Evolution equations beyond one loop from conformal symmetry

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based on

V. Braun, A. Manashov, Eur. Phys. J. C **73** *(2013) 2544 [arXiv:1306.5644 [hep-th]]*
V. Braun, A. Manashov, arXiv:1404.0863 [hep-ph]

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Conformal symmetry of QCD is broken by quantum corrections

— is it lost completely or still implies smth useful ?

Usual (vague) conjecture:

$$\mathcal{Q} = \mathcal{Q}^{\mathrm{conformal}} + \frac{\beta(g)}{g}\Delta\mathcal{Q}$$

- **Generalized Crewther relation** Broadhurst, Kataev, ...
- **NLO GPD evolution equations** Müller, Belitsky& Müller

  — off-diagonal elements of the NLO mixing matrix are determined by one-loop special conformal anomaly
- ...

Our motivation:

- Make precise the separation of "conformal part" and "corrections"
- Emphasize "conformal symmetry friendly" representation of the results
- Explore the road to NNLO (three-loop) evolution equations for GPDs

Difference to D.Müller:

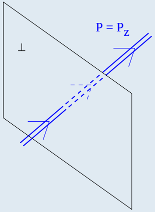
Instead of considering consequences of broken conformal symmetry in QCD we make use of exact conformal symmetry of a modified theory: QCD in $4-2\epsilon$ dimensions at critical coupling

$$\beta^{QCD}(a_s) = 2a_s\left[-\epsilon - \beta_0 a_s + \ldots\right] \qquad a_s^* = -4\pi\epsilon/\beta_0 + \ldots \qquad \beta^{QCD}(a_s^*) = 0$$

Premium:

- Exact symmetry $\Rightarrow$ algebraic group-theory methods
- Answer obtained directly in $\overline{\mathrm{MS}}$ scheme
- Light-ray operator basis, do not need to restore evolution eqs. from local operators.

Collinear Subgroup



$$p_+ = \frac{1}{\sqrt{2}}(p_0 + p_z) \to \infty$$
$$p_- = \frac{1}{\sqrt{2}}(p_0 - p_z) \to 0$$
$$px \to p_+ x_-$$

- Special conformal transformation
$$x_- \to x'_- = \frac{x_-}{1+2ax_-}$$
- translations $x_- \to x'_- = x_- + c$
- dilatations $x_- \to x'_- = \lambda x_-$

form the so-called **collinear subgroup** $SL(2,R)$

$$\alpha \to \alpha' = \frac{a\,\alpha + b}{c\,\alpha + d}, \quad ad - bc = 1$$
$$\Phi(\alpha) \to \Phi'(\alpha) = (c\,\alpha + d)^{-2j}\Phi\left(\frac{a\alpha + b}{c\alpha + d}\right)$$

where $\Phi(x) \to \Phi(x_-) = \Phi(\alpha n_-)$ is the quantum field with scaling dimension ℓ and spin projection s "living" on the light-ray

Conformal spin:

$$j = (l + s)/2$$

Light-ray operators

- There are two possibilities to specify leading twist operators
  1. by an operator function of field coordinates on the light cone $n^2 = 0$

$$[\mathcal{O}](z_1, z_2) \quad \equiv \quad [\bar{q}(z_1 n) \not{n} q(z_2 n)] \; \equiv \sum_{m,k} \frac{z_1^m z_2^k}{m!k!} [(D_n^m \bar{q})(0) \not{n} (D_n^k q)(0)]$$

  2. by a polynomial that details the number of derivatives in local operators

$$[(D_n^m \bar{q})(0) \not{n} (D_n^k q)(0)] \quad = \quad P(\partial_1, \partial_2)[\mathcal{O}](z_1, z_2)|_{z_1 = z_2 = 0}\,, \qquad P(u_1, u_2) = u_1^m u_2^k$$

- Generators of conformal transformations act differently in these representations

$$\begin{aligned} S_+^{(0)} &= z_1^2 \partial_{z_1} + z_2^2 \partial_{z_2} + 2(z_1 + z_2) \\ S_0^{(0)} &= z_1 \partial_{z_1} + z_2 \partial_{z_2} + 2 \\ S_-^{(0)} &= -\partial_{z_1} - \partial_{z_2} \end{aligned}$$

$$\begin{aligned} \widetilde{S}_+^{(0)} &= -u_1 - u_2 \\ \widetilde{S}_0^{(0)} &= u_1 \partial_{u_1} + u_2 \partial_{u_2} + 2 \\ \widetilde{S}_-^{(0)} &= u_1 \partial_{u_1}^2 + u_2 \partial_{u_2}^2 + 2(\partial_{u_1} + \partial_{u_2}) \end{aligned}$$

- Conformal light-ray operators

$$S_-^{(0)} f = 0 \;\Rightarrow\; f = (z_1 - z_2)^n$$

- Conformal local operators

$$\widetilde{S}_-^{(0)} \tilde{f} = 0 \;\Rightarrow\; \tilde{f} = C_n^{3/2}(u_1 - u_2)$$

- Light-ray operator representation is “natural” for conformal symmetry

Light-ray operators satisfy the RG equation Balitsky, Braun '89

$$\Big(M\partial_M + \beta(g)\partial_g + \mathbb{H}\Big)\,[\mathcal{O}(z_1,z_2)] = 0$$

where $\mathbb{H}$ is an integral operator acting on the light-cone coordinates of the fields:

$$\mathbb{H}[\mathcal{O}](z_1,z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h(\alpha,\beta)\,[\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

$$z_{12}^\alpha \equiv z_1\bar\alpha + z_2\alpha$$
$$\bar\alpha = 1-\alpha$$

One can show that the powers $[\mathcal{O}](z_1,z_2) \mapsto (z_1-z_2)^N$ are eigenfunctions of $\mathbb{H}$, and the corresponding eigenvalues are the anomalous dimensions of local operators of spin N (with $N-1$ derivatives)

$$\gamma_N = \int d\alpha d\beta\, h(\alpha,\beta)(1-\alpha-\beta)^{N-1}$$

One-loop evolution equations

- **Expect**

$$[\mathbb{H}^{(1)}, S_\alpha^{(0)}] = 0 \qquad \Longrightarrow \qquad h^{(1)}(\alpha,\beta) = \bar{h}(\tau)\,, \qquad \boxed{\tau = \frac{\alpha\beta}{\bar\alpha\bar\beta}}$$

i.e. function of **two** variables reduces to a function of **one** variable

$\longrightarrow$ can be restored from anomalous dimensions

- **Confirmed by a direct calculation**

$$h^{(1)}(\alpha,\beta) = -4C_F\left[\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\right]$$

where

$$\int d\alpha d\beta\, \delta_+(\tau) f(z_{12}^\alpha, z_{21}^\beta) \equiv \int_0^1 d\alpha \int_0^1 d\beta\, \delta(\tau)\left[f(z_{12}^\alpha, z_{21}^\beta) - f(z_1,z_2)\right]$$
$$= -\int_0^1 d\alpha\, \frac{\bar\alpha}{\alpha}\left[2f(z_1,z_2) - f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\right]$$

- **Combines DGLAP, ERBL and GPD evolution equations in the most compact form**

Beyond one loop in $d = 4 - 2\epsilon$ at critical coupling

- **Exact conformal symmetry, but the generators are modified by quantum corrections**

$$
\begin{aligned}
S_- &= S_-^{(0)}\,,\\
S_0 &= S_0^{(0)} - \epsilon + \frac{1}{2}\mathbb{H}(a_s^*)\,, \qquad \mathbb{H}(a_s^*) = a_s^*\,\mathbb{H}^{(1)} + \ldots\\
S_+ &= S_+^{(0)} + (z_1+z_2)\Big(-\epsilon + \frac{1}{2}a_s^*\mathbb{H}^{(1)}\Big) + a_s^*(z_1-z_2)\Delta_+ + \mathcal{O}(\epsilon^2)\,,
\end{aligned}
$$

$$a_s = \frac{\alpha_s^*}{4\pi}$$

where

arXiv:1404.0863

$$\Delta_+[\mathcal{O}](z_1,z_2) = -2C_F\int_0^1 d\alpha\Big(\frac{\bar\alpha}{\alpha} + \ln\alpha\Big)\Big[[\mathcal{O}](z_{12}^\alpha, z_2) - [\mathcal{O}](z_1, z_{21}^\alpha)\Big]$$

- **Modified generators satisfy usual SL(2) commutation relations**

$$[S_0, S_\pm] = \pm S_\pm\,, \qquad [S_+, S_-] = 2S_0$$

Beyond one loop in $d = 4 - 2\epsilon$ at critical coupling (II)

- Expanding the commutation relations in powers of a_s^*

$$[S_+^{(0)}, \mathbb{H}^{(1)}] = 0\,,$$
$$[S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, S_+^{(1)}]\,,$$
$$[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(1)}, S_+^{(2)}] + [\mathbb{H}^{(2)}, S_+^{(1)}]\,, \qquad \text{etc.}$$

- A nested set of inhomogenious first order differential equations for $\mathbb{H}^{(1)}$
  Their solution determines $\mathbb{H}^{(k)}$ up to an $SL(2)$-invariant term
- The r.h.s. involves $\mathbb{H}^{(k)}$ and $S_+^{(m)}$ at one oder less compared to the l.h.s. D.Müller

- Back to the $d = 4$ world: In MS scheme $\mathbb{H}$ does not depend on ϵ

$$\mathbb{H}(a_s^*) = a_s^*\, \mathbb{H}^{(1)} + (a_s^*)^2\, \mathbb{H}^{(2)} + \ldots \qquad \Big(\mu\partial_\mu + \mathbb{H}(a_s^*)\Big)[\mathcal{O}](z_1, z_2) = 0$$

$\Longleftrightarrow$

$$\mathbb{H}(a_s) = a_s\, \mathbb{H}^{(1)} + a_s^2\, \mathbb{H}^{(2)} + \ldots \qquad \Big(\mu\partial_\mu + \beta(g)\partial_g + \mathbb{H}(a_s)\Big)[\mathcal{O}(z_1, z_2)] = 0$$

Reexpanding $a_s^* = a_s^*(\epsilon) \to \epsilon = \epsilon(a_s^*)$ the kernel in $d = 4$ for arbitrary coupling can be restored from the kernel in $d = 4 - 2\epsilon$ at the critical coupling

- **Procedure tested**
  - Scalar $O(N)$ ϕ^4 theory in $d=4$ in three loops arXiv:1306.5644
  - Scalar $SU(N)$ ϕ^3 theory in $d=6$ in two loops arXiv:1306.5644
  - QCD flavor-nonsinglet in two loops arXiv:1404.0863

- **For a NLO calculation one needs**
  - **Calculate** $S_+ = S_+^{(0)} + a_s^* S_+^{(1)}$ **(one loop) from the conformal Ward identity**
  - **Find a particular solution to the differential equation**

$$[S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, S_+^{(1)}]$$

$\hookleftarrow$ **non-invariant part of the evolution equation**

  - **Restore invariant part (solution of** $[S_+^{(0)}, \mathbb{H}^{(2)}] = 0$**) from known anomalous dimensions**

In QCD

$$h^{(2)}(\alpha,\beta) = 8C_F^2 h_1^{(2)}(\alpha,\beta) + 4C_F C_A h_2^{(2)}(\tau) + 4b_0 C_F h_3^{(2)}(\alpha,\beta)$$

Special conformal Ward identity

- **Conformal symmetry at the critical coupling implies**

$$\Big(S_+^{(z)} - \frac{1}{2}x^2(\bar n\partial_x)\Big)\Big\langle [\mathcal{O}_n](z_1,z_2), [\mathcal{O}_{\bar n}](x,w_1,w_2)\Big\rangle = 0 \qquad \boxed{(nx) = (\bar n x) = 0}$$

- **To find explicit expression for S_+, consider Ward identity**

$$\Big\langle \delta_+[\mathcal{O}^{(n)}](z)\,[\mathcal{O}^{(\bar n)}](x,w)\Big\rangle + \Big\langle [\mathcal{O}^{(n)}](z)\,\delta_+[\mathcal{O}^{(\bar n)}](x,w)\Big\rangle - \Big\langle \delta_+ S_R\,[\mathcal{O}^{(n)}](z)\,[\mathcal{O}^{(\bar n)}](x,w)\Big\rangle = 0$$

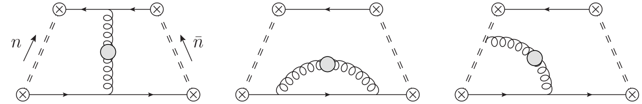
Then

$$\delta_+[\mathcal{O}^{(\bar n)}](x;w_1,w_2) = -x^2(\bar n\partial_x)[\mathcal{O}^{(\bar n)}](x;w_1,w_2)$$

$$\delta_+[\mathcal{O}^{(n)}](0;z_1,z_2) = 2(n\bar n)\Big(S_+^{(0)} - \epsilon(z_1+z_2) - \frac{a_s}{2}[\mathbb{H}^{(1)}, z_1+z_2)]\Big)[\mathcal{O}^{(n)}](0;z_1,z_2)$$

and for the last term

$$\delta_+ S^{QCD} = 4\epsilon\int d^dx(x\bar n)L^{QCD} + 2(d-2)\bar n^\mu\int d^dx\,\delta_{BRST}(\bar c^a A^a_\mu).$$



Two-loop evolution kernels

$$
\begin{aligned}
h_1^{(2)}(\alpha,\beta) &= -\delta_+(\tau)\Big(\phi(\alpha)+\phi(\beta)\Big) + \varphi(\alpha,\beta) + \theta(\bar\tau)\left[2\,\mathrm{Li}_2(\tau) + \ln^2\bar\tau + \ln\tau - \frac{1+\bar\tau}{\tau}\ln\bar\tau\right] \\
&\quad + \theta(-\bar\tau)\left[\ln^2(-\bar\tau/\tau) - \frac{2}{\tau}\ln(-\bar\tau/\tau)\right] + \left(-6\zeta(3) + \frac{1}{3}\pi^2 + \frac{21}{8}\right)\delta(\alpha)\delta(\beta)\,, \\
h_2^{(2)}(\alpha,\beta) &= \frac{1}{3}\left(\pi^2-4\right)\delta_+(\tau) - 2\theta(\bar\tau)\left[\mathrm{Li}_2(\tau) - \mathrm{Li}_2(1) + \frac{1}{2}\ln^2\bar\tau - \frac{1}{\tau}\ln\bar\tau + \frac{5}{3}\right] \\
&\quad - \theta(-\bar\tau)\left[\ln^2(-\bar\tau/\tau) - \frac{2}{\tau}\ln(-\bar\tau/\tau)\right] + \left(6\zeta(3) - \frac{2}{3}\pi^2 + \frac{13}{6}\right)\delta(\alpha)\delta(\beta)\,, \\
h_3^{(2)}(\alpha,\beta) &= -\delta_+(\tau)\left[\ln\bar\alpha + \ln\bar\beta + \frac{5}{3}\right] - \theta(\bar\tau)\left[\ln(1-\alpha-\beta) + \frac{11}{3}\right] + \frac{13}{12}\delta(\alpha)\delta(\beta)\,.
\end{aligned}
$$

where

$$
\phi(\alpha) = -\ln\bar\alpha\left[\frac{3}{2} - \ln\bar\alpha + \frac{1+\bar\alpha}{\bar\alpha}\ln\alpha\right],
$$

$$
\begin{aligned}
\varphi(\alpha,\beta) = -\theta(1-\tau)\Big[&\frac{1}{2}\ln^2(1-\alpha-\beta) + \frac{1}{2}\ln^2\bar\alpha + \frac{1}{2}\ln^2\bar\beta - \ln\alpha\ln\bar\alpha - \ln\beta\ln\bar\beta \\
&- \frac{1}{2}\ln\alpha - \frac{1}{2}\ln\beta + \frac{\bar\alpha}{\alpha}\ln\bar\alpha + \frac{\bar\beta}{\beta}\ln\bar\beta\Big],
\end{aligned}
$$

Conclusions

- QCD evolution equations possess a "hidden" conformal symmetry
- n-loop evolution kernels for twist-two operators in MS scheme can be restored from the $(n-1)$-loop calculation of the special conformal anomaly and n-loop anomalous dimensions
- SL(2) symmetry properties manifest in the light-ray operator representation