Measuring Two Angularities on a Single Jet

Andrew Larkoski, Ian Moult, Duff Neill

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Angular and energy resolutions of modern detectors unprecedented.

With high luminosity one can probe increasingly detailed questions about the structure of QCD radiation.

Such detailed questions often require take the form of multi-differential cross-sections, often on a single "sector."
Drell-Yan: $\vec{q}_T$ and Beam Thrust

Process: $pp \rightarrow l^+l^- + X$ and measure:

- Lepton-Pair Transverse Momentum $\vec{q}_T$.
- Lepton-Pair Invariant Mass $Q^2$.
- Beam thrust: $\tau = \sum_i |\vec{p}_{Ti}|e^{-|y_i|}$

\[pp \rightarrow l^+l^- + X \quad \frac{d\sigma}{d\tau \, dq_T^2 \, dQ^2} \quad q_T, \tau \ll Q^2\]
Drell-Yan: $\vec{q}_T$ and Beam Thrust

Factorization in region $\vec{q}_T^2 \sim Q \tau \ll Q^2$:

$$\frac{d\sigma}{d\tau d\vec{q}_T^2 dQ^2} = \sigma_0 \int dY dt_a dt_b d^2 k_{Ta} d^2 k_{Tb} H(Q^2) \delta^{(2)}(\vec{q}_T - \vec{k}_{Ta} - \vec{k}_{Tb})$$

$$B_n(x_a, \vec{k}_{Ta}, t_a) B_{\bar{n}}(x_b, \vec{k}_{Tb}, t_b) S\left(\tau - \frac{e^{-Y} t_a + e^Y t_b}{Q}\right)$$

$$x_{a,b} = \frac{Q}{E_{cm}} e^{\pm Y}$$

1110.0839: Jain, Procura, Waalewijn

See also:
0708.2833 J. Collins, T. Rogers, and A. Stasto,
0807.2430 T. C. Rogers.
Drell-Yan and Fully Unintegrated PDFs

Operators and phase space of region $q_T^2 \sim Q \tau \ll Q^2$:

$$B_n(x, \vec{k}_T, t) = \theta(p^-)\langle p(p^-)\vert \bar{\chi}_n(0)\delta(xp^- - \mathcal{P}^-)\delta^{(2)}(\vec{k}_T - \vec{\mathcal{P}}_T)\delta(t - xp^- \mathcal{P}^+)\chi_n(0)\vert p(p^-)\rangle$$

$$S(\tau) = \frac{1}{N_c} \text{tr} \langle 0 \vert T\{S_n(0)S_n^\dagger(0)\} \delta(\tau - \hat{\tau})\hat{T}\{S_n^\dagger(0)S_n(0)\} \vert 0 \rangle$$

- Resums $\alpha_s^n \ln^m\left(\frac{\tau}{Q}\right)$ terms in perturbation theory.
- $B_n(x, \vec{k}_T, t)$ correctly computes lower boundary only.
Now factorize in region $|\vec{q}_T| \sim \tau \ll Q$

$$\frac{d\sigma}{d\tau d\vec{q}_T^2 dQ^2} = \sigma_0 \int dY d^2k_{Ta}d^2k_{Tb}d^2k_{Ts} H(Q^2)\delta^{(2)}(\vec{q}_T - \vec{k}_{Ta} - \vec{k}_{Tb})$$

$$B_n(x_a, \vec{k}_{Ta})B_{\bar{n}}(x_b, \vec{k}_{Tb})S(\tau, \vec{k}_{Ts})$$

$$x_{a,b} = \frac{Q}{E_{cm}} e^{\pm Y}$$
Drell-Yan and TMD-PDFs

Operators and phase space of region $|\vec{q}_T| \sim \tau \ll Q$:

\[ B_n(x, \vec{k}_T) = \theta(p^-) \langle p(p^-) | \bar{\chi}_n(0) \delta(xp^- - \mathcal{P}^-) \delta(2)(\vec{k}_T - \vec{P}_T) \chi_n(0) | p(p^-) \rangle \]

\[ S(\tau, \vec{k}_T) = \frac{1}{N_c} \text{tr} \langle 0 | T \{ S_n(0) S_n^{\dagger}(0) \} \delta(\tau - \hat{\tau}) \delta(2)(\vec{k}_T - \vec{P}_T) T \{ S_n^{\dagger}(0) S_n(0) \} | 0 \rangle \]

- Resums $\alpha_s^n \ln^m \left( \frac{|\vec{q}_T|}{Q} \right)$ terms in perturbation theory.
- $S(\tau, \vec{k}_T)$ correctly computes upper boundary.

This is an SCET$_{II}$ factorization.

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Drell-Yan: Beam thrust and $\vec{q}_T$ phase space

How do we interface the different factorization descriptions?

$$\frac{d\sigma}{d\tau d^2\vec{q}_T} = H \left\{ B_n(x_a, \vec{q}_T, \tau) \otimes \tau \ B_n(x_b, \vec{q}_T, \tau) \otimes \tau \ S(\tau) \right\}$$

$$= B_n(x_a, \vec{q}_T) \otimes \vec{q}_T \ B(x_b, \vec{q}_T) \otimes \vec{q}_T \ S(\tau, \vec{q}_T)$$
Drell-Yan: Beam thrust and $q_T$ phase space

$\tau - q_T$ phase space interpolates between EFT power countings:

**SCET$_I$**

\[
\begin{align*}
    p_n &\sim Q(1, \lambda^2, \lambda) \\
    p_{us} &\sim Q(\lambda^2, \lambda^2, \lambda^2)
\end{align*}
\]

**SCET$_{II}$**

\[
\begin{align*}
    p_n &\sim Q(1, \lambda^2, \lambda) \\
    p_s &\sim Q(\lambda, \lambda, \lambda)
\end{align*}
\]

- Wrong soft or collinear function in a region $\rightarrow$ wrong phase space boundaries:

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p_n &\sim Q(1, \lambda^2, \lambda) \\
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\end{align*}

$\text{SCET}_{II}$

\begin{align*}
p_n &\sim Q(1, \lambda^2, \lambda) \\
p_s &\sim Q(\lambda, \lambda, \lambda)
\end{align*}

- $\text{SCET}_I$ has only UV renormalization group.
- $\text{SCET}_{II}$ also has rapidity resummation, e.g., CS eqn. 1202.0814, Chiu, DN, Jain, Rothstein

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Measuring Two Angularities on a Single Jet
Towards Multi-Differential Phase Space: Jet Angularities

Consider $e^+e^- \rightarrow \text{hadrons}$, with event shape:

$$e_\beta = \sum_i \frac{E_i}{Q} \left( \sin \left( \frac{\theta_{\hat{n}i}}{2} \right) \right)^\beta \sim \sum_i z_i \theta^\beta_{\hat{n}i}$$

- $\hat{n}$ “recoil-free” jet axes, e.g., axis min. $e_1$
- $e_\beta \ll 1$ selects jet like structure.

1401.2158: Andrew Larkoski, D.N., Jesse Thaler
Minimize axis for $e_1$. Then measure angularity $e_\beta$

$$\frac{d\sigma}{de_\beta} = H \int de_n de_\bar{n} de_s \delta(e_\beta - e_n - e_\bar{n} - e_s) J_n(e_n) J_\bar{n}(e_\bar{n}) S(e_s)$$

No recoil convolution for all $\beta$. 
Modes Found in Factorization Theorem

\[ \frac{d\sigma}{de_\beta} = H \int de_n de_\bar{n} de_s \delta(e_\beta - e_n - e_\bar{n} - e_s) J_n(e_n) J_\bar{n}(e_\bar{n}) S(e_s) \]

\[ \lambda \sim e_\beta \text{ and } p = (\bar{n} \cdot p, n \cdot p, p_\perp) \]

\[ p_n \sim Q(1, \lambda^{\frac{2}{\beta}}, \lambda^{\frac{1}{\beta}}) \]

\[ p_s \sim Q(\lambda, \lambda, \lambda) \]

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Measuring Two Angularities on a Single Jet
In Laplace space \((e^\beta \rightarrow s^\beta)\):

\[
\mu \frac{d}{d\mu} \ln J_n(s^\beta, \frac{\mu}{Q}) = -\frac{\Gamma[\alpha_s(\mu)]}{1 - \beta} \ln \left(\frac{s^\beta \mu^\beta}{Q^{1 - \beta}}\right) + \gamma J
\]

\[
\mu \frac{d}{d\mu} \ln S(s^\beta, \frac{\mu}{Q}) = 2\frac{\Gamma[\alpha_s(\mu)]}{1 - \beta} \ln \left(\frac{s^\beta \mu}{Q}\right) + \gamma S
\]

Canonical Scales:

\[
\mu_J^2 \sim Q^2 \lambda^2
\]

\[
\mu_S^2 \sim Q^2 \lambda^2
\]
In Laplace space \((e^\beta \rightarrow s^\beta)\):

\[
\mu \frac{d}{d\mu} \ln J_n(s^\beta, \frac{\mu}{Q}) = -\frac{\Gamma[\alpha_s(\mu)]}{1-\beta} \ln \left( s^\beta \frac{\mu^\beta}{Q^\beta} \right) + \gamma_J
\]

\[
\mu \frac{d}{d\mu} \ln S(s^\beta, \frac{\mu}{Q}) = 2\frac{\Gamma[\alpha_s(\mu)]}{1-\beta} \ln \left( s^\beta \frac{\mu}{Q} \right) + \gamma_S
\]

Neighborhood of Canonical Scales:

\[
\mu_J = Q \left( \frac{\nu}{\mu_S} \right)^{\frac{1-\beta}{\beta}} \left( \frac{\mu S}{Q} \right)^{\frac{1}{\beta}}
\]

\[
\frac{\nu}{\mu_S} \sim O(1)
\]
\[ \ln \left( \frac{S(s_{\beta}, \frac{\mu_J}{Q})}{S(s_{\beta}, \frac{\mu_S}{Q})} \right) = \frac{2}{1 - \beta} \int_{\mu_s}^{\mu_J} \frac{d\mu}{\mu} \Gamma[\alpha_s(\mu)] \ln \left( s_{\beta} \frac{\mu}{Q} \right) \]

\[ =_{\beta \to 1} 2 \Gamma[\alpha_s(\mu_S)] \ln \left( s_{\beta} \frac{\mu_S}{Q} \right) \ln \left( \frac{\nu}{Q} \right) \]

\[ \mu_J = Q \left( \frac{\nu}{\mu_S} \right)^{\frac{1-\beta}{\beta}} \left( \frac{\mu_S}{Q} \right)^{\frac{1}{\beta}} \]
Rapidity Resummation from UV-Soft running!

\[
\ln \left( \frac{S(s_\beta, \frac{\mu_J}{Q})}{S(s_\beta, \frac{\mu_S}{Q})} \right) = \frac{2}{1 - \beta} \int_{\mu_s}^{\mu_J} \frac{d\mu}{\mu} \Gamma[\alpha_s(\mu)] \ln \left( s_\beta \frac{\mu}{Q} \right)
\]

\[
= \beta \to 1 \ 2 \Gamma[\alpha_s(\mu_S)] \ln \left( s_\beta \frac{\mu_S}{Q} \right) \ln \left( \frac{\nu}{Q} \right)
\]

CS Kernel

\[
\mu_J = Q \left( \frac{\nu}{\mu_S} \right)^{\frac{1 - \beta}{\beta}} \left( \frac{\mu_S}{Q} \right)^{\frac{1}{\beta}}
\]
Measuring two Angularities

- Measure two angularities $e_\alpha, e_\beta, \alpha > \beta$.
- What is the phase space?

Single emission contributes as:

\[
e_\alpha \sim z \theta^\alpha \\
e_\beta \sim z \theta^\beta
\]

Soft: $z$ sets $e_\alpha \sim e_\beta \ll 1$, and $\theta \sim 1$

Collinear: $\theta$ sets $\left( e_\alpha \right)^{\frac{1}{\alpha}} \sim \left( e_\beta \right)^{\frac{1}{\beta}} \ll 1$, and $z \sim 1$
Two Boundaries of phase space, two factorization regions:

Region $\beta$: $e_\alpha \sim e_\beta$.
Region $\alpha$: $e_\alpha \sim e_\beta^{\alpha \beta}$.
Region $\alpha$: $e_\alpha \sim e_\beta^{\alpha/\beta}$.

- Set by conservation of energy: $\frac{e_\alpha^{1/\alpha}}{e_\beta^{1/\beta}} \sim z^{1/\alpha - 1/\beta}$, $0 < z < 1$
Region $\beta$: $e_{\alpha} \sim e_{\beta}$.

- Set by angular size of jets: $\frac{e_{\alpha}}{e_{\beta}} \sim \theta^{\alpha - \beta} < \frac{\pi}{2}$

![Graph showing the boundaries of phase-space regions for $\alpha$ and $\beta$.](image-url)
Soft and Collinear contributions:

\[ e_{\alpha} \sim z_s \theta_s^\alpha + z_c \theta_c^\alpha \]
\[ e_{\beta} \sim z_s \theta_s^\beta + z_c \theta_c^\beta \]

Since \( e_{\alpha} \approx e_{\beta} \) and \( \theta_c^\alpha \ll \theta_c^\beta \), Collinears power suppressed in \( e_{\alpha} \):

\[
\frac{d\sigma}{de_{\alpha} de_{\beta}} = HJ(e_{\beta}) \otimes_\beta S(e_{\alpha}, e_{\beta})
\]
Region $\beta$: $e_\alpha \sim e_\beta$

Soft and Collinear contributions:

\[
e_\alpha \sim z_s \theta^\alpha_s + O(z_c \theta^\alpha_c)
\]

\[
e_\beta \sim z_s \theta^\beta_s + z_c \theta^\beta_c
\]

Since $e_\alpha \sim e_\beta$ and $\theta^\alpha_c \ll \theta^\beta_c$,

Collinears power suppressed in $e_\alpha$:

\[
\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\beta) \otimes_\beta S(e_\alpha, e_\beta)
\]
Soft and Collinear contributions:

\[ e_\alpha \sim z_s \theta_s^\alpha + z_c \theta_c^\alpha \]
\[ e_\beta \sim z_s \theta_s^\beta + z_c \theta_c^\beta \]

Since \( e_\alpha \sim e_\beta^\alpha \rightarrow e_\alpha \ll e_\beta \),

Softs power suppressed in \( e_\beta \):

\[
\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\alpha, e_\beta) \otimes_\alpha S(e_\alpha)
\]
Region $\alpha$: $e_\alpha \sim e_\beta^\alpha$

Soft and Collinear contributions:

$$e_\alpha \sim z_s \theta_s^\alpha + z_c \theta_c^\alpha$$

$$e_\beta \sim z_c \theta_c^\beta + O(z_s \theta_s^\beta)$$

Since $e_\alpha \sim e_\beta^\alpha \rightarrow e_\alpha \ll e_\beta$,
Softs power suppressed in $e_\beta$:

$$\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\alpha, e_\beta) \otimes_\alpha S(e_\alpha)$$
Jet and soft functions

Single differential functions:

\[ J(e_\beta) = \frac{(2\pi)^3}{N_c} \langle 0 | \bar{\chi}_\bar{n} \delta(n \cdot \hat{P} - Q) \delta(\hat{e}_\beta - e_\beta) \delta^{(2)}(\hat{P}_\perp) \frac{\hbar}{2} \chi_\bar{n} | 0 \rangle \]

\[ S(e_\alpha) = \frac{1}{N_c} \text{tr} \langle 0 | T \{ S^\dagger_{\bar{n}} S_{\bar{n}} \} \delta(\hat{e}_\alpha - e_\alpha) \bar{T} \{ S^\dagger_{\bar{n}} S_{\bar{n}} \} | 0 \rangle \]

Double Differential functions:

\[ J(e_\alpha, e_\beta) = \frac{(2\pi)^3}{N_c} \langle 0 | \bar{\chi}_\bar{n} \delta(n \cdot \hat{P} - Q) \delta(\hat{e}_\alpha - e_\alpha) \delta(\hat{e}_\beta - e_\beta) \delta^{(2)}(\hat{P}_\perp) \frac{\hbar}{2} \chi_\bar{n} | 0 \rangle \]

\[ S(e_\alpha, e_\beta) = \frac{1}{N_c} \text{tr} \langle 0 | T \{ S^\dagger_{\bar{n}} S_{\bar{n}} \} \delta(\hat{e}_\alpha - e_\alpha) \delta(\hat{e}_\beta - e_\beta) \bar{T} \{ S^\dagger_{\bar{n}} S_{\bar{n}} \} | 0 \rangle \]
Double differential functions renormalize exactly the same as single differential:

\[
\mu \frac{d}{d\mu} \ln J(e_\alpha, e_\beta) = \delta(e_\beta) \mu \frac{d}{d\mu} \ln J(e_\alpha)
\]

\[
\mu \frac{d}{d\mu} \ln S(e_\alpha, e_\beta) = \delta(e_\alpha) \mu \frac{d}{d\mu} \ln S(e_\beta)
\]

\[
\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\alpha, e_\beta) \otimes_\alpha S(e_\alpha)
\]

\[
\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\beta) \otimes_\beta S(e_\alpha, e_\beta)
\]
Region $\beta$: Factorization resums $\alpha_s^n \ln^m(e_\beta)$ like $\frac{d\sigma}{de_\beta}$.
Region $\alpha$: Factorization resums $\alpha_s^n \ln^m(e_\alpha)$ like $\frac{d\sigma}{de_\alpha}$. 
Double Differential Angularities: Interpolating Resummation

- No naive factorization description covering all phase space.
- How do we move from one boundary to the other smoothly, while resumming all large logarithms?
Use Two Facts:

- Boundary conditions on the double cumulative distribution.
- The structure of single differential x-sec resummation.
Consider cumulative distribution:

\[
\Sigma(e_\alpha, e_\beta) = \int_{e_\alpha}^{e_\alpha'} \int_{e_\beta}^{e_\beta'} \frac{d\sigma}{de_\alpha' de_\beta'}
\]
Double Differential Angularities: Interpolating Resummation

Boundary conditions on cumulative distribution:

\[ \Sigma(e_\alpha, e_\beta = e_\alpha^{\beta/\alpha}) = \Sigma(e_\alpha) \], \[ \Sigma(e_\alpha = e_\beta, e_\beta) = \Sigma(e_\beta) \]

\[ \frac{\partial}{\partial e_\alpha} \Sigma(e_\alpha, e_\beta) \bigg|_{e_\beta = e_\alpha^{\beta/\alpha}} = \frac{d\sigma}{de_\alpha}(e_\alpha) \], \[ \frac{\partial}{\partial e_\beta} \Sigma(e_\alpha, e_\beta) \bigg|_{e_\alpha = e_\beta} = \frac{d\sigma}{de_\beta}(e_\beta) \]

\[ \frac{\partial}{\partial e_\alpha} \Sigma(e_\alpha, e_\beta) \bigg|_{e_\beta = e_\alpha} = 0 \], \[ \frac{\partial}{\partial e_\beta} \Sigma(e_\alpha, e_\beta) \bigg|_{e_\beta = e_\beta^{\alpha/\beta}} = 0 \]
Structure of Resummation of Single Differential

Using RG equations of factorized cross-section:

\[
\Sigma(e_\beta) = \int_0^{e_\beta} de'_\beta \frac{d\sigma}{de'_\beta} = \frac{e^{-\gamma E R'(e_\beta)}}{\Gamma(1 + R'(e_\beta))} e^{-R(e_\beta) - T(e_\beta)}
\]

- \( R(e_\beta) \) is the radiator.
- \( R(e_\beta) \) has an exact (Laplace space) description in terms of the integrals over cusp parts of hard, jet, soft anomalous dimensions.
- Fact. Theorems predict \( \Sigma(e_\alpha, e_\beta) \) to have exact same form near a boundary.
Ansatz for Resummation of Double Differential (At Least to NLL)

Using single differential x-sec form, make ansatz:

\[ \Sigma(e_\alpha, e_\beta) = e^{-\gamma E \tilde{R}(e_\alpha,e_\beta)} e^{-R(e_\alpha,e_\beta)-T(e_\alpha,e_\beta)} \frac{e^{-\gamma E \tilde{R}(e_\alpha,e_\beta)}}{\Gamma(1 + \tilde{R}(e_\alpha, e_\beta))} \]

- Demand for satisfy boundary conditions.
- Thus reduces exactly to single differential resummation on boundary.
Deriving expressions for $\tilde{R}(e_\alpha, e_\beta)$ and $R(e_\alpha, e_\beta)$:

- Start from $R(e_\beta)$ as sum of cusp pieces of hard, jet and soft anom. dim.

- Re-arrange jet and soft parts as linear combinations with $O(1)$ changes of canonical RG destinations.

- e.g., $\mu J \rightarrow \mu J \left(\frac{e_\beta}{e_\alpha}\right)^c$ in region-$\alpha$

- Apply boundary conditions.
One-Loop Result for Double Differential Radiator Ansatz

\[ R^{(1)}(e_\alpha, e_\beta) = \frac{C_i}{2\pi\alpha_s\beta_0^2} \left[ \frac{1}{\alpha - 1} U \left( 2\alpha_s\beta_0 \log e_\alpha \right) - \frac{\beta}{\beta - 1} U \left( 2\alpha_s\beta_0 \frac{\log e_\beta}{\beta} \right) \right. \]
\[ \left. + \frac{\alpha - \beta}{(\alpha - 1)(\beta - 1)} U \left( 2\alpha_s\beta_0 \frac{\log e_\alpha^{1-\beta} e_\beta^{\alpha-1}}{\alpha - \beta} \right) \right] , \]

\[ U(x) = x \ln x \]
New logarithmic structure not appearing in either factorization theorem:

\[ \log \frac{e^{1-\beta} e^{\alpha-1}}{\alpha - \beta} \sim \log(z\theta) \sim \log \left( \frac{k_T}{Q} \right) \]

Interpolates between double differential soft and jet functions:

- \( e_\beta \sim e^{\beta/\alpha} \)
- \( e_\alpha \sim e_\beta \)

\[ J(e_\alpha, e_\beta) \quad S(e_\alpha) \quad J(e_\beta) \quad S(e_\alpha, e_\beta) \]
New logarithmic structure not appearing in either factorization theorem:

\[
\log \frac{e^{1-\beta} e^{\alpha-1}}{e^{\alpha-\beta}} \sim \log(z\theta) \sim \log \left( \frac{k_T}{Q} \right)
\]

Interpolates between double differential soft and jet functions:

- \( J(e_{\alpha}, e_{\beta}) \)
- \( S(e_{\alpha}) \)
- \( e_{\beta} \sim e_{\alpha}^{\beta/\alpha} \)
- \( e_{\alpha} \sim e_{\beta} \)
- \( \log e_{\beta}^{1/\beta} \)
- \( \log e_{\alpha}^{\frac{1-\beta}{\alpha}} e_{\beta}^{\frac{\alpha-1}{\beta}} \)
- \( \log e_{\alpha} \)

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Note: Pure resummation (NLL) not sufficient to reproduce boundary behavior, need low scale matrix elements.
Possible Future Directions

- Justify/Disprove/Correct Ansatz. (Progress in this direction has been made...)
- Include low-scale matrix elements with resummation (NLL').
- Perform similar analysis for TMD-PDF versus FU-PDF factorizations.
- Analyze all phase-space factorizations for more complicated multiple differential observables.