Feynman Rules for Piecewise Linear Wilson Lines

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Outline

Path-Ordered Exponentials

Wilson Line

$$\mathcal{U}[\mathcal{C}] = \mathcal{P}\,\mathsf{exp}\left(-\mathrm{i}g\int_{\mathcal{C}}\mathrm{d}z^{\mu}\;\;A_{\mu}(z)\right)$$

$$= \mathcal{P}\,\mathsf{exp}\left(-\mathrm{i}g\int_{a}^{b}\mathrm{d}\lambda\;\;\left(z^{\mu}\right)'A_{\mu}(\lambda)\right)$$

Path-Ordered Exponentials

Path-Ordering for linear lines

$$z^\mu = r^\mu + \hat{n}^\mu \lambda \qquad \lambda = a \dots b$$

$$\frac{1}{m!} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} \mathrm{d}z_1 \cdots \mathrm{d}z_m = \int_a^b \int_{\lambda_1}^b \cdots \int_{\lambda_{m-1}}^b \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_m$$

$$= \int_a^b \int_a^{\lambda_m} \cdots \int_a^{\lambda_2} \mathrm{d}\lambda_m \cdots \mathrm{d}\lambda_1$$

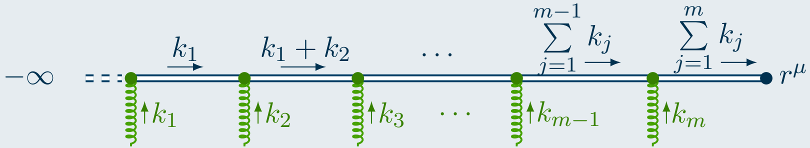
Feynman Rules

Wilson Line Bounded From Above

$$\mathcal{U}_{(r\,;\,-\infty)} = \sum_{m=0}^{\infty} (-\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(k_m)\cdots\hat{n}\cdot A(k_1) \times \cdots$$

$$K(j) = \sum_{l=1}^{j} k_l \qquad \cdots \times \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_{j=1}^{m} \frac{\mathrm{i}}{\hat{n}\cdot K(j) + \mathrm{i}\eta}$$

Feynman Diagram



Feynman Rules

Feynman Rules for Linear Wilson Lines

1) Wilson line propagator:

$$= \frac{\mathrm{i}}{\hat{n}\cdot k + \mathrm{i}\eta}$$

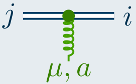
2) external point:

$$= e^{-\mathrm{i} r\cdot k}$$

3) infinite point:

$$= 1 \qquad (k = 0)$$

4) Wilson vertex:



$$= -\mathrm{i} g\, \hat{n}^\mu \, (t^a)_{ij}$$

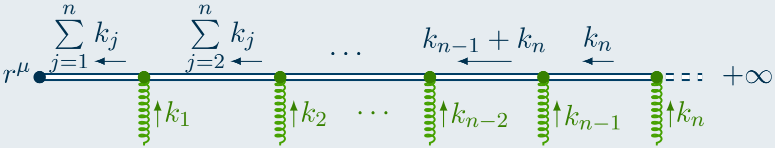
Feynman Rules

Wilson Line Bounded From Below

$$\mathcal{U}_{(+\infty\,;\,r)} = \sum_{m=0}^{\infty} (-\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4} \right)^m \hat{n}\cdot A(k_m) \cdots \hat{n}\cdot A(k_1) \times \cdots$$

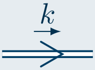
$$\widetilde{K}(j) = \sum_{l=1}^{j} k_{m-l+1} \qquad \cdots \times \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_{j=1}^{m} \frac{-\mathrm{i}}{\hat{n}\cdot\widetilde{K}(j) - \mathrm{i}\eta}$$

Feynman Diagram

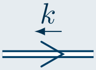


Feynman Rules

Reversals



$$= \frac{\mathrm{i}}{\hat{n}\cdot k + \mathrm{i}\eta}$$



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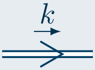
Feynman Rules

Reversals



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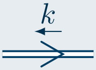

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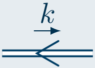

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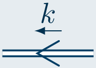

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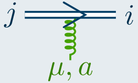
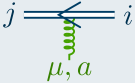


$$= -\mathrm{i}\, g\, \hat{n}^{\mu}\, (t^{a})_{ij}$$



$$= \mathrm{i}\, g\, \hat{n}^{\mu}\, (t^{a})_{ij}$$

More Types

Finite Line

$$\mathcal{U}_{(b\,;\,a)} = \sum_{n=0}^{\infty} (-\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(k_m) \cdots \hat{n}\cdot A(k_1) \times \cdots$$

$$\cdots \times \sum_{l=0}^{m} \mathrm{e}^{-\mathrm{i}a\cdot K(l)} \mathrm{e}^{-\mathrm{i}b\cdot K(m-l)} \prod_{j=1}^{l} \frac{-\mathrm{i}}{\hat{n}\cdot \widetilde{K}(j)} \prod_{j=l+1}^{m} \frac{\mathrm{i}}{n\cdot K(j)}$$

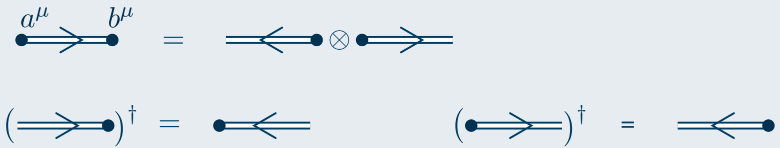
Hermitian Conjugate

$$\mathcal{U}^{\dagger}_{(r\,;\,-\infty)} = \sum_{n=0}^{\infty} (\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(k_1) \cdots \hat{n}\cdot A(k_m) \times \cdots$$

$$\cdots \times \mathrm{e}^{-\mathrm{i}b\cdot K} \prod_{j=1}^{m} \frac{\mathrm{i}}{\hat{n}\cdot K(j) + \mathrm{i}\eta}$$

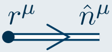
More Types

Finite Lines and Hermitian Conjugates



Different Types

Semi-Infinite Lines and Path Reversals



$$(-\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_j^m \frac{-\mathrm{i}}{\hat{n}\cdot\widetilde{K}(j) - \mathrm{i}\eta} \stackrel{\mathrm{N}}{=} A^m(r,\hat{n})$$

$$(\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_j^m \frac{-\mathrm{i}}{\hat{n}\cdot K(j) - \mathrm{i}\eta} \stackrel{\mathrm{N}}{=} B^m(r,\hat{n})$$

$$(\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_j^m \frac{\mathrm{i}}{\hat{n}\cdot\widetilde{K}(j) + \mathrm{i}\eta} = A^m(r,-\hat{n})$$

$$(-\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{-\mathrm{i}r\cdot K} \prod_j^m \frac{\mathrm{i}}{\hat{n}\cdot K(j) + \mathrm{i}\eta} = B^m(r,-\hat{n})$$

Relation Between A^m and B^m

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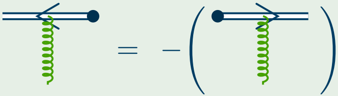
$$B^m(r,\hat{n}) = (-)^m A^m(r,\hat{n})\Big|_{k_1\to k_n,\ldots,k_n\to k_1}$$



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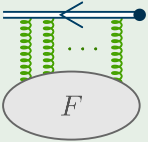


$$= (-)^m \int \left(\frac{\mathrm{d}k_i}{16\pi^4}\right)^m A^m(r,\hat{n}) F^{\mu_1\cdots\mu_m}_{a_1\cdots a_m}(k_m,\ldots,k_1)$$

(absorb gluon propagators in F)

Relation Between A^m and B^m

Symmetrising the Blob

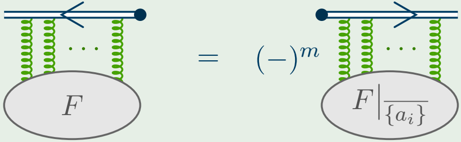
- symmetrise F simultaneously in k_i, μ_i and a_i

  (identical to making all crossings)
- because all Lorentz indices are contracted with the same $\hat{n}^\mu$,

  F is automatically symmetric in μ_i
- interchanging k_i and k_j is thus same as interchanging a_i and a_j
- sometimes F will have straightforward color symmetry

Relation Between A^m and B^m

Symmetrising the Blob



Easy Blob Example: 3-Gluon Vertex

Outline

Piecewise Path-Ordered Exponentials

Wilson Line With M Segments

$$\mathcal{U}(\lambda) = \begin{cases} \mathcal{U}^A(\lambda) & \lambda = a_1 \dots a_2 \\ \mathcal{U}^B(\lambda) & \lambda = a_2 \dots a_3 \\ \vdots & \\ \mathcal{U}^M(\lambda) & \lambda = a_M \dots a_{M+1} \end{cases}$$

Result for Full Wilson Line

$$\mathcal{U}_1 = \sum_{J=1}^{M} \mathcal{U}_1^J$$

$$\mathcal{U}_2 = \sum_{J=1}^{M} \mathcal{U}_2^J + \sum_{K=2}^{M} \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J$$

Piecewise Path-Ordered Exponentials

Result for Full Wilson Line

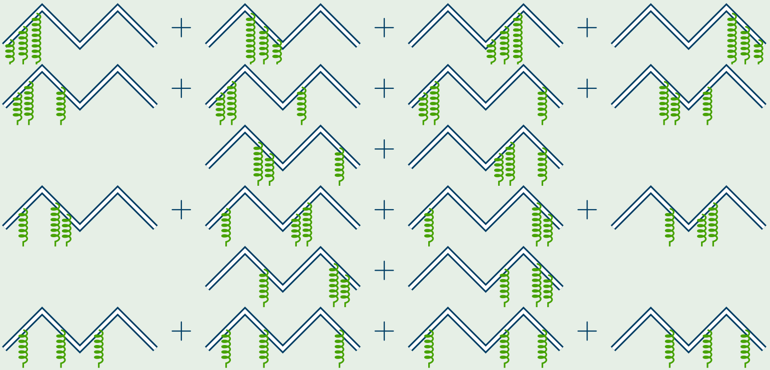
$$\mathcal{U}_3 = \sum_{J=1}^{M} \mathcal{U}_3^J + \sum_{J=2}^{M} \sum_{K=1}^{J-1} \left[\mathcal{U}_1^J \mathcal{U}_2^K + \mathcal{U}_2^J \mathcal{U}_1^K\right] + \sum_{J=3}^{M} \sum_{K=2}^{J-1} \sum_{L=1}^{K-1} \mathcal{U}_1^J \mathcal{U}_1^K \mathcal{U}_1^L$$

$$\vdots$$

$$\mathcal{U}_m = \sum_{i=1}^{m} \left[\left(\prod_{j=1}^{i} \sum_{J_j = i-j+1}^{J_{j-1}-1} \right)_{J_0 - 1 = M} \begin{pmatrix} \text{All terms of the form } \prod_{j=1}^{i} \mathcal{U}_{l_j}^{J_j} \\ \text{such that } \sum_{j=1}^{i} l_j = m \end{pmatrix} \right]$$

Piecewise Path-Ordered Exponentials

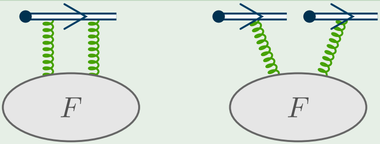
Illustration for $\mathcal{U}_3$



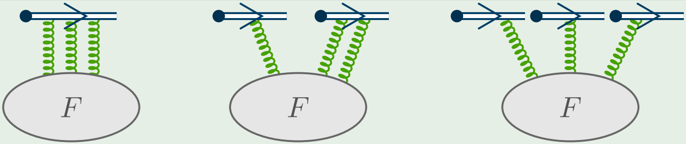
Outline

Basic Diagrams

$m = 2$

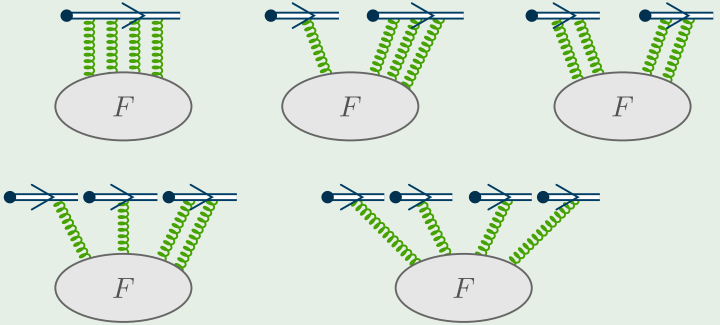


$m = 3$



Basic Diagrams

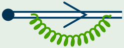
$m = 4$

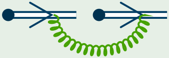
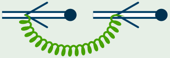
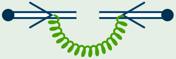


Blob Examples

$m = 2$

 $= \delta^{ab}\,\delta^{(4)}(k_1 - k_2)\,D_{\mu\nu}(k_1)$

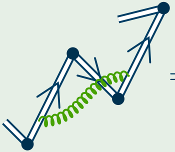
 $=$ 

 $=$  $= -$ 

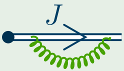
Blob Examples

$m = 2$

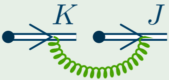
$$\mathcal{U}_2 = \sum_{J=1}^{M} \mathcal{U}_2^J + \sum_{K=2}^{M} \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J$$



$$= \sum_J^M$$



$$+ \sum_K^M \sum_J^{K-1} (-)^{\phi_J + \phi_K}$$

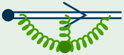
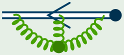


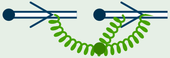
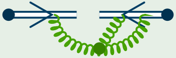
$$\phi_J = \begin{cases} 0 \\ 1 \end{cases}$$

Blob Examples

$m = 3$

 $= g f^{abc} D^{k_1}_{\mu_1\nu_1} D^{k_2}_{\mu_2\nu_2} D^{k_3}_{\mu_3\nu_3} \, g^{\nu_1\nu_2} \, (k_1 - k_2)^{\nu_3}$ +cross.

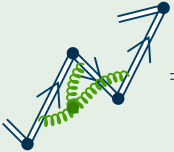
 $=$ 

 $= -$  etc.

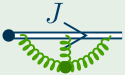
Blob Examples

$m = 3$

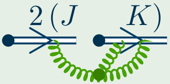
$$\mathcal{U}_3 = \sum_{J=1}^{M} \mathcal{U}_3^J + \sum_{J=2}^{M} \sum_{K=1}^{J-1} \left[\mathcal{U}_1^J \mathcal{U}_2^K + \mathcal{U}_2^J \mathcal{U}_1^K \right] + \sum_{J=3}^{M} \sum_{K=2}^{J-1} \sum_{L=1}^{K-1} \mathcal{U}_1^J \mathcal{U}_1^K \mathcal{U}_1^L$$



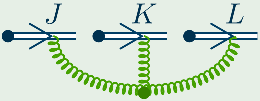
$$= \sum_{J}^{M}$$



$$+ \sum_{K}^{M} \sum_{J}^{K-1} (-)^{\phi_J + \phi_K}$$



$$+ \sum_{J=3}^{M} \sum_{K=2}^{J-1} \sum_{L=1}^{K-1} (-)^{\phi_J + \phi_K + \phi_L}$$

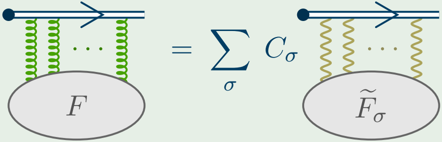


Non-Trivial Color Structure

Blob With Non-Trivial Color Structure

$$F^{a_1 \cdots a_m}_{\mu_1 \cdots \mu_m}(k_1, \ldots, k_m) = \sum_{\text{perm}} C^{\sigma(a_1 \cdots a_m)} \widetilde{F}_{\sigma(\mu_1 \cdots \mu_m)}(\sigma(k_1, \ldots, k_m))$$

Factorize Out Color



$$C_\sigma = t^{a_n} \ldots t^{a_1} C^{\sigma(a_1 \cdots a_m)}$$

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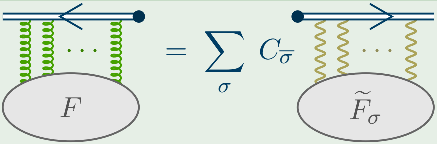

$$C_{\overline{\sigma}} = t^{a_1}\ldots t^{a_n}C^{\sigma(a_1\cdots a_m)}$$

Blob Example With Non-Trivial Color Structure

$m = 4$

$$= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \widetilde{F}_{\mu_1 \cdots \mu_4}(k_1, \ldots, k_4)$$

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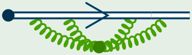
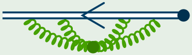
 $=$ 

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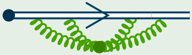
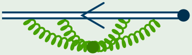
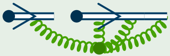
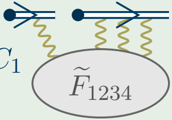
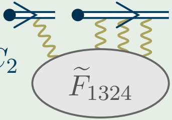
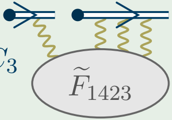
 $= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \widetilde{F}_{\mu_1 \cdots \mu_4}(k_1, \ldots, k_4)$

 $=$ 

 $= C_1$ 

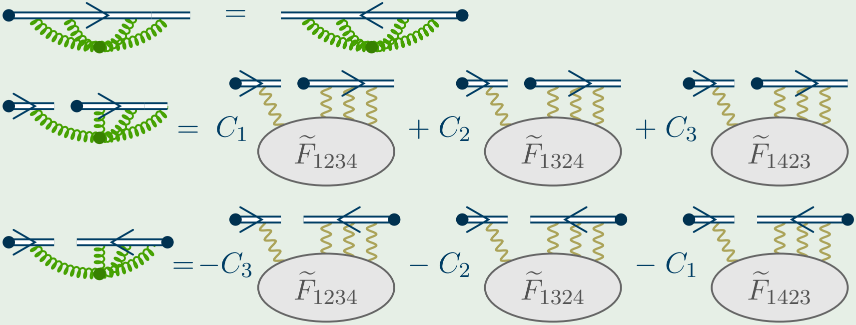
$+ C_2$ 

$+ C_3$ 

Blob Example With Non-Trivial Color Structure

$m = 4$

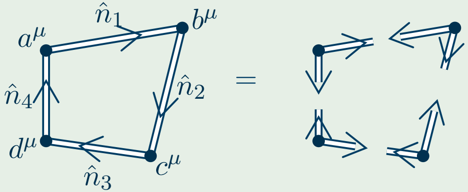
$$= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \widetilde{F}_{\mu_1 \cdots \mu_4}(k_1, \ldots, k_4)$$



Outline

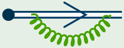
Quadrilateral Wilson Loop

Quadrilateral Wilson Loop

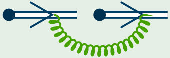


Quadrilateral Wilson Loop

First Order



$$= F(r_J, \hat{n}_J)$$



$$= G(r_J, r_K, \hat{n}_J, \hat{n}_K)$$

Quadrilateral Wilson Loop

First Order

$$\mathcal{U}_2 = \sum_{J=1}^{M} \mathcal{U}_2^J + \sum_{K=2}^{M} \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J$$
$$= \sum_{J}^{M} F(r_J, \hat{n}_J) + \sum_{K=2}^{M} \sum_{J=1}^{K-1} (-)^{J+K} G(r_J, r_K, \hat{n}_J, \hat{n}_K)$$

Result for Light-Like Loop

$$\mathcal{U}_2 = \frac{\alpha_s C_F}{\pi} (-2\pi\mu^2)^\epsilon \, \Gamma(1-\epsilon) \times \cdots$$
$$\cdots \times \left[\frac{1}{\epsilon^2} \left((b-d)^2 - \mathrm{i}\eta\right)^\epsilon + \frac{1}{\epsilon^2} \left((c-a)^2 - \mathrm{i}\eta\right)^\epsilon \right]$$

Conjecture

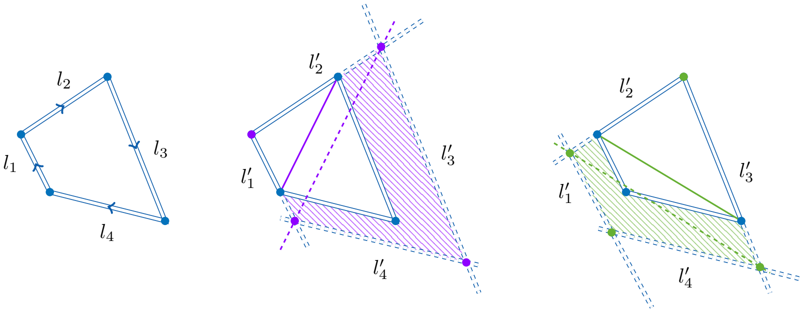
Geometric Evolution of Light-Like Quadrilateral

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\left\langle\frac{\delta}{\delta\ln\Sigma}\right\rangle\ln\mathcal{W}_{\gamma}=-\sum_{\text{cusps}}\Gamma_{\text{cusp}}$$

Gamma cusp at NLO:

$$\Gamma_{\text{cusp}}(g)=\frac{\alpha_s}{\pi}C_F+\left(\frac{\alpha_s}{\pi}\right)^2C_F\left(C_A\left(\frac{67}{36}-\frac{\pi^2}{12}\right)-N_f\frac{5}{18}\right)$$

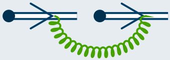
Conjecture



Reusability

Example

$$= \mathrm{i}\, g^2 \frac{C_F}{16\pi} \frac{\hat{n}^2}{\eta} \left(-4\pi\mu^2\right)^{\epsilon} \Gamma(\epsilon) X(\hat{n}^2, \epsilon)$$



$$= g^2 \frac{C_F}{16\pi} \frac{\hat{n}_1 \cdot \hat{n}_2}{\hat{n}_1 \sqrt{\widetilde{n}_2 \cdot \hat{n}_2}} \left(2\pi \mathrm{i}\, \mu^2 \frac{1}{\eta} \sqrt{\frac{R}{N_2}}\right)^{\epsilon} Y_{\epsilon}(\mathrm{i}\eta\sqrt{RN_2})$$

Conclusions

Conclusions & Outlook

- framework to minimize number of diagrams for piecewise linear Wilson lines
- lesser diagrams in exchange for more general (and thus more complicated) integrals
- only interesting for $M > 2$

- calculate higher orders
- include final-state cut
- try framework for TMD Wilson line structure