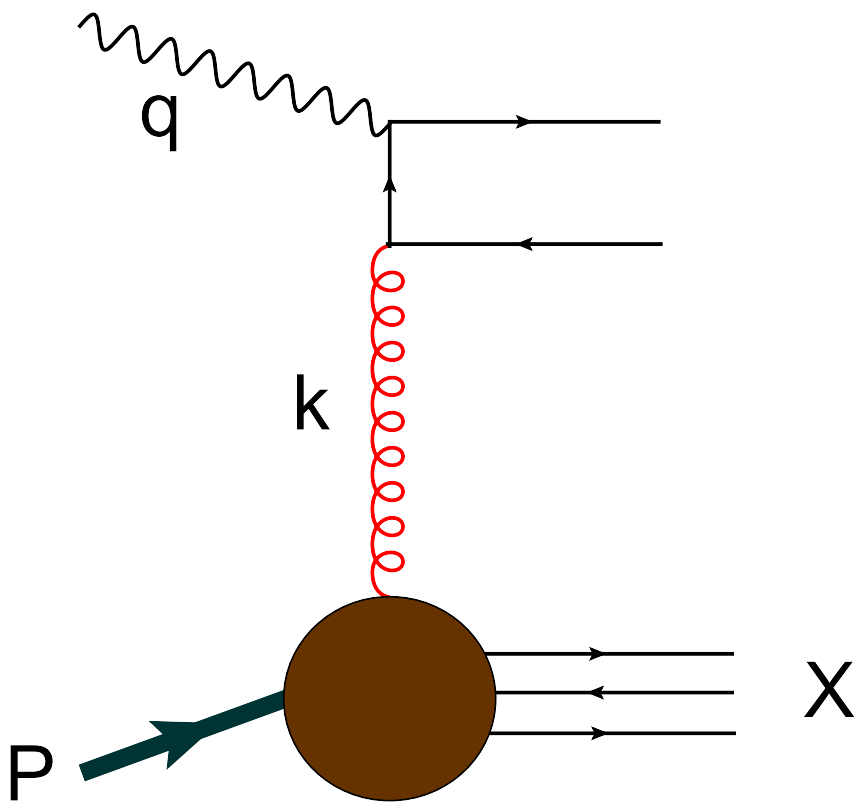
Regge Limit

$$Q^2 \;\; \text{fixed}, \; s \to \infty$$

$$x_B = \frac{Q^2}{s} \to 0$$

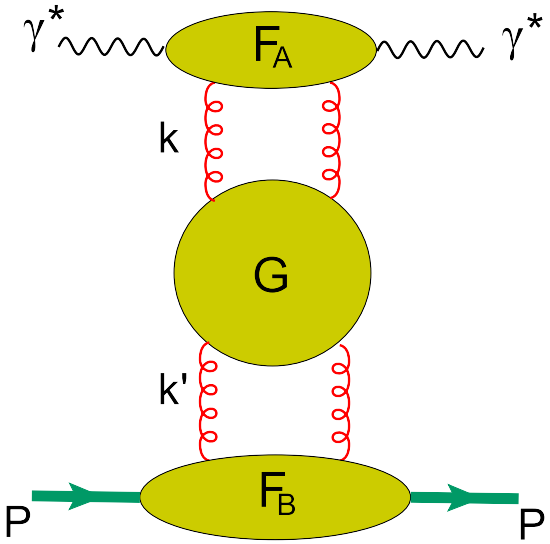
resum $\alpha_s \ln \frac{1}{x_B}$

Collinear vs. kt factorization

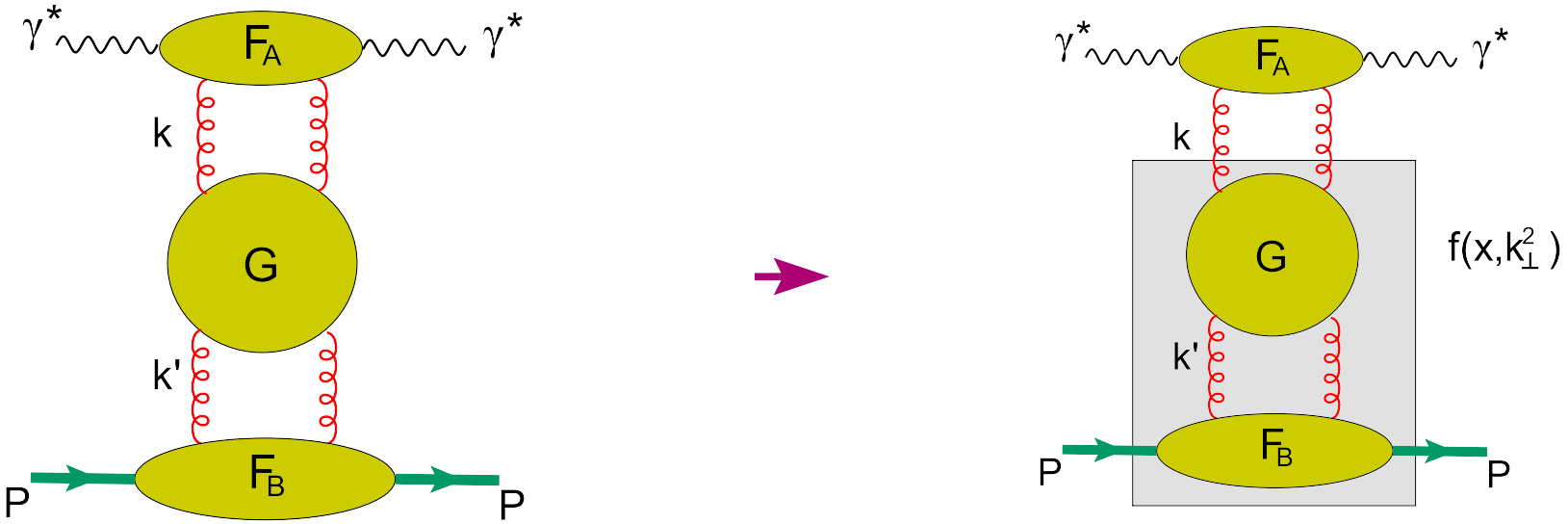


- Collinear factorization: emitted gluon is on-shell.
- $k_\perp$ factorization: emitted gluon is off-shell.

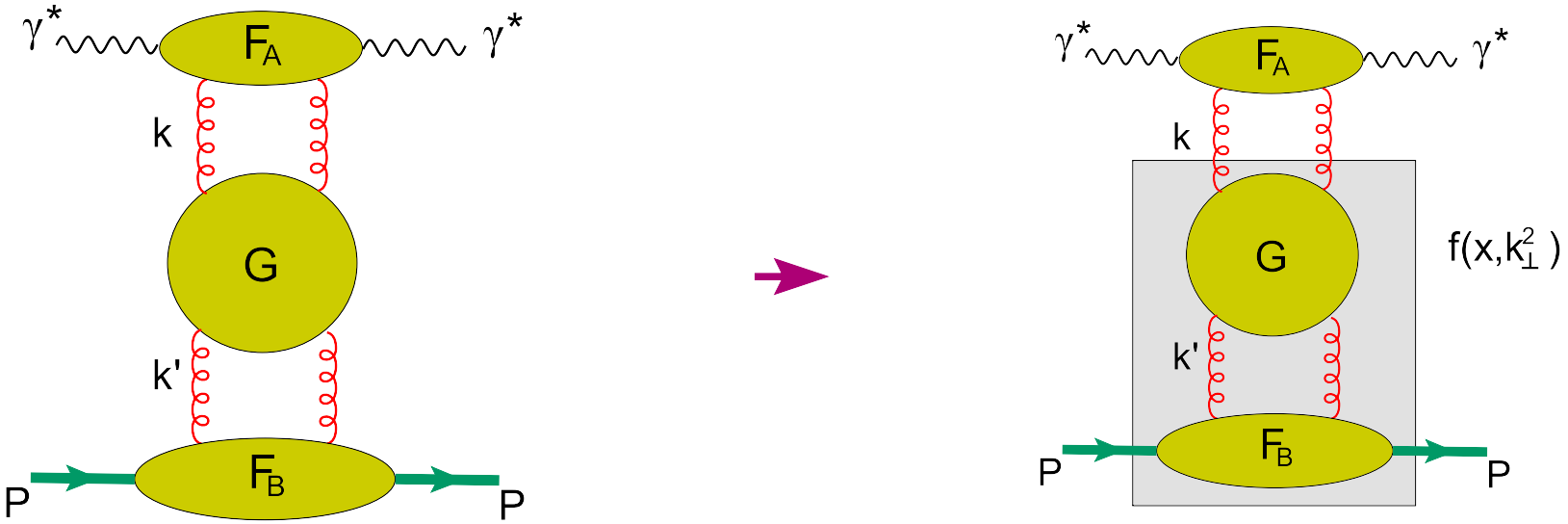
Photon Impact Factor for BFKL pomeron



Photon Impact Factor for BFKL pomeron



Photon Impact Factor for BFKL pomeron



$$\sigma^{\gamma^* p}(x,Q^2) \propto \int \frac{d^2k}{k_\perp^2} \int \frac{d^2k'}{k_\perp'^2} F_A(x,k_\perp,Q^2) F_B(x,k_\perp,k'_\perp) G(x,k_\perp,k'_\perp)$$

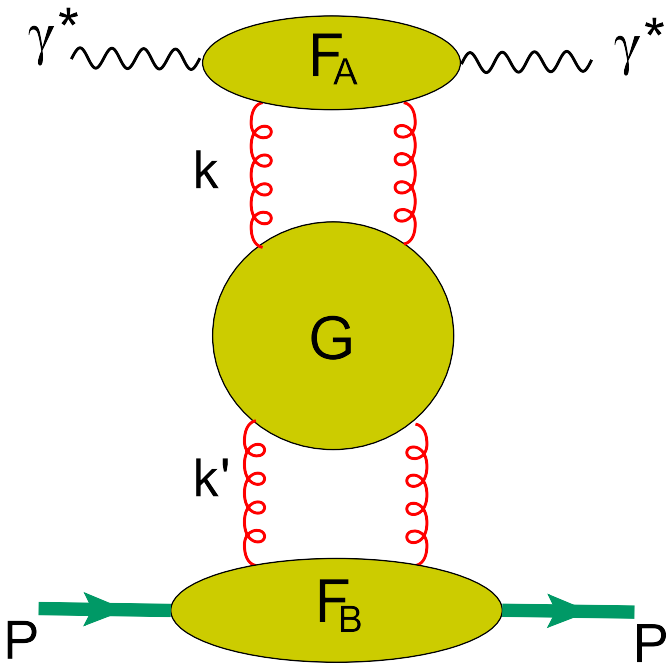
$\longrightarrow$

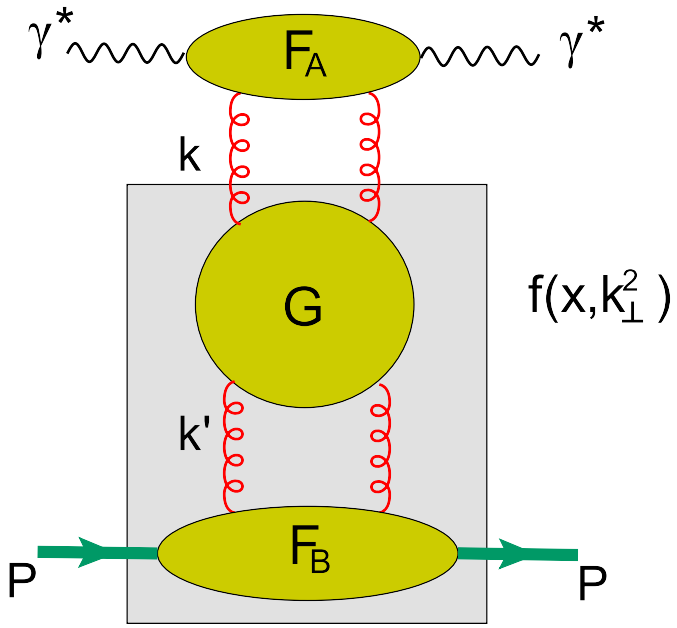
$$\sigma^{\gamma^* p}(x,Q^2) \propto \int \frac{d^2k}{k_\perp^2} \int_x^1 f(\frac{x}{x'},k_\perp^2) \sigma^{\gamma^* g}(x',k_\perp^2,Q^2)$$

$$g(x,Q^2) = \int^{Q^2} \frac{dk_\perp^2}{k_\perp^2} f(x,k_\perp^2)$$

$f(x,k_\perp^2)$: unintegrated gluon distribution

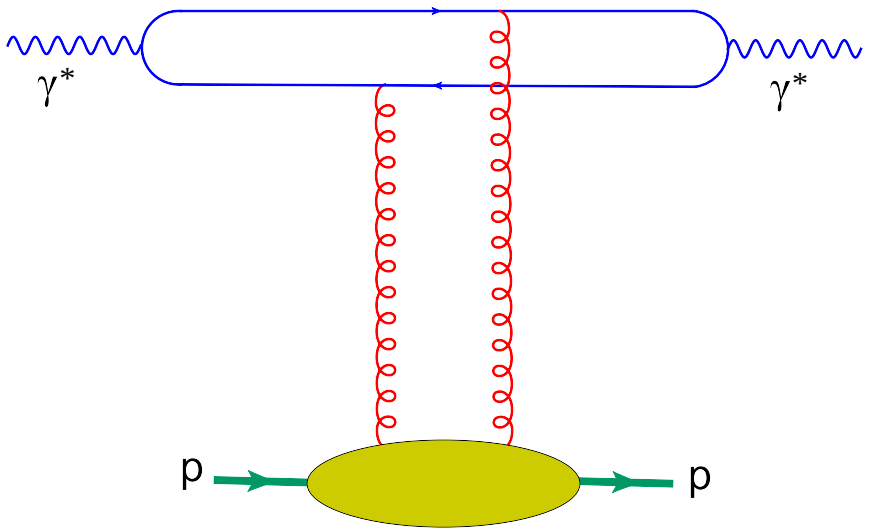
Photon Impact Factor for BFKL pomeron





- $f(x,k_\perp^2) \propto \int \frac{d^2k'}{k_\perp'^2} F_B(k_\perp'^2)\, k_\perp^2\, G(x,k_\perp,k_\perp')$

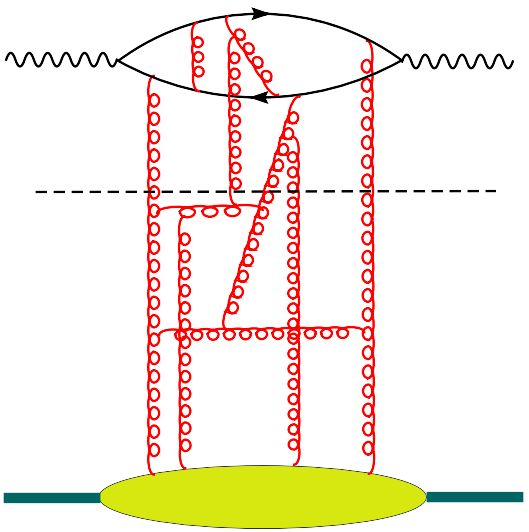
Dipole model



- At LO: equivalence of the $k_\perp$-factorization and dipole approach.
- At NLO?

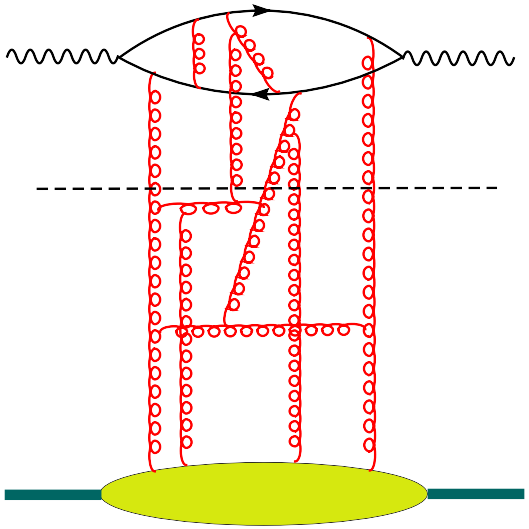
$$\sigma_{L,T}^{\gamma^* p} = \int_0^1 dz \int d^2b \, |\psi_{L,T}(z, b_\perp|^2 \sigma(x, b_\perp)$$

High-energy expansion in color dipoles at the NLO



$$\langle B|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle \to \langle \mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A$$

High-energy expansion in color dipoles at the NLO



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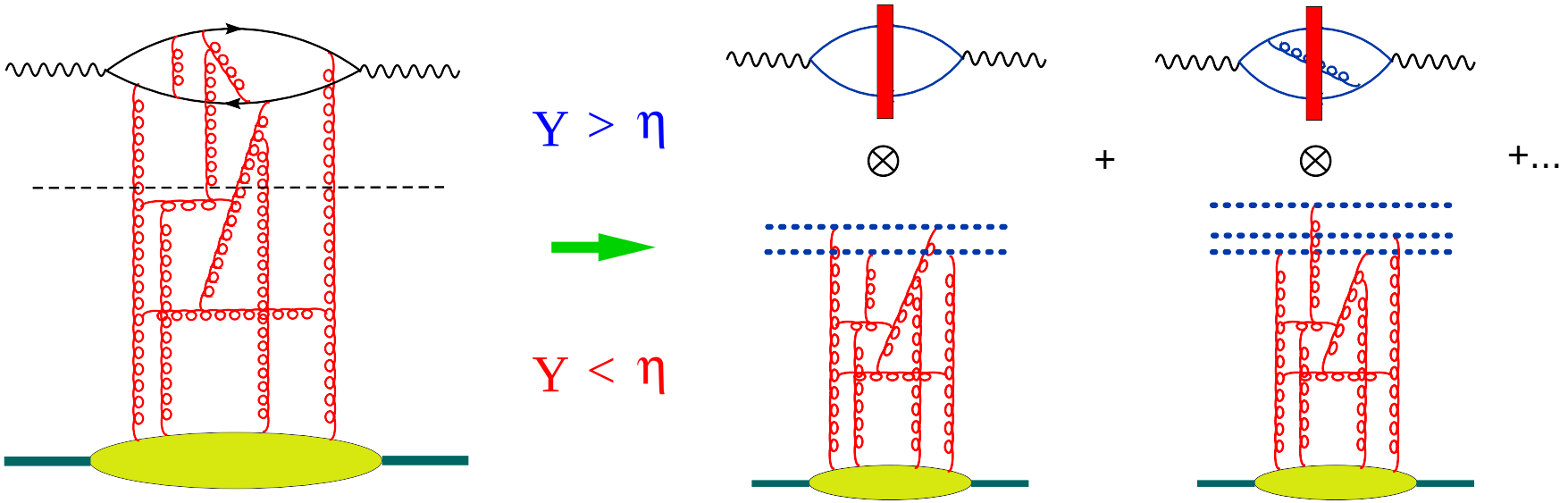

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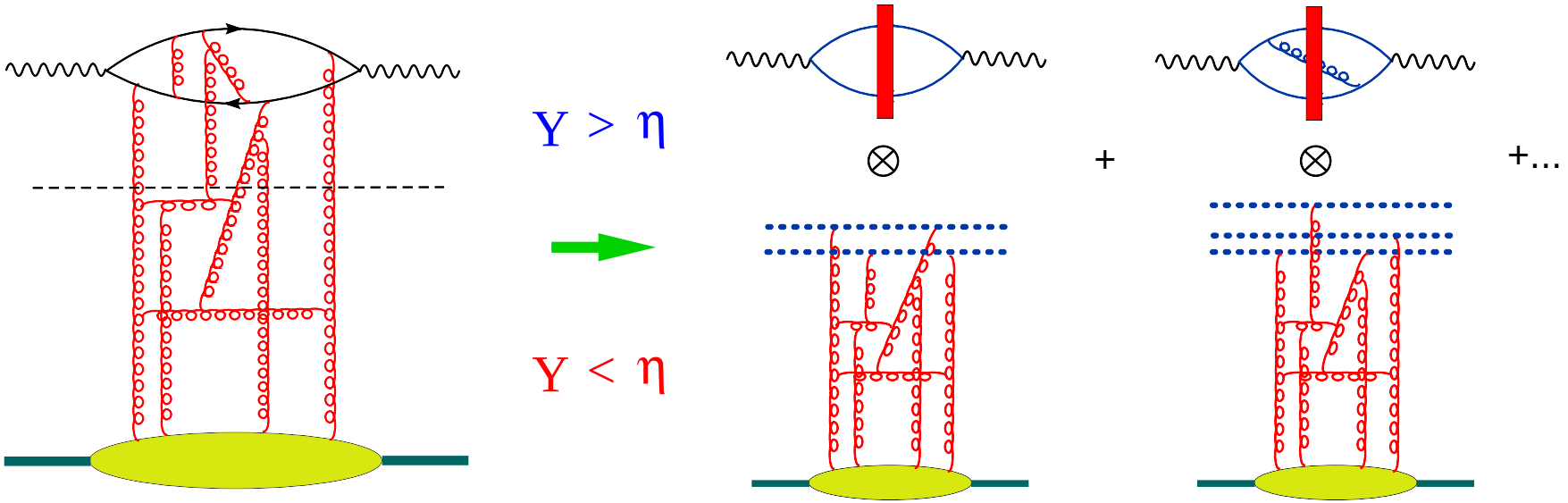


η - rapidity factorization scale

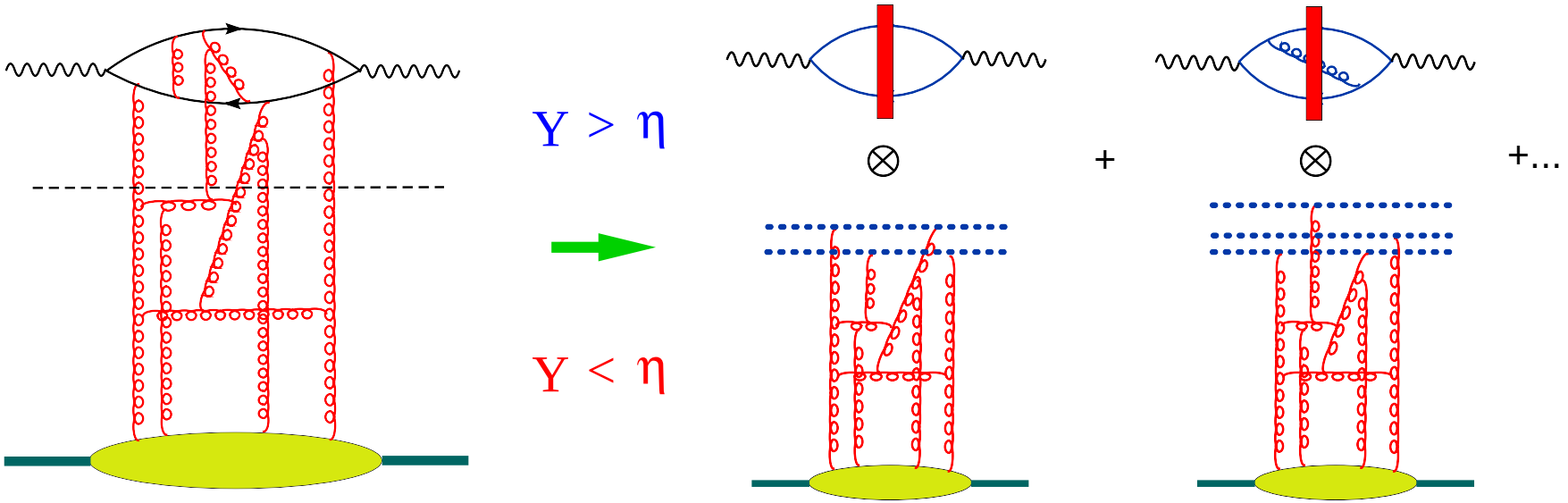
Rapidity Y > η - coefficient function ("impact factor")

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \mathrm{Pexp}\left[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\right]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

High-energy expansion in color dipoles at the NLO

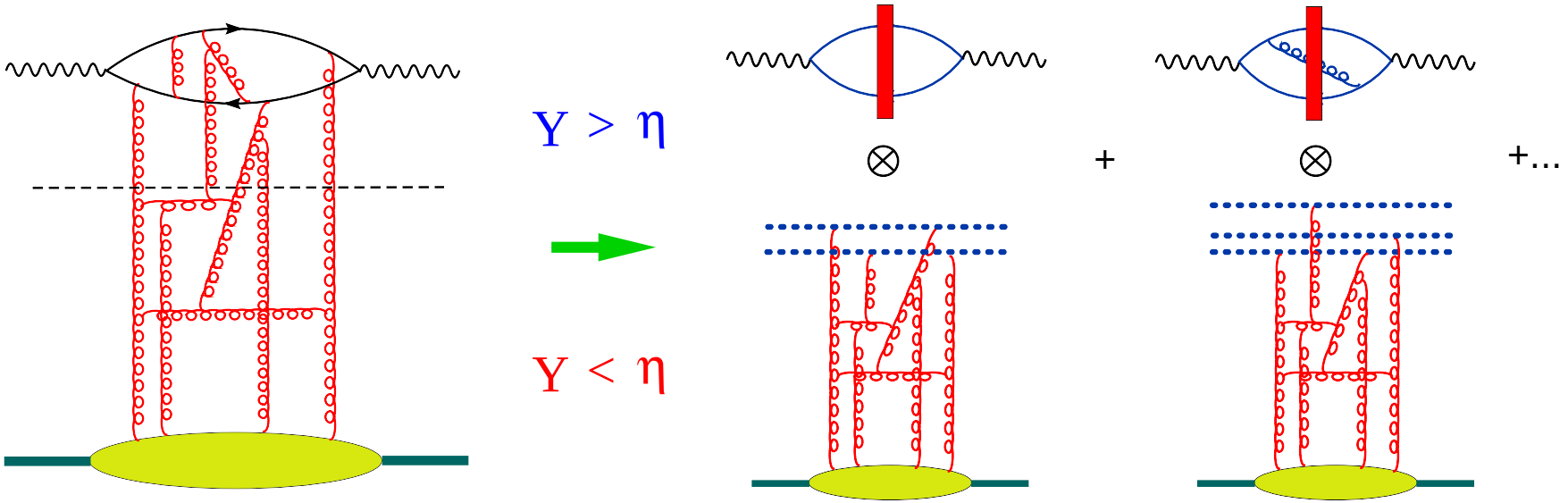


The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

High-energy expansion in color dipoles at the NLO



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$$\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\langle B|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle - N_c\langle B|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle]$$

Propagation in the shock wave: Wilson line (Spectator frame)

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x,y] = Pe^{ig\int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)} \qquad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

Propagation in the shock wave: Wilson line (Spectator frame)

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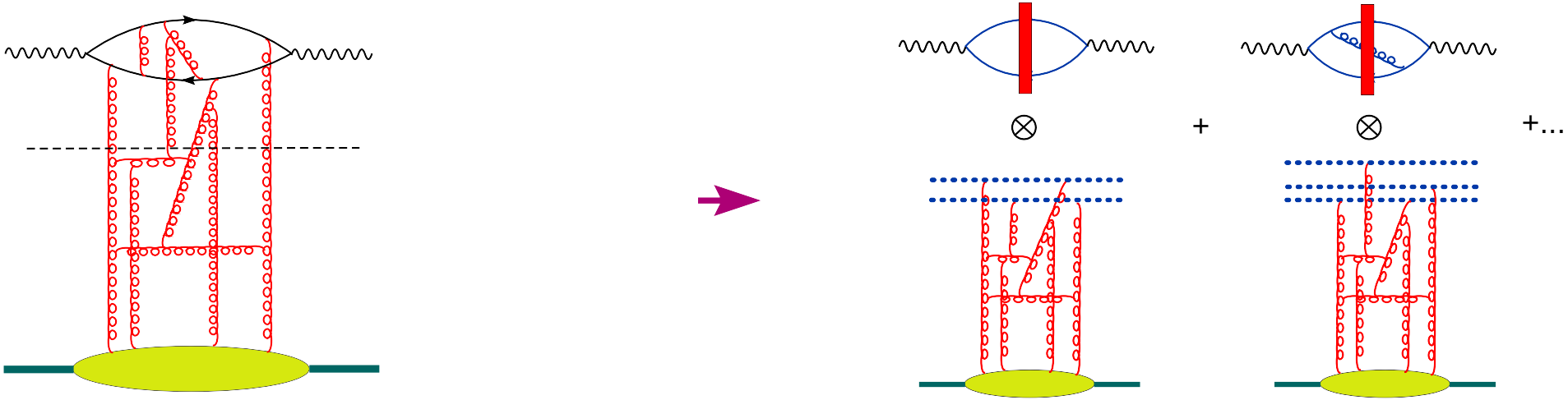
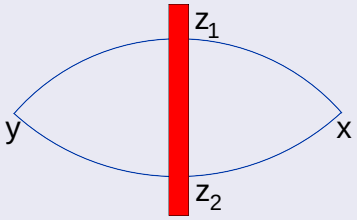


LO and NLO Impact Factor

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

LO Impact Factor diagram: $I^{\rm LO}$



LO and NLO Impact Factor

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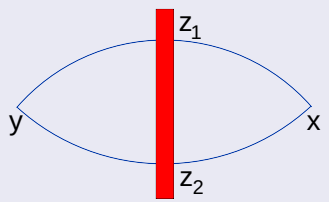
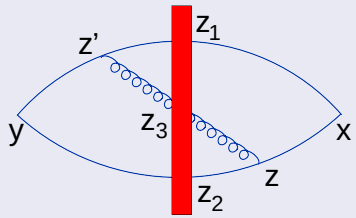
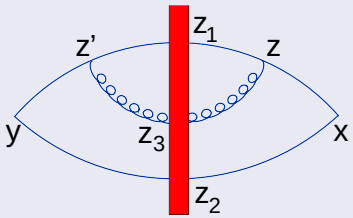
$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

LO Impact Factor diagram: $I^{\rm LO}$



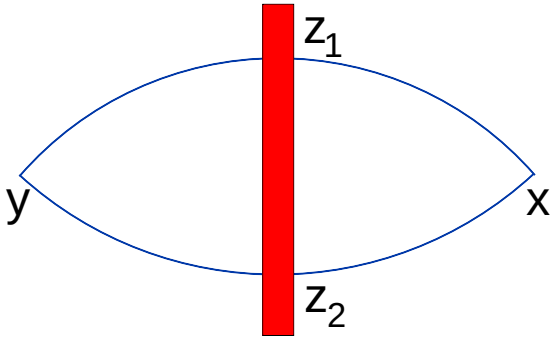
NLO Impact Factor diagrams: $I^{\rm NLO}$





LO Impact Factor

Conformal invariance: $(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2$



Conformal vectors:

$$\kappa = \frac{\sqrt{s}}{2x_*}\left(\frac{p_1}{s} - x^2 p_2 + x_\perp\right) - \frac{\sqrt{s}}{2y_*}\left(\frac{p_1}{s} - y^2 p_2 + y_\perp\right)$$

$$\zeta_1 = \left(\frac{p_1}{s} + z_{1\perp}^2 p_2 + z_{1\perp}\right), \qquad \zeta_2 = \left(\frac{p_1}{s} + z_{2\perp}^2 p_2 + z_{2\perp}\right)$$

Here $x^2 = -x_\perp^2$, $x_* \equiv x_\mu p_2^\mu$ (similarly for y); $\mathcal{R} = \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$

$$I_{\mu\nu}^{\rm LO}(z_1, z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)} \frac{\partial^2}{\partial x^\mu \partial y^\nu}\left[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2) - \frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\right]$$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm LO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A$$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm NLO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3 \left[I_1^{\mu\nu}(z_1,z_2,z_3) + I_2^{\mu\nu}(z_1,z_2,z_3)\right]$$
$$\times\left[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}\right]$$

where $I_2^{\mu\nu}(z_1,z_2,z_3)$ is finite and conformal, while

$$I_1^{\mu\nu}(z_1,z_2,z_3) = \frac{\alpha_s}{2\pi^2} I^{\rm LO}_{\mu\nu}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{z}_3}$$

is rapidity divergent.

How to get the NLO Impact factor

$$\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A$$

$$+ \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\, I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}] + \ldots$$

$\Rightarrow$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm NLO} - \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\left[\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\right]^{\rm LO}$$

$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\, I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

$$\left[\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\right]^{\rm LO} = \frac{\alpha_s}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]\int_0^{e^\eta}\frac{d\alpha}{\alpha}$$

How to get the NLO Impact factor

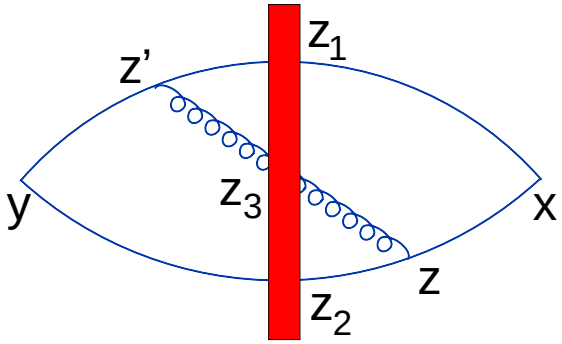
$$\int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\; I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

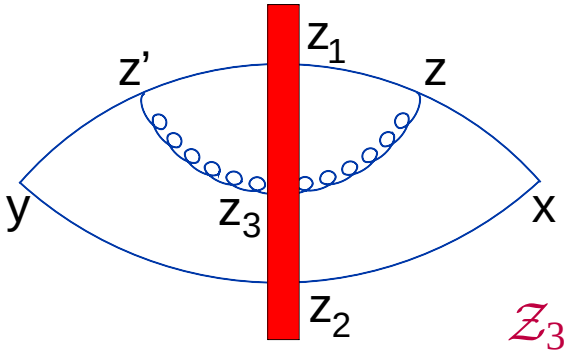
$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\; \left\{ I_2^{\mu\nu}(z_1,z_2,z_3) + \frac{\alpha_s}{2\pi^2} I^{\rm LO}_{\mu\nu} \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\int_0^{+\infty} \frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3} - \int_0^{e^\eta} \frac{d\alpha}{\alpha} \right] \right\}$$

$$\times [{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

$$\left[\int_0^{+\infty} \frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3} - \int_0^{e^\eta} \frac{d\alpha}{\alpha} \right] \to -\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C$$

NLO Impact Factor



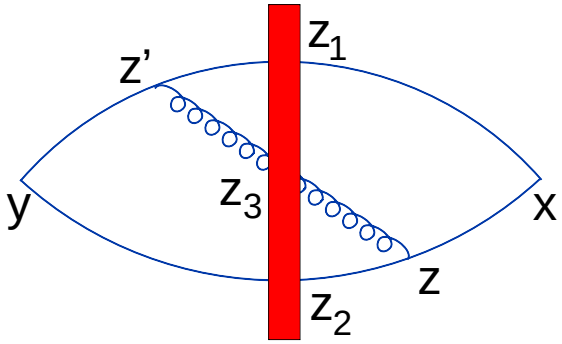


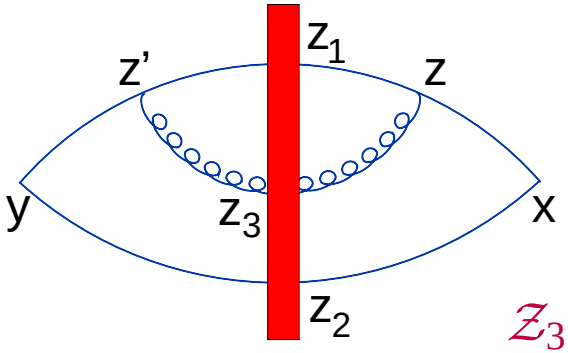
$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I_{\mu\nu}^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I_{\mu\nu}^{\rm LO} \times \frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

NLO Impact Factor





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$$I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}_{\mu\nu} \times \frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} \;=\; \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\,\frac{\alpha_s}{4\pi}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\frac{1}{N_c}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\Big]\ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2} \;+\; O(\alpha_s^2)$$

the impact factor becomes conformal at the NLO.

Conformal Composite Operator

$$[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\mathrm{conf}}_{a,\eta}$$
$$= \mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{sz_{13}^2z_{23}^2} + O(\alpha_s^2)$$

choose a rapidity-dependent constant $a \to ae^{-2\eta} \Rightarrow [\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]^{\mathrm{conf}}_{a}$
does not depend on $\eta = \ln\sigma$ and all the rapidity dependence is encoded into a-dependence:

$$[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\mathrm{conf}}_{a}$$
$$= \mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{\sigma^2sz_{13}^2z_{23}^2} + O(\alpha_s^2)$$

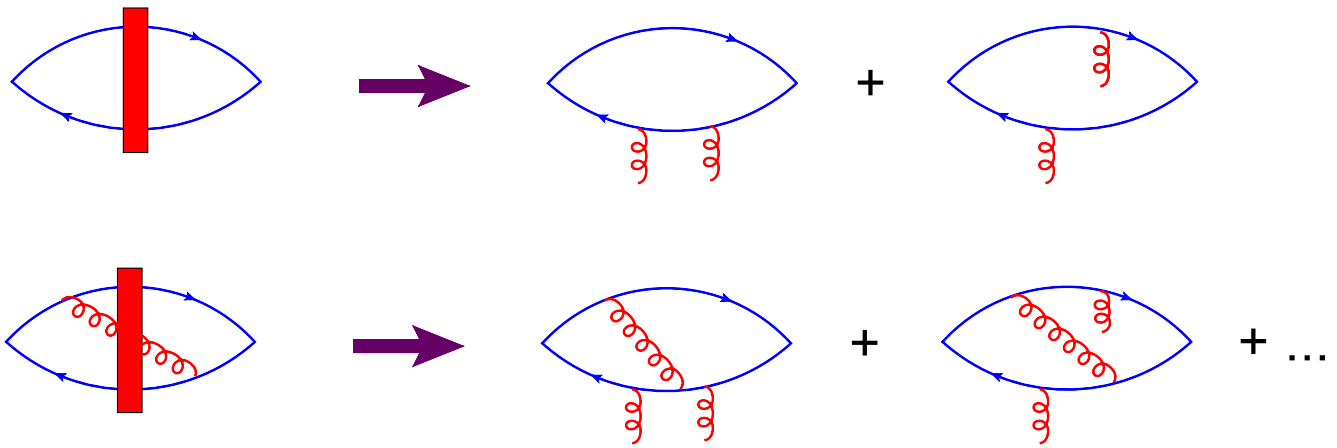
Using the leading-order evolution equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\} = \sigma\frac{d}{d\sigma}\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\} = \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]$$

$\Rightarrow \frac{d}{d\eta}[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\mathrm{conf}}_{a} = 0$ (with $O(\alpha_s^2)$ accuracy).

$$2a\frac{d}{da}[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\mathrm{conf}}_{a} = \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]$$

2-gluon approx. and BFKL pomeron in DIS



$$I^{\mathrm{LO}}\hat{\mathcal{U}}(x_\perp, y_\perp)$$

$$I^{\mathrm{NLO}}\left\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y)\right\}$$

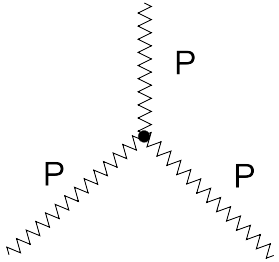
where $\mathcal{U}(x,y) = 1 - \frac{1}{N_c}\mathrm{tr}\{U_x U_y^\dagger\}$ and we neglected the non-linear term $\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)$

The triple Pomeron vertex: Fan Diagrams

The Balitsky equation becomes the BK equation when

$$\langle \mathrm{tr}\{1 - U_x U_z^\dagger\}\mathrm{tr}\{1 - U_z U_y^\dagger\}\rangle$$
$$= \frac{N_c^2}{2(N_c^2-1)}\left\{2\langle \mathcal{U}_{xz}\rangle\langle \mathcal{U}_{zy}\rangle + \frac{1}{N_c^2}\left[2\langle \mathcal{U}_{xy}\rangle\left(\langle \mathcal{U}_{xy}\rangle - \langle \mathcal{U}_{xz}\rangle - \langle \mathcal{U}_{yz}\rangle\right) + \langle \mathcal{U}_{zy}\rangle\langle \mathcal{U}_{zy}\rangle + \langle \mathcal{U}_{xz}\rangle\langle \mathcal{U}_{xz}\rangle - \langle \mathcal{U}_{xy}\rangle\langle \mathcal{U}_{xy}\rangle\right]\right\}$$

We extract the non planar (next-to-leading in N_c) contribution from $\langle \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\rangle$ for diffractive processes and for "fan" diagrams.
G.A.C, L.Szymanowski and S.Wallon 2010



We get

$$\int d^2\rho_a d^2\rho_b\; 16\; h_\alpha(h_\alpha-1)\bar{h}_\alpha(\bar{h}_\alpha-1)E_{h_\alpha\bar{h}_\alpha}(\rho_{a\alpha},\rho_{b\alpha})\left[\int d^2\rho_c\; \frac{1}{|\rho_{ab}|^2|\rho_{ac}|^2|\rho_{bc}|^2}E_{h_\beta\bar{h}_\beta}(\rho_{a\beta},\rho_{c\beta})E_{h_\gamma\bar{h}_\gamma}(\rho_{b\gamma},\rho_{c\gamma})\right.$$
$$\left.-\frac{2\pi}{N_c^2}\frac{1}{|\rho_{ab}|^4}\mathrm{Re}\{\psi(1)+\psi(h_\alpha)-\psi(h_\beta)-\psi(h_\gamma)\}E_{h_\beta\bar{h}_\beta}(\rho_{a\beta},\rho_{b\beta})E_{h_\gamma\bar{h}_\gamma}(\rho_{b\gamma},\rho_{c\gamma})\right]$$

which agrees with Bartels and Wusthoff (1995)

Regularization of the rapidity divergence

$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle \simeq \int d^2z_1 d^2z_2\; I^{LO}(z_1,z_2)\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\,\eta}_{z_2}\}|B\rangle$$

$$+\frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3\; I^{NLO}(z_1,z_2,z_3)\Big[\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U_{z_3}U^{\dagger\eta}_{z_2}\}|B\rangle - N_c\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\,\eta}_{z_2}\}|B\rangle\Big]$$

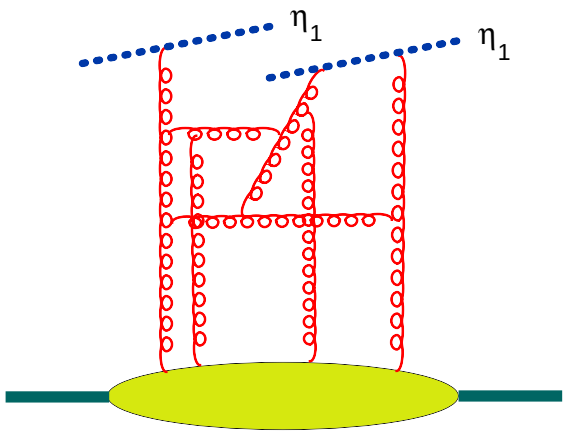
$$\eta = \ln\frac{1}{x_B}$$

Regularization of the rapidity divergence

$$\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A \simeq \int d^2z_1 d^2z_2\; I^{LO}(z_1,z_2)\langle \mathrm{tr}\{U^{\eta_1}_{z_1} U^{\dagger\;\eta_1}_{z_2}\}\rangle_A$$

$$+\;\frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3\; I^{NLO}(z_1,z_2,z_3)\Big[\langle \mathrm{tr}\{U^{\eta_1}_{z_1} U^{\dagger\eta_1}_{z_3}\}\mathrm{tr}\{U_{z_3} U^{\dagger\eta_1}_{z_2}\}\rangle_A - N_c\langle \mathrm{tr}\{U^{\eta_1}_{z_1} U^{\dagger\;\eta_1}_{z_2}\}\rangle_A\Big]$$

Matrix elements of Wilson lines: $\langle \mathrm{Tr}\{U(x)U^\dagger(y)\}\rangle_A$ are divergent

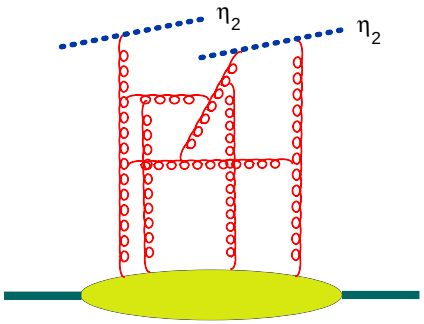


For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^{\infty} d\eta = \infty$$

Regularization of the rapidity divergence

Matrix elements of Wilson lines: $\langle \mathrm{Tr}\{U(x)U^\dagger(y)\rangle_A\}$ are divergent



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

Regularization by: slope

$$U^\eta(x_\perp) = \mathrm{Pexp}\Big\{ig\int_{-\infty}^{\infty} du\; n_\mu\, A^\mu(un + x_\perp)\Big\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

Regularization by: Rigid cut-off (used in NLO)

$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\; p_1^\mu A_\mu^\eta(up_1 + x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

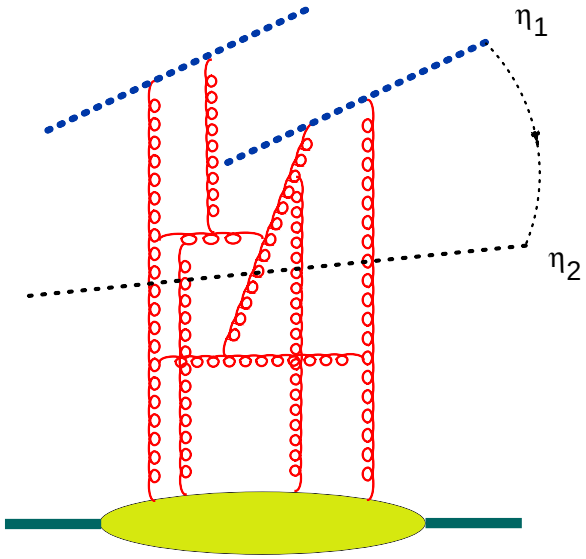
$k^\mu = \alpha_k\, p_1^\mu + \beta_k\, p_2^\mu + k_\perp^\mu$

Evolution Equation

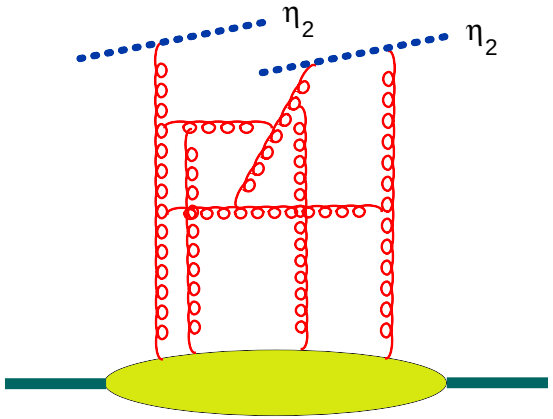
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} \quad \Rightarrow \quad \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle$$

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to η_2).

In the frame || to η_1 the gluons with $\eta < \eta_1$ are seen as pancake.

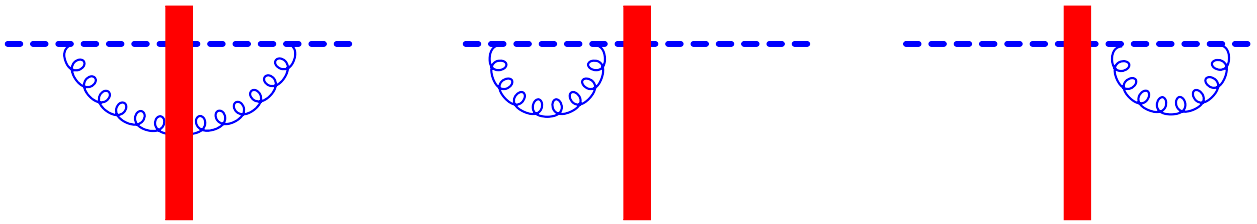


$$\alpha_s(\eta_1 - \eta_2)K_{\mathrm{evol}} \quad \otimes$$



Particles with different rapidity perceive each other as Wilson lines.

Non-linear evolution equation



$$\langle \{U_x^{\eta_1}\}_{ij} \rangle_A = \frac{\alpha_s}{2\pi^2} \Delta\eta \int \frac{d^2 z_\perp}{(x-z)_\perp^2} \left[\langle \mathrm{tr}\{U_x^{\eta_2} U_z^{\eta_2 \dagger}\} \{U_z^{\eta_2}\}_{ij} \rangle_A - \langle \frac{1}{N_c} \{U_x^{\eta_2}\}_{ij} \rangle_A \right]$$

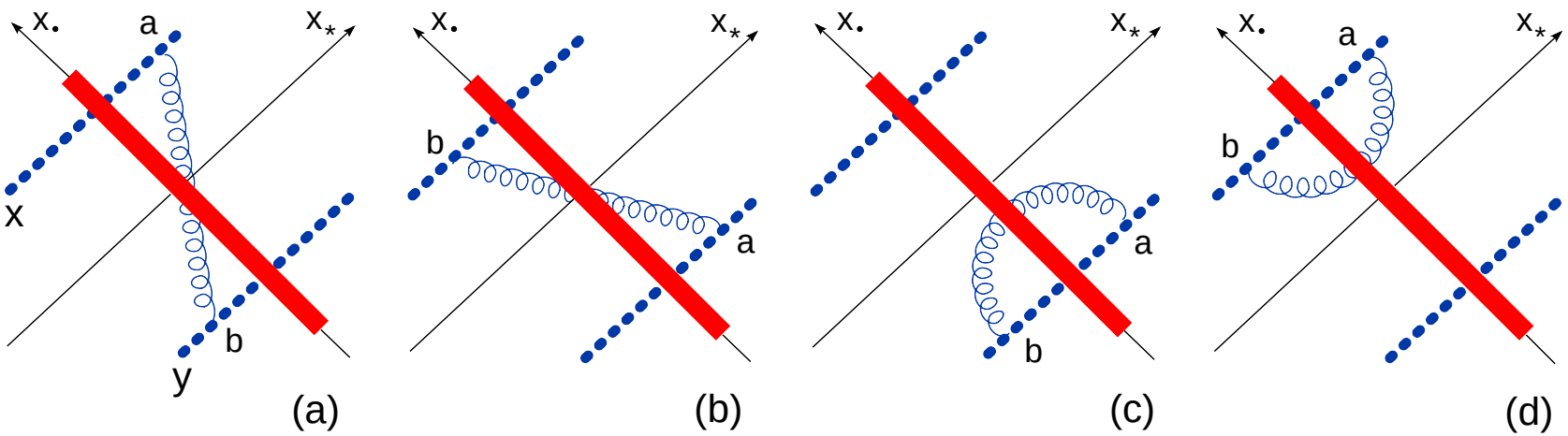
$\Delta = \eta_1 - \eta_2$

$$\{U_x^{\dagger \eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\dagger \eta_1}\}_{ij}, \quad \{U_x^{\dagger \eta_1} U_y^{\dagger \eta_1}\}_{ij}$$

Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

Non-linear evolution equation: BK equation

$$U_z^{ab} = 2\mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

Non-linear evolution equation: BK equation

$$U_z^{ab} = 2\mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\left\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\}$$

Alternative approach: JIMWLK (1997-2000)

Non-linear evolution equation: BK equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

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$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\left\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\}$$

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

Non linear evolution equation: BK equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\left\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\}$$

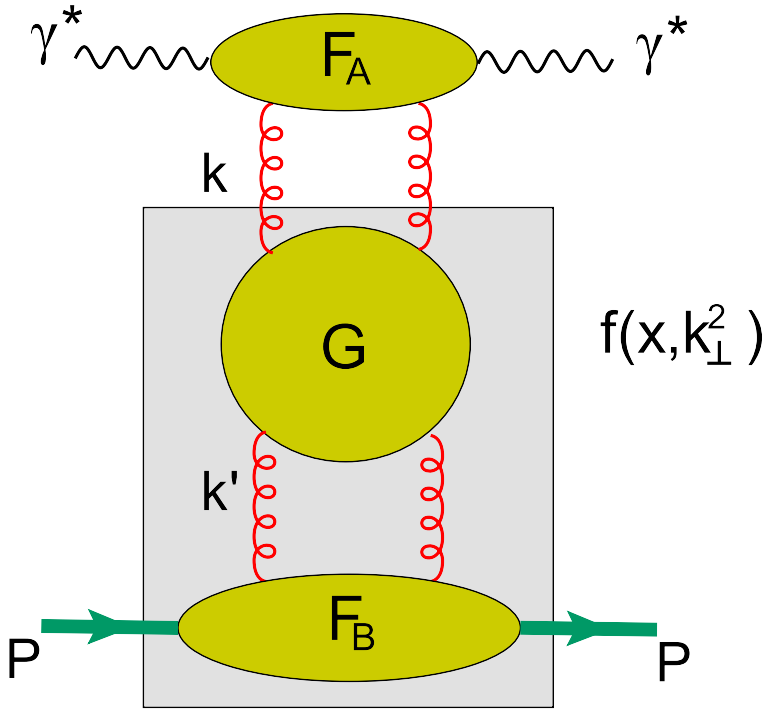
Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1,\ \alpha_s^2 A^{1/3} \sim 1$)

(s for semi-classical)

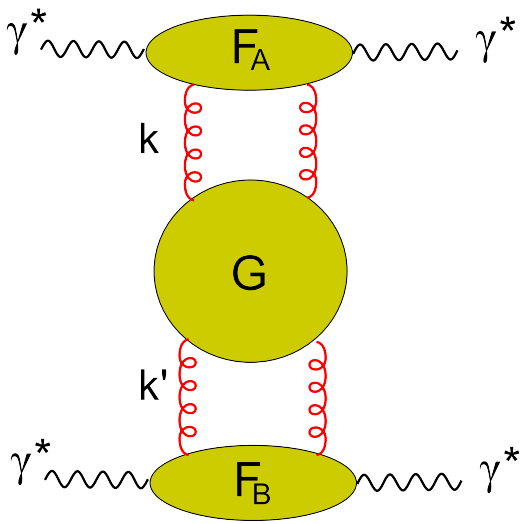
Evolution of the unintegrated gluon distribution



- $f(x,k_\perp^2) \propto \int \frac{d^2k'}{k_\perp'^2} F_B(k_\perp'^2)\, k_\perp^2\, G(x,k_\perp,k'_\perp)$
- $\mathcal{V}(z) \equiv z^{-2}\mathcal{U}(z)$

$$2a\frac{d}{da}\mathcal{V}_a(z) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int d^2z' \left[\frac{2\mathcal{V}_a(z')}{(z-z')^2} - \frac{z^2\mathcal{V}_a(z)}{z'^2(z-z')^2}\right]$$

$\gamma^* - \gamma^*$ scattering at NLO



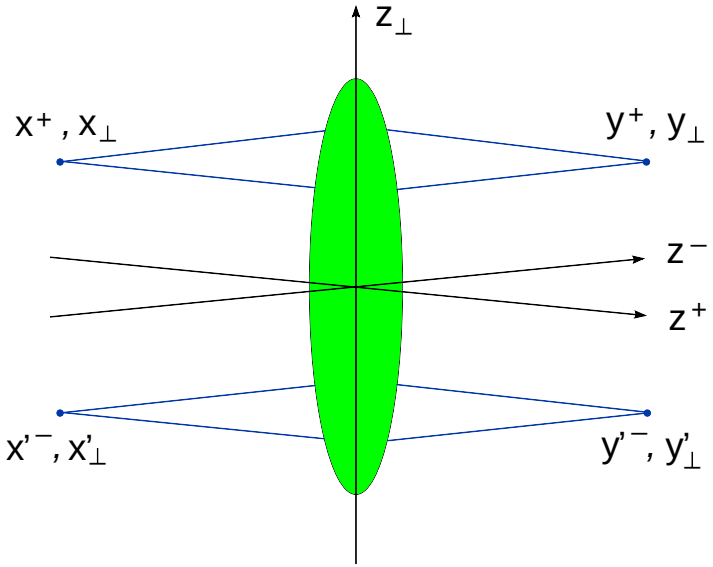
$$2a\frac{d}{da}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\left\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)\right\} + K_{\rm NLO}$$

- NLO photon impact factor has been calculated:
  G.A.C. and I. Balitsky see Balitsky talk.
- NLO BK has been calculated: G.A.C. and I. Balitsky (2007).
- Due to the running of the coupling we do not have an analytic expression for the NLO amplitude.
- In $\mathcal{N}$=4 we can have a full analytic amplitude at NLO:
  G.A.C. and I. Balitsky (2009)

Small-x (Regge) limit in the coordinate space

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

Regge limit: $x^+ \to \rho x^+,\ x'^+ \to \rho x'^+, y^- \to \rho' y^-,\ y'^- \to \rho' y^-$ $\quad \rho, \rho' \to \infty$



LLA: $\alpha_s \ll 1$, $\alpha_s \ln\rho \sim 1$, $\Rightarrow \sum(\alpha_s \ln\rho)^n \equiv$ BFKL pomeron.
LLA $\Leftrightarrow$ tree diagrams $\Rightarrow$ the BFKL pomeron is Möbius invariant .

NLO LLA: extra α_s: $\sum \alpha_s(\alpha_s \ln\rho)^n \equiv$ NLO BFKL

In conformal theory ($\mathcal{N}=4$ SYM) the NLO BFKL for composite conformal dipole operator is Möbius invariant.

NLO Amplitude in $\mathcal{N}$=4 SYM theory

The pomeron contribution in a conformal theory can be represented as an integral over one real variable ν Cornalba (2007)

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$
$$= \frac{i}{2}\int d\nu\, \tilde{f}_+(\nu)\frac{\tanh\pi\nu}{\nu}F(\nu)\Omega(r,\nu)R^{\frac{1}{2}\omega(\nu)}$$

$\omega(\nu)\equiv\omega(0,\nu)$ is the pomeron intercept,
$\tilde{f}_+(\omega)=(e^{i\pi\omega}-1)/\sin\pi\omega$ is the signature factor in the coordinate space.
$F(\nu)$ is the "pomeron residue".
The conformal function $\Omega(r,\nu)$ is given by a representation in terms of the two-dimensional integral

$$\Omega(r,\nu) = \frac{\nu^2}{\pi^3}\int d^2z\left(\frac{\kappa^2}{(2\kappa\cdot\zeta)^2}\right)^{\frac{1}{2}+i\nu}\left(\frac{\kappa'^2}{(2\kappa'\cdot\zeta)^2}\right)^{\frac{1}{2}-i\nu}$$

$$\zeta = p_1+\frac{z_\perp^2}{s}p_2+z_\perp, \qquad p_1^2=p_2^2=0,\quad 2(p_1,p_2)=s$$
$$\kappa = \frac{1}{2x^+}(p_1-\frac{x^2}{s}p_2+x_\perp)-\frac{1}{2y^+}(p_1-\frac{y^2}{s}p_2+y_\perp), \qquad \kappa^2\kappa'^2 = \frac{1}{R}$$
$$\kappa' = \frac{1}{2x'^-}(p_1-\frac{x'^2}{s}p_2+x'_\perp)-\frac{1}{2y'^-}(p_1-\frac{y'^2}{s}p_2+y'_\perp, \qquad 4(\kappa\cdot\kappa')^2 = \frac{r}{R}$$

Pomeron in the conformal theory

$$A(x,y;x',y') \overset{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

Pomeron intercept $\omega(\nu,\lambda)$ is known in two limits:

1. $\lambda \to 0$: $\omega(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu) + \lambda^2\omega_1(\nu) + ...$

$\chi(\nu) = 2\psi(1) - \psi(\frac{1}{2}+i\nu) - \psi(\frac{1}{2}-i\nu)$ - BFKL intercept,

$\omega_1(\nu)$ - NLO BFKL intercept Lipatov, Kotikov (2000)

2. $\lambda \to \infty$: AdS/CFT $\Rightarrow$ $\omega(\nu,\lambda) = 2 - \frac{\nu^2+4}{2\sqrt{\lambda}} + ...$

2 = gravition spin , next term - Brower, Polchinski, Strassler, Tan (2006)

Pomeron in the conformal theory

$$A(x,y;x',y') \overset{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

The function $F(\nu,\lambda)$ in two limits:

1. $\lambda \to 0$: $F(\nu,\lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + ...$

$F_0(\nu) = \frac{\pi \sinh \pi\nu}{4\nu \cosh^3 \pi\nu}$ Cornalba, Costa, Penedones (2007)

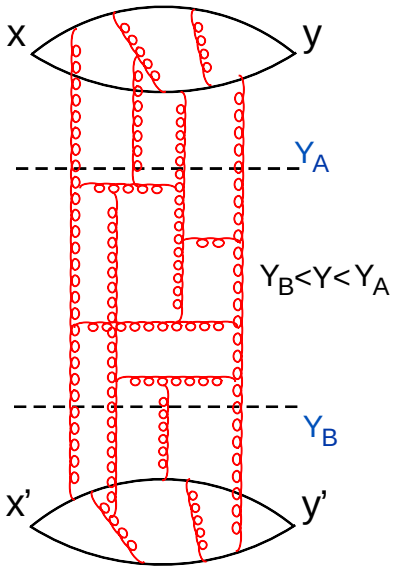
$F_1(\nu)$ = see below I. Balitsky and G. A. C. (2009)

2. $\lambda \to \infty$: $AdS/CFT \Rightarrow F(\nu) = \pi^3\nu^2 \frac{1+\nu^2}{\sinh^2 \pi\nu} + ...$

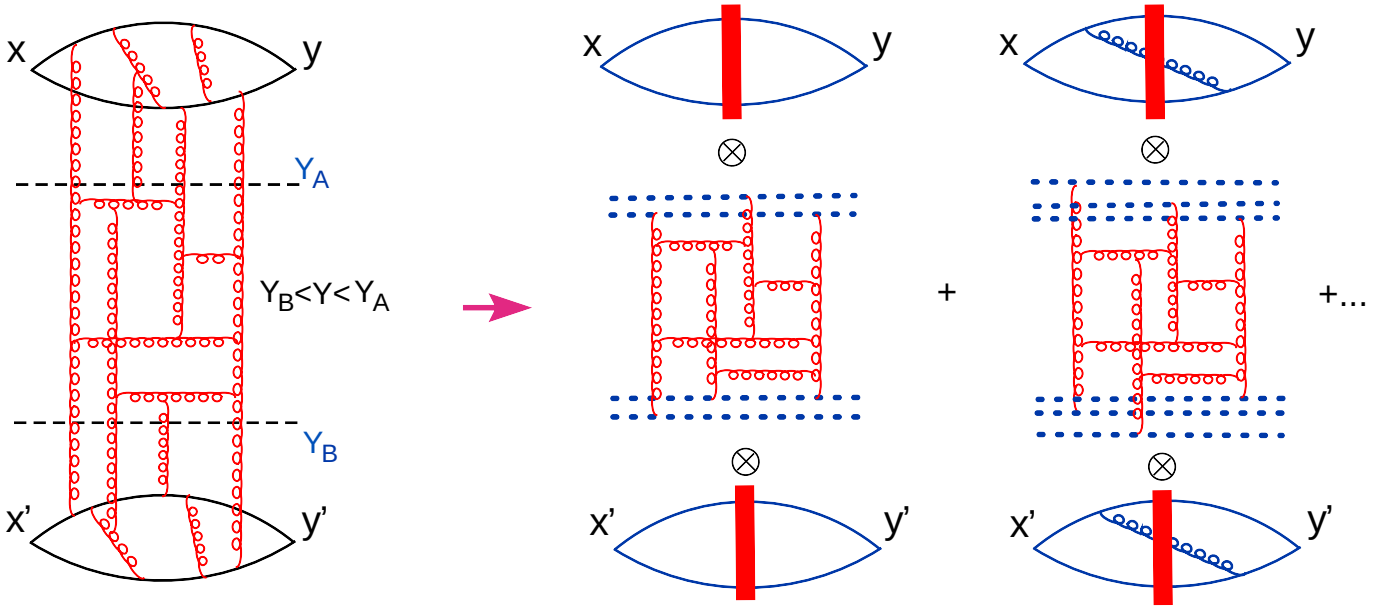
L.Cornalba (2007)

We calculate $F_1(\nu)$ (and confirm $\omega_1(\nu)$) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)

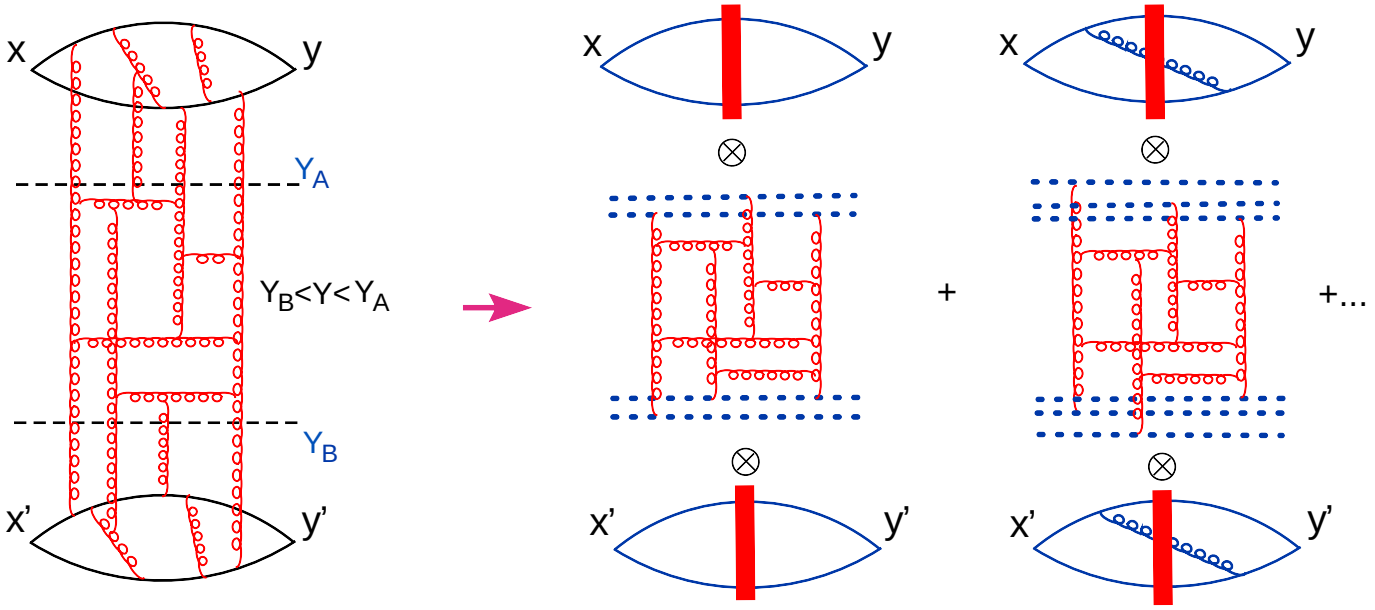
Factorization in rapidity



Factorization in rapidity



Factorization in rapidity



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0=\frac{x^+y^+}{(x-y)^2}$, $b_0=\frac{x'^-y'^-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy

$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$

Operator expansion in conformal dipoles in $\mathcal{N}=4$ SYM

$\mathcal{O} \equiv \frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2-1}}\mathrm{Tr}\{Z^2\}$ $\quad (Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2))$ renorm-invariant chiral primary operator

$$
\begin{aligned}
(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\} \;=\; & \frac{(x-y)^4}{\pi^2(N_c^2-1)}\int d^2z_1 d^2z_2 \frac{(x_* y_*)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\,[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} \\
& - \frac{\alpha_s(x-y)^4}{2\pi^4(N_c^2-1)}\int d^2z_1 d^2z_2 d^2z_3 \frac{z_{12}^2(x_* y_*)^{-2}}{z_{13}^2 z_{23}^2 \mathcal{Z}_1^2\mathcal{Z}_2^2} \\
& \times\Big(\ln\frac{x_* y_* z_{12}^2 e^{2\eta}}{16(x-y)_\perp^2 z_{13}^2 z_{23}^2}\Big[\frac{(x-z_3)^2}{x_*} - \frac{(y-z_3)^2}{y_*}\Big]^2 - i\pi + 2C\Big) \\
& \qquad\qquad \times\,[\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]
\end{aligned}
$$

The impact factor is Möbius invariant and does not scale with the energy.

$$
\begin{aligned}
&(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)^\dagger\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle \\
&= -\frac{1}{\pi^4}\int d\nu\int d^2z_0\,\frac{1+4\nu^2}{8\pi}\frac{\Gamma^2\left(\frac{1}{2}-i\nu\right)}{\Gamma(1-2i\nu)}\left(\frac{\kappa^2}{4(\kappa\cdot\zeta_0)^2}\right)^{\frac{1}{2}+i\nu} \\
&\times\frac{\left(-a_0b_0+i\epsilon\right)^{\frac{1}{2}\omega(\nu)}-(a_0b_0+i\epsilon)^{\frac{1}{2}\omega(\nu)}}{\pi\omega}I_0(\nu)\left[1-\frac{\alpha_sN_c}{4\pi}\Phi_1(\nu)\right] \\
&\times\frac{1+4\nu^2}{8\pi}\frac{\Gamma^2\left(\frac{1}{2}+i\nu\right)}{\Gamma(1+2i\nu)}\left(\frac{\kappa'^2}{4(\kappa'\cdot\zeta_0')^2}\right)^{\frac{1}{2}-i\nu}I_0(\nu)\left[1-\frac{\alpha_sN_c}{4\pi}\Phi_1(\nu)\right] \\
&\times\left[1-\frac{\alpha_sN_c}{2\pi}\left(\chi(\gamma)\left[4C+\frac{2}{\gamma(1-\gamma)}\right]+\frac{\pi^2}{3}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
F(\nu) &= \frac{N_c^2}{N_c^2-1}\frac{16\pi^4\alpha_s^2}{\cosh^2\pi\nu}\left[1-\frac{\alpha_sN_c}{4\pi}\Phi_1(\nu)\right]\left[1-\frac{\alpha_sN_c}{4\pi}\Phi_1(\nu)\right] \\
&\left[1-\frac{\alpha_sN_c}{2\pi}\left(\chi(\gamma)\left\{4C+\frac{2}{\gamma(1-\gamma)}\right\}+\frac{\pi^2}{3}\right)\right]+O(\alpha_s^2)
\end{aligned}
$$

which gives the pomeron residue in the next-to-leading order.

Evolution of the non-linear term of the BK: First B-hierarchy

$$\frac{d}{d\eta}\mathrm{tr}\{U_x U_z^\dagger\}\mathrm{tr}\{U_z U_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2 w_\perp \Bigg\{$$

$$\mathrm{tr}\{U_z U_y^\dagger\}\Big[\mathrm{tr}\{U_x U_w^\dagger\}\mathrm{tr}\{U_w U_z^\dagger\} - N_c \mathrm{tr}\{U_x U_z^\dagger\}\Big]\frac{(x-z)_\perp^2}{(x-w)^2(z-w)^2}$$

$$+\frac{1}{2}\Big[\mathrm{tr}\{U_x U_z^\dagger U_w U_y^\dagger U_z U_w^\dagger\} - \mathrm{tr}\{U_x U_y^\dagger\}\Big]\left(\frac{(x-w,z-w)_\perp^2}{(x-w)^2(z-w)^2} - \frac{(x-w,y-w)_\perp^2}{(x-w)^2(y-w)^2}\right.$$

$$\left.+\frac{(z-w,y-w)_\perp^2}{(z-w)^2(y-w)^2} - \frac{1}{(z-w)^2}\right) + (x \leftrightarrow y)^\dagger\Bigg\}$$

Evolution of the non-linear term of the BK: First B-hierarchy

$$S^{(2)}(x_\perp, y_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\{U_x U_y^\dagger\}$$

$$S^{(4)}(x_\perp, z_\perp, y_\perp) \equiv \frac{1}{N_c^2}\mathrm{tr}\{U_x U_z^\dagger\}\mathrm{tr}\{U_z U_y^\dagger\}$$

$$S^{(6)}(x_\perp, w_\perp, z_\perp, y_\perp) \equiv \frac{1}{N_c^3}\mathrm{tr}\{U_x U_w^\dagger\}\mathrm{tr}\{U_w U_z^\dagger\}\mathrm{tr}\{U_z U_y^\dagger\}$$

$$\tilde{S}^{(6)}(x_\perp, z_\perp, w_\perp, y_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\{U_x U_z^\dagger U_w U_y^\dagger U_z U_w^\dagger\}$$

Evolution of the non-linear term of the BK: First B-hierarchy

$$\frac{d}{d\eta}S^{(4)}(x_\perp, z_\perp, y_\perp)$$

$$= \frac{\alpha_s}{2\pi^2}\int d^2w_\perp \Bigg\{ N_c \left[S^{(6)}(x_\perp, w_\perp, z_\perp, y_\perp) - S^{(4)}(x_\perp, y_\perp)\right] \frac{(x-w, z-w)_\perp^2}{(x-w)^2(z-w)^2}$$

$$+\frac{1}{2N_c}\left[\tilde{S}^{(6)}(x_\perp, z_\perp, w_\perp, y_\perp) - S^{(2)}(x_\perp, y_\perp)\right]\left(\frac{(x-w, z-w)_\perp^2}{(x-w)^2(z-w)^2} - \frac{(x-w, y-w)_\perp^2}{(x-w)^2(y-w)^2}\right.$$

$$\left.+\frac{(z-w, y-w)_\perp^2}{(z-w)^2(y-w)^2} - \frac{1}{(z-w)^2}\right) + (x \leftrightarrow y)^\dagger \Bigg\}$$

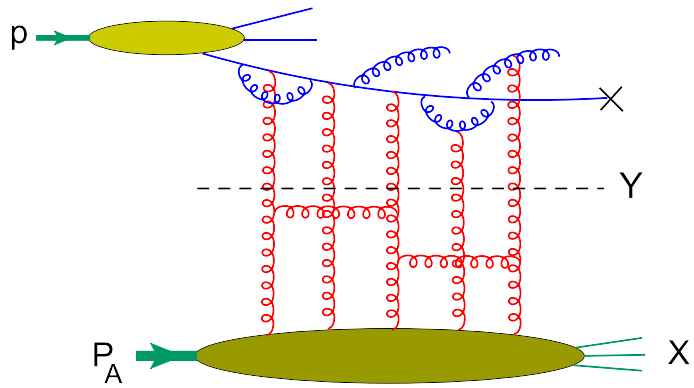
Evolution of the non-linear term of the BK: First B-hierarchy

$$\frac{d}{d\eta}S^{(4)}(x_\perp, z_\perp, y_\perp)$$
$$= \frac{\alpha_s}{2\pi^2}\int d^2w_\perp \Bigg\{ N_c \left[S^{(6)}(x_\perp, w_\perp, z_\perp, y_\perp) - S^{(4)}(x_\perp, y_\perp)\right] \frac{(x-w, z-w)^2_\perp}{(x-w)^2(z-w)^2}$$
$$+\frac{1}{2N_c}\left[\tilde{S}^{(6)}(x_\perp, z_\perp, w_\perp, y_\perp) - S^{(2)}(x_\perp, y_\perp)\right]\Bigg(\frac{(x-w, z-w)^2_\perp}{(x-w)^2(z-w)^2} - \frac{(x-w, y-w)^2_\perp}{(x-w)^2(y-w)^2}$$
$$+\frac{(z-w, y-w)^2_\perp}{(z-w)^2(y-w)^2} - \frac{1}{(z-w)^2}\Bigg) + (x \leftrightarrow y)^\dagger \Bigg\}$$

- $S^{(6)}(x_\perp, w_\perp, z_\perp, y_\perp)$ relevant for $\mathrm{P} \to 3\mathrm{P}$ vertices.
- Pomeron vertices from B-hierarchy.

One-loop Factorization for Inclusive Hadron Production in pA

G.A.C, B-W. Xiao, F. Yuan (2011)



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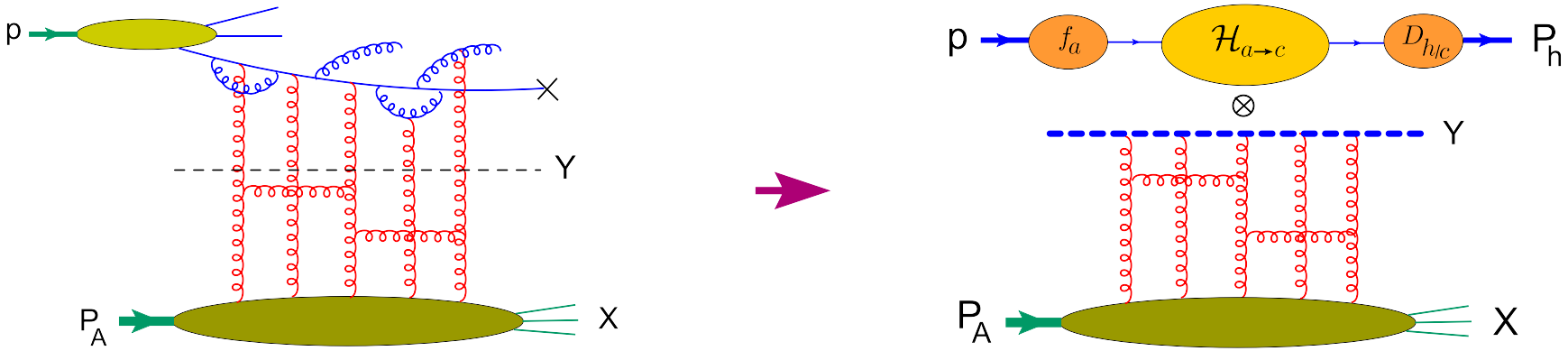

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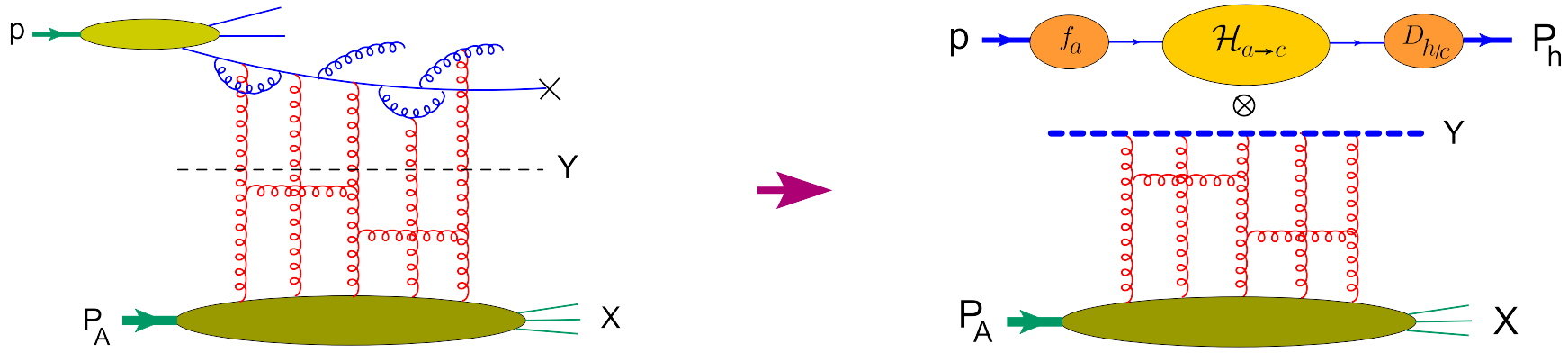

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \sum_a \int \frac{dz}{z^2}\frac{dx}{x}\,\xi\, x\, f_a(x,\mu) D_{h/c}(z,\mu) \int [dx_\perp]\, S^Y_{a,c}([x_\perp])\, \mathcal{H}_{a\to c}(\alpha_s,\xi,[x_\perp]\mu)$$

- $s = (p+P_A)^2 \longrightarrow \infty$
- x is the momentum fraction of the nucleon carried by the parton a; z the momentum fraction of parton c carried by the final state hadron h.
- $\xi = \frac{\tau}{xz}, \quad \tau = p_\perp \frac{e^y}{\sqrt{s}}$.
- y and $p_\perp$: rapidity and transverse momentum for the final state hadron.

One-loop Factorization for Inclusive Hadron Production in pA

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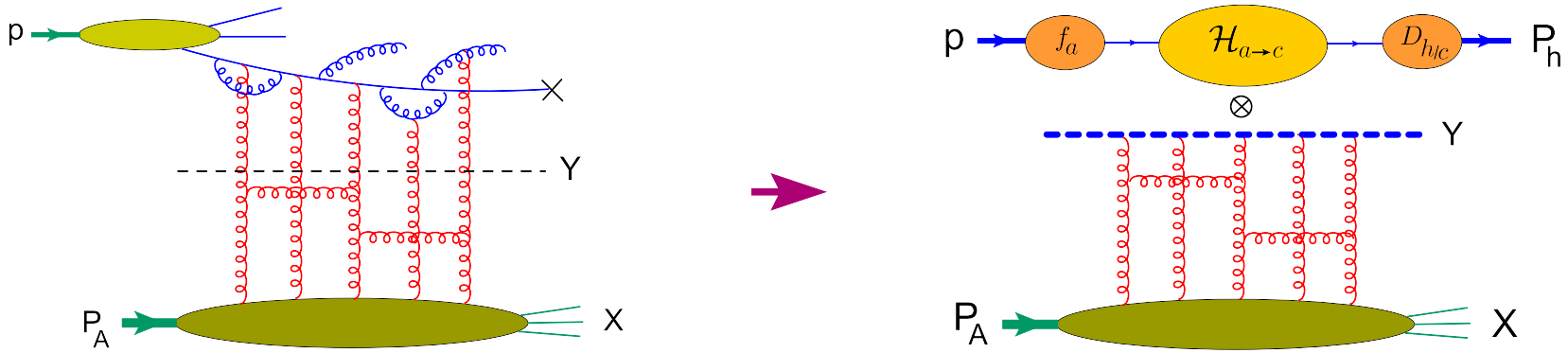

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \left[\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp}\right]^{\rm LO} + \left[\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp}\right]^{\rm NLO} + \ldots$$

One-loop Factorization for Inclusive Hadron Production in pA

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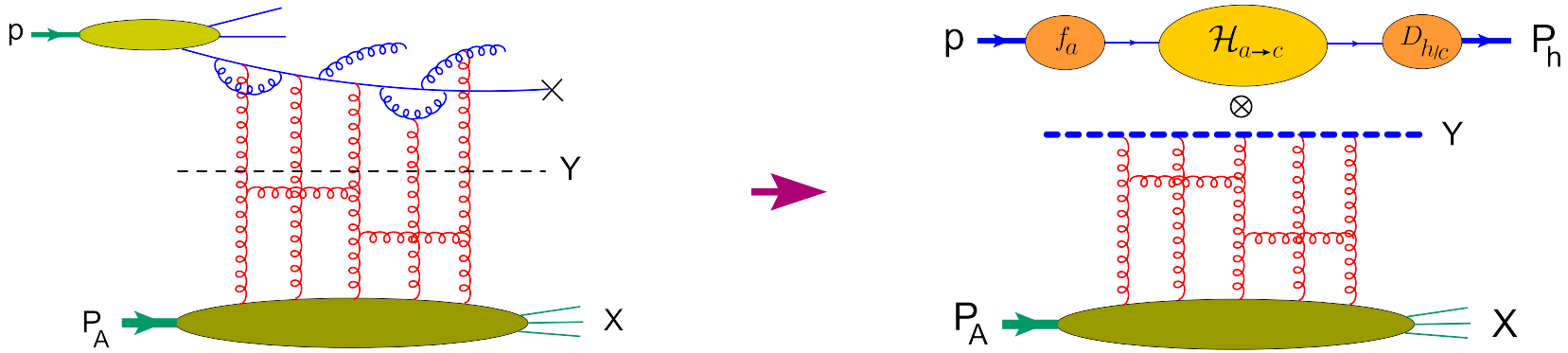


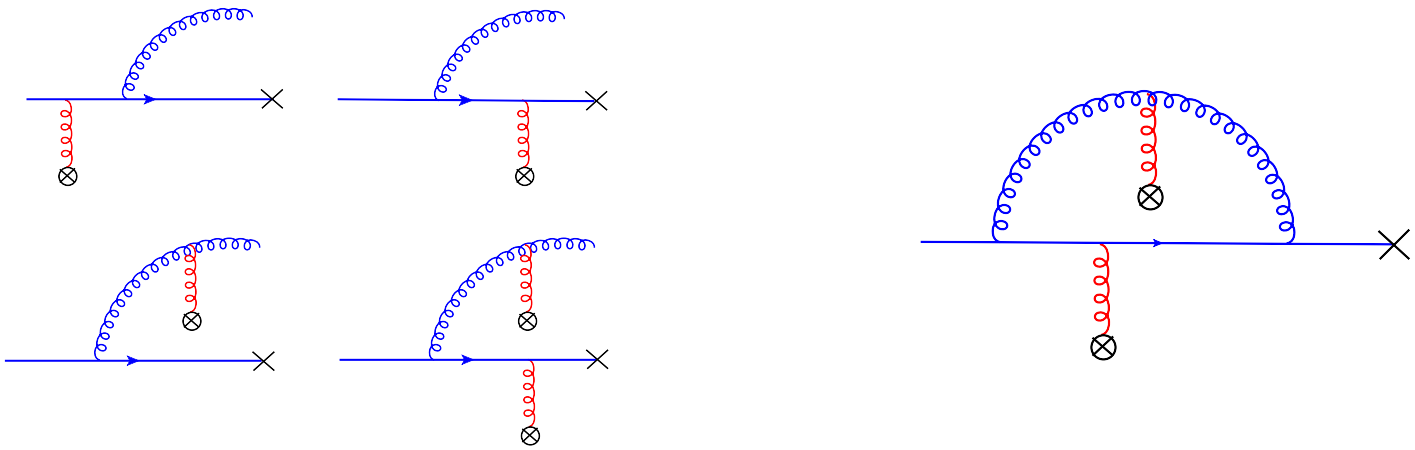
$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \left[\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp}\right]^{\rm LO} + \left[\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp}\right]^{\rm NLO} + \ldots$$

$$\Rightarrow$$

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} - \left[\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp}\right]^{\rm LO} = \left[\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp}\right]^{\rm NLO}$$

One-loop Factorization for Inclusive Hadron Production in pA

Quark channel contribution: $qA \longrightarrow q + X$ at one loop order



$$-\frac{\alpha_s N_c}{2\pi^2}\int_0^1 \frac{d\xi'}{1-\xi'}\int \frac{d^2 b_\perp}{(2\pi)^2} e^{-ik_\perp\cdot r_\perp}\frac{(x-y)_\perp^2}{(x-b)_\perp^2 (y-b)_\perp^2}\left[S^{(2)}(x_\perp,y_\perp) - S^{(4)}(x_\perp,b_\perp,y_\perp)\right]$$

$$+\frac{\alpha_s C_F}{2\pi}\int_{\tau/z}^1 d\xi \left(-\frac{1}{\epsilon}\right)\left[\mathcal{P}_{qq}(\xi)e^{-ik_\perp\cdot r_\perp} + \mathcal{P}_{qq}(\xi)\frac{1}{\xi^2}e^{-i\frac{k_\perp}{\xi}\cdot r_\perp}\right]\frac{1}{(2\pi)^2}S^{(2)}(x_\perp,y_\perp)$$

$$S^{(2)}(x_\perp,y_\perp) = \frac{1}{N_c}\langle U(x_\perp)U^\dagger(y_\perp)\rangle_Y$$

$$S^{(4)}(x_\perp,b_\perp,y_\perp) = \frac{1}{N_c^2}\langle \mathrm{Tr}[U(x_\perp)U^\dagger(b_\perp)]\mathrm{Tr}[U(b_\perp)U^\dagger(y_\perp)]\rangle_Y$$

One-loop Factorization for Inclusive Hadron Production in pA

Quark channel contribution: $qA \longrightarrow q + X$ at one loop order

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \int \frac{dz}{z^2}\frac{dx}{x}\,\xi\, x\,[q(x,\mu)]^{1-\text{loop}}[D_{h/q}(z,\mu)]^{1-\text{loop}} \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} \times$$
$$\left\{[S^{(2)}(x_\perp,y_\perp)]^{1-\text{loop}}\left[\mathcal{H}^{(0)}_{2qq} + \frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{2qq}\right] + \int \frac{d^2b_\perp}{(2\pi)^2} S^{(4)}(x_\perp,b_\perp,y_\perp)\frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{4qq}\right\}$$

- The collinear divergence is absorbed into the renormalization of the quark distribution and fragmentation functions
  - $[q(x,\mu)]^{1-\text{loop}}$ and $[D_{h/q}(z,\mu)]^{1-\text{loop}}$: DGLAP evolution equation
- The rapidity divergence can be absorbed into the renormalization of the dipole gluon distribution
  - $[S^{(2)}(x_\perp,y_\perp)]^{1-\text{loop}}$: Balitsky-BK evolution equation
- kt Factorization is violated in pA at NLO level.

One-loop Factorization for Inclusive Hadron Production in pA

Hard coefficients at one loop for quark channel: Infra-red and Ultra-violet finite.

$$\mathcal{H}^{(1)}_{2qq} = C_F \mathcal{P}_{qq}(\xi) \ln \frac{c_0^2}{r_\perp^2 \mu^2} \left(e^{-ik_\perp \cdot r_\perp} + \frac{1}{\xi^2} e^{-i\frac{k_\perp}{\xi} \cdot r_\perp} \right) - 3C_F \delta(1-\xi) e^{-ik_\perp \cdot r_\perp} \ln \frac{c_0^2}{r_\perp^2 k_\perp^2}$$
$$- (2C_F - N_c)\, e^{-ik_\perp \cdot r_\perp} \left[\frac{1+\xi^2}{(1-\xi)_+} \widetilde{I}_{21} - \left(\frac{\left(1+\xi^2\right) \ln (1-\xi)^2}{1-\xi} \right)_+ \right]$$

$$\mathcal{H}^{(1)}_{4qq} = -4\pi N_c e^{-ik_\perp \cdot r_\perp} \left\{ e^{-i\frac{1-\xi}{\xi} k_\perp \cdot (x_\perp - b_\perp)} \frac{1+\xi^2}{(1-\xi)_+} \frac{1}{\xi} \frac{x_\perp - b_\perp}{(x_\perp - b_\perp)^2} \cdot \frac{y_\perp - b_\perp}{(y_\perp - b_\perp)^2} \right.$$
$$\left. -\delta(1-\xi) \int_0^1 d\xi' \frac{1+\xi'^2}{(1-\xi')_+} \left[\frac{e^{-i(1-\xi')k_\perp \cdot (y_\perp - b_\perp)}}{(b_\perp - y_\perp)^2} - \delta^{(2)}(b_\perp - y_\perp) \int d^2 r'_\perp \frac{e^{ik_\perp \cdot r'_\perp}}{r'^2_\perp} \right] \right\}$$

where $c_0 = 2e^{-\gamma_E}$ with γ_E the Euler constant, and

$$\widetilde{I}_{21} = \int \frac{d^2 b_\perp}{\pi} \left\{ e^{-i(1-\xi)k_\perp \cdot b_\perp} \left[\frac{b_\perp \cdot (\xi b_\perp - r_\perp)}{b_\perp^2 \, (\xi b_\perp - r_\perp)^2} - \frac{1}{b_\perp^2} \right] + e^{-ik_\perp \cdot b_\perp} \frac{1}{b_\perp^2} \right\}$$

What to expect at NNLO in pA collisions

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \int \frac{dz}{z^2}\frac{dx}{x}\,\xi\, x\,[q(x,\mu)]^{2-\text{loop}}[D_{h/q}(z,\mu)]^{2-\text{loop}} \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} \times$$

$$\Bigg\{ [S^{(2)}(x_\perp,y_\perp)]^{2-\text{loop}} \left[\mathcal{H}^{(0)}_{2qq} + \frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{2qq} + \frac{\alpha_s^2}{(2\pi)^2}\mathcal{H}^{(2)}_{2qq}\right]$$

$$+ \int \frac{d^2b_\perp}{(2\pi)^2}[S^{(4)}(x_\perp,b_\perp,y_\perp)]^{1-\text{loop}} \left[\frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{4qq} + \frac{\alpha_s^2}{(2\pi)^2}\mathcal{H}^{(2)}_{4qq}\right]$$

$$+ \int \frac{d^2b}{(2\pi)^2}\frac{d^2\omega}{(2\pi)^2} S^{(6)}(x_\perp,b_\perp,\omega_\perp,y_\perp)\frac{\alpha_s^2}{(2\pi)^2}\mathcal{H}^{(2)}_{6qq} \Bigg\}$$

What to expect at NNLO in pA collisions

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \int \frac{dz}{z^2}\frac{dx}{x}\,\xi\, x\,[q(x,\mu)]^{2-\text{loop}}[D_{h/q}(z,\mu)]^{2-\text{loop}} \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} \times$$
$$\Bigg\{ [S^{(2)}(x_\perp,y_\perp)]^{2-\text{loop}} \left[\mathcal{H}^{(0)}_{2qq} + \frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{2qq} + \frac{\alpha_s^2}{(2\pi)^2}\mathcal{H}^{(2)}_{2qq}\right]$$
$$+ \int \frac{d^2b_\perp}{(2\pi)^2}[S^{(4)}(x_\perp,b_\perp,y_\perp)]^{1-\text{loop}} \left[\frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{4qq} + \frac{\alpha_s^2}{(2\pi)^2}\mathcal{H}^{(2)}_{4qq}\right]$$
$$+ \int \frac{d^2b}{(2\pi)^2}\frac{d^2\omega}{(2\pi)^2} S^{(6)}(x_\perp,b_\perp,\omega_\perp,y_\perp)\frac{\alpha_s^2}{(2\pi)^2}\mathcal{H}^{(2)}_{6qq}\Bigg\}$$

- B-hierarchy at work.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The planar (leading N_c) and non-planar (next-to-leading N_c) contribution to the triple Pomeron vertex has been derived through the Wilson line formalism.
- Pomeron vertices from B-hierarchy.
- In $\mathcal{N}$=4 we obtained the analytic amplitude at NLO.
- Factorization for Inclusive Hadron Production in pA holds at one loop level: the collinear divergence reproduces DGLAP eq. for parton distribution and fragmentation functions, while the rapidity divergence reproduces the Balitsky-BK evolution equation.
- B-hierarchy plays the same role in small-x regime that DGLAP plays at large-x.
- kt Factorization is violated in pA at NLO level.